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Some new fractional integral inequalities for s-convex functions

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Abstract

In this paper, a similar equality which is given in [Ç. Yıldız, M. E. Özdemir, M. Z. Sarıkaya, Kyungpook Math. J., **56** (2016), 161–172] is proved by using different symbols and impressions. By using this equality, some new fractional integral inequalities for s-convex functions are obtained. Also, some applications to special means of positive real numbers are given. If the $\alpha = 1$ is taken, our results coincide with the results given in [E. Set, M. E. Özdemir, M. Z. Sarıkaya, Facta Unv. Ser. Math. Inform., **27** (2012), 67–82] so our results are more general from the results given there.

Keywords: Ostrowski type inequalities, midpoint type inequalities, Riemann-Liouville fractional integrals, s-convex functions.
2010 MSC: 26A51, 26A33, 26D10.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [14].

Theorem 1.1 ([6]). *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with the property that* $|f'(t)| \leqslant M$ *for all* $t \in (a, b)$. *Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leqslant (b-a) M \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \tag{1.1}$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible, it means that it cannot be replaced by a smaller constant.
The inequality (1.1) can be expressed in the following form:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leqslant \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]. \tag{1.2}$$

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These inequalities give an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)\,dt$ by the value $f(x)$ at point $x \in [a, b]$. In recent years, many researchers have studied inequalities (1.1) and (1.2), for instance see [1, 2, 6, 7, 9, 10, 12, 13, 15–17].

In [8], Hudzik and Maligranda introduced the class of s-convex functions as follows.

Definition 1.2. A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \leqslant t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$.

It is easily seen that for $s = 1$, s-convexity becomes the ordinary convexity of functions defined on $[0, \infty)$.

In [5], Dragomir and Fitzpatrick presented the Hermite-Hadamard inequalities for s-convex functions in the second sense as follows.

Theorem 1.3. *Suppose that $f : [0, \infty) \to [0, \infty)$ is a s-convex in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f' \in L^1([a, b])$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

For other recent result concerning s-convex functions, see [2, 5, 8, 16, 17].

We will now give definitions of the right-hand side and left-hand side Riemann-Liouville fractional integrals which are used throughout this paper.

Definition 1.4. Let $f \in L[a, b]$. The right-hand side and left-hand side Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $b > a \geqslant 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$
$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^\infty e^{-t}t^{\alpha-1}dt$ (see [11]).

"As a result of many investigations in different areas of applied sciences and engineering and as a consequence of the relationship between CTRWs and diffusion-type pseudo-differential equations, new fractional differential models were used in a great number of different applied fields. We can mention material science , physics, astrophysics, optics, signal processing and control theory, chemistry, transport phenomena, geology, bioengineering and medicine, finance, wave and diffusion phenomena, dissemination of atmospheric pollutants, flux of contaminants transported by subterranean waters through different strata, chaos, and so on", Baleanu et al. [4]. This shows how important and widespread used the fractional calculus is.

Because of the wide application of integral inequalities and fractional integrals, many researchers extend their studies to integral inequalities involving fractional integrals not limited to integer integrals. Recently, more and more integral inequalities involving fractional integrals have been obtained for different classes of functions; see [7, 12, 13, 15, 16].

In [18], Yıldız et al. proved a very similar equality with the following equality for differentiable mapping.

Lemma 1.5. *Let* $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$. *If* $f' \in L[a, b]$, *then for all* $x \in [a, b]$ *and* $\alpha > 0$ *we have*

$$\begin{aligned}\frac{(b-x)^{\alpha}-(a-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{x+}f(b)+J^{\alpha}_{x-}f(a)\right]\\ =(a-b)\int_0^1 p(t)^{\alpha}f'(ta+(1-t)b)dt,\end{aligned} \tag{1.4}$$

where

$$p(t)=\begin{cases} t & , \quad t\in\left[0,\frac{b-x}{b-a}\right], \\ t-1 & , \quad t\in\left(\frac{b-x}{b-a},1\right].\end{cases}$$

Proof. Calculating the integral in equation (1.4), we have

$$\begin{aligned}(a-b)\int_0^1 p(t)^{\alpha}f'(ta+(1-t)b)dt &= (a-b)\int_0^{\frac{b-x}{b-a}} t^{\alpha}f'(ta+(1-t)b)dt\\ &\quad +(a-b)\int_{\frac{b-x}{b-a}}^1 (t-1)^{\alpha}f'(ta+(1-t)b)dt\\ &= I_1+I_2.\end{aligned} \tag{1.5}$$

Using integration by parts and substitution of $ta+(1-t)b=u$, $dt=\frac{du}{a-b}$ we have

$$\begin{aligned} I_1 &= (a-b)\int_0^{\frac{b-x}{b-a}} t^{\alpha}f'(ta+(1-t)b)dt\\ &=(a-b)\left[\left.\frac{t^{\alpha}f(ta+(1-t)b)}{a-b}\right|_0^{\frac{b-x}{b-a}}-\alpha\int_0^{\frac{b-x}{b-a}} t^{\alpha-1}\frac{f(ta+(1-t)b)}{a-b}dt\right]\\ &=\frac{(b-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\alpha}{(b-a)^{\alpha}}\int_x^b (b-u)^{\alpha-1}f(u)du\\ &=\frac{(b-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{x+}f(b),\end{aligned} \tag{1.6}$$

and

$$\begin{aligned} I_2 &= (a-b)\int_{\frac{b-x}{b-a}}^1 (t-1)^{\alpha}f'(ta+(1-t)b)dt\\ &=(a-b)\left[\left.\frac{(t-1)^{\alpha}f(ta+(1-t)b)}{a-b}\right|_{\frac{b-x}{b-a}}^1-\alpha\int_{\frac{b-x}{b-a}}^1 (t-1)^{\alpha-1}\frac{f(ta+(1-t)b)}{a-b}dt\right]\\ &=-\frac{(a-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\alpha}{(b-a)^{\alpha}}\int_a^x (u-a)^{\alpha-1}f(u)du\\ &=-\frac{(a-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{x-}f(a).\end{aligned} \tag{1.7}$$

A combination of (1.5), (1.6), (1.7) we have (1.4). This completes the proof. □

2. Main results

We will use Lemma 1.5 to prove some new integral inequalities for *s*-convex functions via fractional integrals.

Theorem 2.1. *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$, *be a differentiable function on* (a,b) *with* $a<b$ *such that* $f'\in L[a,b]$. *If* $|f'|$ *is s-convex in the second sense on* $[a,b]$ *for some fixed* $s\in(0,1]$ *and* $\alpha>0$, *then we have the following fractional inequality*

$$\left|\frac{(b-x)^{\alpha}-(a-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)\right]\right| \leqslant (b-a)\left[\begin{array}{c}|f'(a)|\left(\frac{\left(\frac{b-x}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}+B_{\frac{x-a}{b-a}}(\alpha+1,s+1)\right)\\ +|f'(b)|\left(B_{\frac{b-x}{b-a}}(\alpha+1,s+1)+\frac{\left(\frac{x-a}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}\right)\end{array}\right] \tag{2.1}$$

for all $x\in[a,b]$, *where* $B_x(a,b)=\int\limits_0^x t^{a-1}(1-t)^{b-1}dt$ *is incomplete beta function.*

Proof. Using Lemma 1.5, we have

$$\left|\frac{(b-x)^{\alpha}-(a-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)\right]\right| \leqslant (b-a)\left[\int_0^{\frac{b-x}{b-a}}t^{\alpha}\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha}\left|f'(ta+(1-t)b)\right|dt\right]. \tag{2.2}$$

Since $|f'|$ is s-convex, we have

$$\left|f'(ta+(1-t)b)\right|\leqslant t^{s}\left|f'(a)\right|+(1-t)^{s}\left|f'(b)\right|. \tag{2.3}$$

Using (2.3) in (2.2), we have

$$\begin{aligned}&\left|\frac{(b-x)^{\alpha}-(a-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)\right]\right|\\ &\leqslant (b-a)\left[\begin{array}{c}\int_0^{\frac{b-x}{b-a}}t^{\alpha}\left[t^{s}|f'(a)|+(1-t)^{s}|f'(b)|\right]dt\\ +\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha}\left|\left[t^{s}|f'(a)|+(1-t)^{s}|f'(b)|\right]\right|dt\end{array}\right]\\ &\leqslant (b-a)\left[\begin{array}{c}|f'(a)|\int_0^{\frac{b-x}{b-a}}t^{\alpha+s}dt+|f'(b)|\int_0^{\frac{b-x}{b-a}}t^{\alpha}(1-t)^{s}dt\\ +|f'(a)|\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha}t^{s}dt+|f'(b)|\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha+s}dt\end{array}\right]\\ &\leqslant (b-a)\left[\begin{array}{c}|f'(a)|\frac{\left(\frac{b-x}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}+|f'(b)|B_{\frac{b-x}{b-a}}(\alpha+1,s+1)\\ +|f'(a)|B_{\frac{x-a}{b-a}}(\alpha+1,s+1)+|f'(b)|\frac{\left(\frac{x-a}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}\end{array}\right]\\ &\leqslant (b-a)\left[\begin{array}{c}|f'(a)|\left(\frac{\left(\frac{b-x}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}+B_{\frac{x-a}{b-a}}(\alpha+1,s+1)\right)\\ +|f'(b)|\left(B_{\frac{b-x}{b-a}}(\alpha+1,s+1)+\frac{\left(\frac{x-a}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}\right)\end{array}\right].\end{aligned}$$

The last inequality gives (2.1). This completes the proof. □

Corollary 2.2. *In addition to the conditions of Theorem 2.1,*

1. *If one takes* $|f'(x)|\leqslant M$ *for all* $x\in[a,b]$, *one has the following fractional Ostrowski type inequality*

$$\left|\frac{(b-x)^{\alpha}-(a-x)^{\alpha}}{(b-a)^{\alpha}}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)\right]\right| \leqslant M(b-a)\left[\begin{array}{c}\left(\frac{\left(\frac{b-x}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}+B_{\frac{x-a}{b-a}}(\alpha+1,s+1)\right)\\ +\left(B_{\frac{b-x}{b-a}}(\alpha+1,s+1)+\frac{\left(\frac{x-a}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}\right)\end{array}\right].$$

2. *If one takes* $x = \frac{a+b}{2}$, *one has the following fractional midpoint type inequality for convex function*

$$\left|\frac{\left(\frac{b-a}{2}\right)^{\alpha} - \left(\frac{a-b}{2}\right)^{\alpha}}{(b-a)^{\alpha}} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}+} f(b) + J^{\alpha}_{\frac{a+b}{2}-} f(a)\right]\right| \leqslant (b-a)\left(\frac{1}{2^{\alpha+s+1}(\alpha+s+1)} + B_{\frac{1}{2}}(\alpha+1, s+1)\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

3. *If one takes* $|f'(x)| \leqslant M$ *for all* $x \in [a,b]$ *and* $\alpha = 1$, *one has the following Ostrowski type inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leqslant M\left[\frac{2(s+1)\left[(b-x)^{s+2} + (x-a)^{s+2}\right] + 2(b-a)^{s+2}}{(s+1)(s+2)(b-a)^{s+1}} - \frac{\left[(b-x)^{s+1} + (x-a)^{s+1}\right]}{(s+1)(b-a)^{s}}\right].$$

4. *If one takes* $x = \frac{a+b}{2}$ *and* $\alpha = 1$, *one has* [17, Corollary 2.1]

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leqslant \frac{(b-a)}{(s+1)(s+2)}\left(1 - \frac{1}{2^{s+1}}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

Remark 2.3. In Theorem 2.1, if one takes $\alpha = 1$, one has [17, Theorem 2.1].

Theorem 2.4. *Let* $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ *be a differentiable function on* (a,b) *with* $a < b$ *such that* $f' \in L[a,b]$. *If* $|f'|^q$ *is* s*-convex in the second sense on* $[a,b]$ *for some fixed* $s \in (0,1]$, $q \geqslant 1$ *and* $\alpha > 0$, *then we have the following fractional inequality*

$$\left|\frac{(b-x)^{\alpha} - (a-x)^{\alpha}}{(b-a)^{\alpha}} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{x+} f(b) + J^{\alpha}_{x-} f(a)\right]\right| \leqslant (b-a)\left[\begin{array}{c} K_1^{1-\frac{1}{q}}(x,a,b,\alpha)\left(\begin{array}{c}\frac{\left(\frac{b-x}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}|f'(a)|^q \\ +B_{\frac{b-x}{b-a}}(\alpha+1,s+1)|f'(b)|^q\end{array}\right)^{\frac{1}{q}} \\ +K_2^{1-\frac{1}{q}}(x,a,b,\alpha)\left(\begin{array}{c}B_{\frac{x-a}{b-a}}(\alpha+1,s+1)|f'(a)|^q \\ +\frac{\left(\frac{x-a}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}|f'(b)|^q\end{array}\right)^{\frac{1}{q}}\end{array}\right], \tag{2.4}$$

where

$$K_1(x,a,b,\alpha) = \int_0^{\frac{b-x}{b-a}} t^{\alpha}dt = \frac{(b-x)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha+1}},$$
$$K_2(x,a,b,\alpha) = \int_{\frac{b-x}{b-a}}^{1} (1-t)^{\alpha}dt = \frac{(x-a)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha+1}}$$

for all $x \in [a,b]$, *where* $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$ *is incomplete beta function.*

Proof. Using Lemma 1.5 and power mean inequality, we have

$$\begin{aligned}&\left|\frac{(b-x)^{\alpha} - (a-x)^{\alpha}}{(b-a)^{\alpha}} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{x+} f(b) + J^{\alpha}_{x-} f(a)\right]\right| \\ &\leqslant (b-a)\left[\int_0^{\frac{b-x}{b-a}} t^{\alpha}\left|f'(ta+(1-t)b)\right|dt + \int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha}\left|f'(ta+(1-t)b)\right|dt\right] \\ &\leqslant (b-a)\left[\begin{array}{c}\left(\int_0^{\frac{b-x}{b-a}} t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{b-x}{b-a}} t^{\alpha}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\end{array}\right].\end{aligned} \tag{2.5}$$

Since $|f'|^q$ is s-convex, we have

$$\left|f'(ta+(1-t)b)\right|^q \leqslant t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q. \tag{2.6}$$

Using (2.6) in (2.5), we have

$$\begin{aligned}
&\left|\frac{(b-x)^\alpha-(a-x)^\alpha}{(b-a)^\alpha}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{x+}f(b)+J^\alpha_{x-}f(a)\right]\right| \\
&\leqslant (b-a)\left[\begin{array}{c}\left(\frac{(b-x)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha+1}}\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{b-x}{b-a}} t^\alpha\left[t^s|f'(a)|^q+(1-t)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\ +\left(\frac{(x-a)^{\alpha+1}}{(\alpha+1)(b-a)^{\alpha+1}}\right)^{1-\frac{1}{q}}\left(\int_{\frac{b-x}{b-a}}^1(1-t)^\alpha\left|\left[t^s|f'(a)|^q+(1-t)^s|f'(b)|^q\right]\right|dt\right)^{\frac{1}{q}}\end{array}\right] \\
&\leqslant (b-a)\left[\begin{array}{c}K_1^{1-\frac{1}{q}}(x,a,b,\alpha)\left(\begin{array}{c}|f'(a)|^q\int_0^{\frac{b-x}{b-a}}t^{\alpha+s}dt \\ +|f'(b)|^q\int_0^{\frac{b-x}{b-a}}t^\alpha(1-t)^s dt\end{array}\right)^{\frac{1}{q}} \\ +K_2^{1-\frac{1}{q}}(x,a,b,\alpha)\left(\begin{array}{c}|f'(a)|^q\int_{\frac{b-x}{b-a}}^1(1-t)^\alpha t^s dt \\ +|f'(b)|^q\int_{\frac{b-x}{b-a}}^1(1-t)^{\alpha+s}dt\end{array}\right)^{\frac{1}{q}}\end{array}\right] \\
&\leqslant (b-a)\left[\begin{array}{c}K_1^{1-\frac{1}{q}}(x,a,b,\alpha)\left(\begin{array}{c}\frac{\left(\frac{b-x}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}|f'(a)|^q \\ +B_{\frac{b-x}{b-a}}(\alpha+1,s+1)|f'(b)|^q\end{array}\right)^{\frac{1}{q}} \\ +K_2^{1-\frac{1}{q}}(x,a,b,\alpha)\left(\begin{array}{c}B_{\frac{x-a}{b-a}}(\alpha+1,s+1)|f'(a)|^q \\ +\frac{\left(\frac{x-a}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}|f'(b)|^q\end{array}\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

The last inequality gives (2.4). This completes the proof. □

Corollary 2.5. *In addition to the conditions of Theorem 2.4,*

1. *If one takes $|f'(x)| \leqslant M$ for all $x \in [a,b]$, one has the following fractional Ostrowski type inequality*

$$\begin{aligned}
&\left|\frac{(b-x)^\alpha-(a-x)^\alpha}{(b-a)^\alpha}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{x+}f(b)+J^\alpha_{x-}f(a)\right]\right| \\
&\leqslant M(b-a)\left[\begin{array}{c}K_1^{1-\frac{1}{q}}(x,a,b,\alpha)\left(\frac{\left(\frac{b-x}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}+B_{\frac{b-x}{b-a}}(\alpha+1,s+1)\right)^{\frac{1}{q}} \\ +K_2^{1-\frac{1}{q}}(x,a,b,\alpha)\left(B_{\frac{x-a}{b-a}}(\alpha+1,s+1)+\frac{\left(\frac{x-a}{b-a}\right)^{\alpha+s+1}}{\alpha+s+1}\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

2. *If one takes $x=\frac{a+b}{2}$, one has the following fractional midpoint type inequality for convex function*

$$\begin{aligned}
&\left|\frac{\left(\frac{b-a}{2}\right)^\alpha-\left(\frac{a-b}{2}\right)^\alpha}{(b-a)^\alpha}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}+}f(b)+J^\alpha_{\frac{a+b}{2}-}f(a)\right]\right| \\
&\leqslant (b-a)\left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\begin{array}{c}\left(\begin{array}{c}\frac{1}{2^{\alpha+s+1}(\alpha+s+1)}|f'(a)|^q \\ +B_{\frac{1}{2}}(\alpha+1,s+1)|f'(b)|^q\end{array}\right)^{\frac{1}{q}} \\ +\left(\begin{array}{c}B_{\frac{1}{2}}(\alpha+1,s+1)|f'(a)|^q \\ +\frac{1}{2^{\alpha+s+1}(\alpha+s+1)}|f'(b)|^q\end{array}\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

3. *If one takes $|f'(x)| \leqslant M$ for all $x \in [a,b]$ and $\alpha=1$, one has the following Ostrowski type inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leqslant M(b-a)\left[\begin{array}{c}\left(\frac{(b-x)^2}{2(b-a)^2}\right)^{1-\frac{1}{q}}\left(\frac{2(b-x)^{s+2}}{(s+2)(b-a)^{s+2}}+\frac{1}{s+1}-\frac{1}{s+2}-\frac{(b-x)^{s+1}}{(s+1)(b-a)^{s+1}}\right)^{\frac{1}{q}} \\ +\left(\frac{(x-a)^2}{2(b-a)^2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}-\frac{1}{s+2}-\frac{(b-x)^{s+1}}{(s+1)(b-a)^{s+1}}\right)^{\frac{1}{q}}\end{array}\right].$$

4. *If one takes* $x = \frac{a+b}{2}$ *and* $\alpha = 1$, *one has the following midpoint type inequality.*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leqslant \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\begin{array}{c} \left(\frac{1}{(s+2)2^s}|f'(a)|^q + \frac{s+3}{(s+1)(s+2)2^s}|f'(b)|^q\right)^{\frac{1}{q}} \\ + \left(\frac{s+3}{(s+1)(s+2)2^s}|f'(a)|^q + \frac{1}{(s+2)2^s}|f'(b)|^q\right)^{\frac{1}{q}} \end{array} \right].$$

Remark 2.6. If one takes $\alpha = 1$, in Theorem 2.4, one has [17, Theorem 2.5].

Theorem 2.7. *Let* $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ *be a differentiable function on* (a, b) *with* $a < b$ *such that* $f' \in L[a, b]$. *If* $|f'|^q$ *is s-convex in the second sense on* $[a, b]$ *for some fixed* $s \in (0, 1]$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\alpha > 0$, *then we have the following fractional inequality*

$$\left|\frac{(b-x)^\alpha - (a-x)^\alpha}{(b-a)^\alpha} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{x+}f(b) + J^\alpha_{x-}f(a)\right]\right| \leqslant (b-a)\left[\begin{array}{c} K_3^{\frac{1}{p}}(x,a,b,\alpha)\left(\frac{\left(\frac{b-x}{b-a}\right)^{s+1}}{s+1}|f'(a)|^q + \frac{1-\left(\frac{x-a}{b-a}\right)^{s+1}}{s+1}|f'(b)|^q\right)^{\frac{1}{q}} \\ +K_4^{\frac{1}{p}}(x,a,b,\alpha)\left(\frac{1-\left(\frac{b-x}{b-a}\right)^{s+1}}{s+1}|f'(a)|^q + \frac{\left(\frac{x-a}{b-a}\right)^{s+1}}{s+1}|f'(b)|^q\right)^{\frac{1}{q}} \end{array}\right], \tag{2.7}$$

where

$$K_3(x,a,b,\alpha) = \int_0^{\frac{b-x}{b-a}} t^{\alpha p}\,dt = \frac{(b-x)^{\alpha p+1}}{(\alpha p+1)(b-a)^{\alpha p+1}},$$
$$K_4(x,a,b,\alpha) = \int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha p}\,dt = \frac{(x-a)^{\alpha p+1}}{(\alpha p+1)(b-a)^{\alpha p+1}}$$

for all $x \in [a, b]$, *where* $B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$ *is incomplete beta function.*

Proof. Using Lemma 1.5 and Hölder inequality, we have

$$\begin{aligned} &\left|\frac{(b-x)^\alpha - (a-x)^\alpha}{(b-a)^\alpha} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{x+}f(b) + J^\alpha_{x-}f(a)\right]\right| \\ &\leqslant (b-a)\left[\int_0^{\frac{b-x}{b-a}} t^\alpha \left|f'(ta+(1-t)b)\right| dt + \int_{\frac{b-x}{b-a}}^1 (1-t)^\alpha \left|f'(ta+(1-t)b)\right| dt\right] \\ &\leqslant (b-a)\left[\begin{array}{c} \left(\int_0^{\frac{b-x}{b-a}} t^{\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\ + \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \end{array}\right]. \end{aligned} \tag{2.8}$$

Since $|f'|^q$ is s-convex, we have

$$\left|f'(ta+(1-t)b)\right|^q \leqslant t^s\left|f'(a)\right|^q + (1-t)^s\left|f'(b)\right|^q. \tag{2.9}$$

Using (2.9) in (2.8), we have

$$\left|\frac{(b-x)^\alpha - (a-x)^\alpha}{(b-a)^\alpha} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{x+}f(b) + J^\alpha_{x-}f(a)\right]\right|$$

$$\leqslant (b-a)\left[\begin{array}{c} \left(\frac{(b-x)^{\alpha p+1}}{(\alpha p+1)(b-a)^{\alpha p+1}}\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left[t^s|f'(a)|^q+(1-t)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\ +\left(\frac{(x-a)^{\alpha p+1}}{(\alpha p+1)(b-a)^{\alpha p+1}}\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left|\left[t^s|f'(a)|^q+(1-t)^s|f'(b)|^q\right]\right|dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leqslant (b-a)\left[\begin{array}{c} K_3^{\frac{1}{p}}(x,a,b,\alpha)\left(\begin{array}{c}|f'(a)|^q\int_0^{\frac{b-x}{b-a}}t^s dt \\ +|f'(b)|^q\int_0^{\frac{b-x}{b-a}}(1-t)^s dt\end{array}\right)^{\frac{1}{q}} \\ +K_4^{\frac{1}{p}}(x,a,b,\alpha)\left(\begin{array}{c}|f'(a)|^q\int_{\frac{b-x}{b-a}}^1 t^s dt \\ +|f'(b)|^q\int_{\frac{b-x}{b-a}}^1(1-t)^s dt\end{array}\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leqslant (b-a)\left[\begin{array}{c} K_3^{\frac{1}{p}}(x,a,b,\alpha)\left(\frac{\left(\frac{b-x}{b-a}\right)^{s+1}}{s+1}|f'(a)|^q+\frac{1-\left(\frac{x-a}{b-a}\right)^{s+1}}{s+1}|f'(b)|^q\right)^{\frac{1}{q}} \\ +K_4^{\frac{1}{p}}(x,a,b,\alpha)\left(\frac{1-\left(\frac{b-x}{b-a}\right)^{s+1}}{s+1}|f'(a)|^q+\frac{\left(\frac{x-a}{b-a}\right)^{s+1}}{s+1}|f'(b)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

The last inequality gives (2.7). This completes the proof. □

Corollary 2.8. *In addition to the conditions of Theorem 2.7,*

1. *If one takes* $|f'(x)|\leqslant M$ *for all* $x\in[a,b]$, *one has the following fractional Ostrowski type inequality*

$$\left|\frac{(b-x)^\alpha-(a-x)^\alpha}{(b-a)^\alpha}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{x+}^\alpha f(b)+J_{x-}^\alpha f(a)\right]\right|$$
$$\leqslant M(b-a)\left[\begin{array}{c} K_3^{\frac{1}{p}}(x,a,b,\alpha)\left(\frac{\left(\frac{b-x}{b-a}\right)^{s+1}}{s+1}+\frac{1-\left(\frac{x-a}{b-a}\right)^{s+1}}{s+1}\right)^{\frac{1}{q}} \\ +K_4^{\frac{1}{p}}(x,a,b,\alpha)\left(\frac{1-\left(\frac{b-x}{b-a}\right)^{s+1}}{s+1}+\frac{\left(\frac{x-a}{b-a}\right)^{s+1}}{s+1}\right)^{\frac{1}{q}}\end{array}\right].$$

2. *If one takes* $x=\frac{a+b}{2}$, *one has the following fractional midpoint type inequality for convex function*

$$\left|\frac{\left(\frac{b-a}{2}\right)^\alpha-\left(\frac{a-b}{2}\right)^\alpha}{(b-a)^\alpha}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{\frac{a+b}{2}+}^\alpha f(b)+J_{\frac{a+b}{2}-}^\alpha f(a)\right]\right|$$
$$\leqslant (b-a)\left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}$$
$$\times\left[\begin{array}{c}\left(\frac{\left(\frac{b-x}{b-a}\right)^{s+1}}{s+1}|f'(a)|^q+\frac{1-\left(\frac{x-a}{b-a}\right)^{s+1}}{s+1}|f'(b)|^q\right)^{\frac{1}{q}} \\ +\left(\frac{1-\left(\frac{b-x}{b-a}\right)^{s+1}}{s+1}|f'(a)|^q+\frac{\left(\frac{x-a}{b-a}\right)^{s+1}}{s+1}|f'(b)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

3. *If one takes* $|f'(x)|\leqslant M$ *for all* $x\in[a,b]$ *and* $\alpha=1$, *one has the following Ostrowski type inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leqslant M(b-a)\left(\frac{(b-x)^{s+1}+(b-a)^{s+1}-(x-a)^{s+1}}{(s+1)(b-a)^{s+1}}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\frac{(b-x)^{p+1}}{(p+1)(b-a)^{p+1}}\right)^{\frac{1}{p}}+\left(\frac{(x-a)^{p+1}}{(p+1)(b-a)^{p+1}}\right)^{\frac{1}{p}}\right].$$

4. *If one takes* $x = \frac{a+b}{2}$ *and* $\alpha = 1$, *one has the following midpoint type inequality.*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leqslant \frac{(b-a)}{4}\frac{1}{2^{\frac{s}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\begin{array}{c} \left(\frac{1}{(s+1)}|f'(a)|^q + \frac{2^{s+1}-1}{(s+1)}|f'(b)|^q\right)^{\frac{1}{q}} \\ +\left(\frac{2^{s+1}-1}{(s+1)}|f'(a)|^q + \frac{1}{(s+1)}|f'(b)|^q\right)^{\frac{1}{q}} \end{array} \right].$$

Remark 2.9. If one takes $\alpha = 1$ in Theorem 2.7, one has [17, Theorem 2.2].

Theorem 2.10. *Let* $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ *be a differentiable function on* (a,b) *with* $a < b$ *such that* $f' \in L[a,b]$. *If* $|f'|^q$ *is s-convex in the second sense on* $[a,b]$ *for some fixed* $s \in (0,1]$, $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ *and* $\alpha > 0$, *then we have the following fractional inequality*

$$\left| \frac{(b-x)^\alpha - (a-x)^\alpha}{(b-a)^\alpha} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{x+}f(b) + J^\alpha_{x-}f(a)\right] \right|$$
$$\leqslant (b-a) \left[\begin{array}{c} \left(\frac{b-x}{b-a}\right)^{\frac{1}{p}} \left(\begin{array}{c} \frac{\left(\frac{b-x}{b-a}\right)^{\alpha q+s+1}}{\alpha q+s+1}|f'(a)|^q \\ +B_{\frac{b-x}{b-a}}(\alpha q+1, s+1)|f'(b)|^q \end{array} \right)^{\frac{1}{q}} \\ +\left(\frac{x-a}{b-a}\right)^{\frac{1}{p}} \left(\begin{array}{c} B_{\frac{x-a}{b-a}}(\alpha q+1, s+1)|f'(a)|^q \\ +\frac{\left(\frac{x-a}{b-a}\right)^{\alpha q+s+1}}{\alpha q+s+1}|f'(b)|^q \end{array} \right)^{\frac{1}{q}} \end{array} \right] \tag{2.10}$$

for all $x \in [a,b]$, *where* $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$ *is incomplete beta function.*

Proof. Using Lemma 1.5 and Hölder inequality, we have

$$\left| \frac{(b-x)^\alpha - (a-x)^\alpha}{(b-a)^\alpha} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{x+}f(b) + J^\alpha_{x-}f(a)\right] \right|$$
$$\leqslant (b-a)\left[\int_0^{\frac{b-x}{b-a}} t^\alpha \left|f'(ta+(1-t)b)\right| dt + \int_{\frac{b-x}{b-a}}^1 (1-t)^\alpha \left|f'(ta+(1-t)b)\right| dt\right]$$
$$\leqslant (b-a) \left[\begin{array}{c} \left(\int_0^{\frac{b-x}{b-a}} 1dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} t^{\alpha q}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{b-x}{b-a}}^1 1dt\right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha q}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \end{array} \right]. \tag{2.11}$$

Since $|f'|^q$ is s-convex, we have

$$\left|f'(ta+(1-t)b)\right|^q \leqslant t^s\left|f'(a)\right|^q + (1-t)^s\left|f'(b)\right|^q. \tag{2.12}$$

Using (2.12) in (2.11), we have

$$\left| \frac{(b-x)^\alpha - (a-x)^\alpha}{(b-a)^\alpha} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{x+}f(b) + J^\alpha_{x-}f(a)\right] \right|$$
$$\leqslant (b-a) \left[\begin{array}{c} \left(\frac{b-x}{b-a}\right)^{\frac{1}{p}} \left(\int_0^{\frac{b-x}{b-a}} t^{\alpha q}\left[t^s|f'(a)|^q + (1-t)^s|f'(b)|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\frac{x-a}{b-a}\right)^{\frac{1}{p}} \left(\int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha q}\left|\left[t^s|f'(a)|^q + (1-t)^s|f'(b)|^q\right]\right| dt\right)^{\frac{1}{q}} \end{array} \right]$$
$$\leqslant (b-a) \left[\begin{array}{c} \left(\frac{b-x}{b-a}\right)^{\frac{1}{p}} \left(\begin{array}{c} |f'(a)|^q \int_0^{\frac{b-x}{b-a}} t^{\alpha q+s} dt \\ +|f'(b)|^q \int_0^{\frac{b-x}{b-a}} t^{\alpha q}(1-t)^s dt \end{array} \right)^{\frac{1}{q}} \\ +\left(\frac{x-a}{b-a}\right)^{\frac{1}{p}} \left(\begin{array}{c} |f'(a)|^q \int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha q} t^s dt \\ +|f'(b)|^q \int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha q+s} dt \end{array} \right)^{\frac{1}{q}} \end{array} \right]$$

$$\leqslant (b-a)\left[\begin{array}{c}\left(\frac{b-x}{b-a}\right)^{\frac{1}{p}}\left(\begin{array}{c}\frac{\left(\frac{b-x}{b-a}\right)^{\alpha q+s+1}}{\alpha q+s+1}|f'(a)|^q\\+B_{\frac{b-x}{b-a}}(\alpha q+1,s+1)|f'(b)|^q\end{array}\right)^{\frac{1}{q}}\\+\left(\frac{x-a}{b-a}\right)^{\frac{1}{p}}\left(\begin{array}{c}B_{\frac{x-a}{b-a}}(\alpha q+1,s+1)|f'(a)|^q\\+\frac{\left(\frac{x-a}{b-a}\right)^{\alpha q+s+1}}{\alpha q+s+1}|f'(b)|^q\end{array}\right)^{\frac{1}{q}}\end{array}\right].$$

The last inequality gives (2.10). This completes the proof. □

Corollary 2.11. *In addition to the conditions of Theorem 2.10,*

1. *If one takes* $|f'(x)| \leqslant M$ *for all* $x \in [a,b]$, *one has the following fractional Ostrowski type inequality*

$$\left|\frac{(b-x)^\alpha-(a-x)^\alpha}{(b-a)^\alpha}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{x+}^\alpha f(b)+J_{x-}^\alpha f(a)\right]\right|$$
$$\leqslant M(b-a)\left[\begin{array}{c}\left(\frac{b-x}{b-a}\right)^{\frac{1}{p}}\left(\frac{\left(\frac{b-x}{b-a}\right)^{\alpha q+s+1}}{\alpha q+s+1}+B_{\frac{b-x}{b-a}}(\alpha q+1,s+1)\right)^{\frac{1}{q}}\\+\left(\frac{x-a}{b-a}\right)^{\frac{1}{p}}\left(B_{\frac{x-a}{b-a}}(\alpha q+1,s+1)+\frac{\left(\frac{x-a}{b-a}\right)^{\alpha q+s+1}}{\alpha q+s+1}\right)^{\frac{1}{q}}\end{array}\right].$$

2. *If one takes* $x=\frac{a+b}{2}$, *one has the following fractional midpoint type inequality for convex function*

$$\left|\frac{\left(\frac{b-a}{2}\right)^\alpha-\left(\frac{a-b}{2}\right)^\alpha}{(b-a)^\alpha}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{\frac{a+b}{2}+}^\alpha f(b)+J_{\frac{a+b}{2}-}^\alpha f(a)\right]\right|$$
$$\leqslant \frac{(b-a)}{2^{\frac{1}{p}}}\left[\begin{array}{c}\left(\frac{1}{2^{\alpha q+s+1}(\alpha q+s+1)}|f'(a)|^q+B_{\frac{1}{2}}(\alpha q+1,s+1)|f'(b)|^q\right)^{\frac{1}{q}}\\+\left(B_{\frac{1}{2}}(\alpha q+1,s+1)|f'(a)|^q+\frac{1}{2^{\alpha q+s+1}(\alpha q+s+1)}|f'(b)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

3. *If one takes* $|f'(x)| \leqslant M$ *for all* $x \in [a,b]$ *and* $\alpha=1$, *one has the following Ostrowski type inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leqslant M(b-a)\left[\begin{array}{c}\left(\frac{b-x}{b-a}\right)^{\frac{1}{p}}\left(\frac{\left(\frac{b-x}{b-a}\right)^{q+s+1}}{q+s+1}+B_{\frac{b-x}{b-a}}(q+1,s+1)\right)^{\frac{1}{q}}\\+\left(\frac{x-a}{b-a}\right)^{\frac{1}{p}}\left(B_{\frac{x-a}{b-a}}(q+1,s+1)+\frac{\left(\frac{x-a}{b-a}\right)^{q+s+1}}{q+s+1}\right)^{\frac{1}{q}}\end{array}\right].$$

4. *If one takes* $x=\frac{a+b}{2}$ *and* $\alpha=1$, *one has the following midpoint type inequality.*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leqslant\frac{(b-a)}{2^{\frac{1}{p}}}\left[\begin{array}{c}\left(\frac{1}{2^{q+s+1}(q+s+1)}|f'(a)|^q+B_{\frac{1}{2}}(q+1,s+1)|f'(b)|^q\right)^{\frac{1}{q}}\\+\left(B_{\frac{1}{2}}(q+1,s+1)|f'(a)|^q+\frac{1}{2^{q+s+1}(q+s+1)}|f'(b)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

3. Applications to special means

Let us recall the following special means of positive numbers a, b with $a < b$.

1. The arithmetic mean:
$$A = A(a,b) := \frac{a+b}{2}.$$
2. The n-logarithmic mean:
$$L_n = L_n(a,b) := \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}},$$
where $n \in \mathbb{Z}\setminus\{-1,0\}$.

In [3], Alomari et al. used the s-convex mapping $f:[0,1]\to[0,1]$, $f(x)=x^s$ where $s\in(0,1)$, to have some application to special means.

Proposition 3.1. *Let $0<a<b$ and $s\in(0,1)$, then we have the following inequality*
$$\left|A^{s+1}(a,b)-L_{s+1}^{s+1}(a,b)\right| \leqslant \frac{(b-a)}{(s+2)}\left(1-\frac{1}{2^{s+1}}\right)[a^s+b^s].$$

Proof. The proof follows by (4) of Corollary 2.2, applied for the function $f:[0,1]\to[0,1]$, $f(x)=\frac{x^{s+1}}{s+1}$ where $s\in(0,1)$ (it is clear that $|f'(x)|=x^s$ is s-convex mapping). □

Proposition 3.2. *Let $0<a<b$, $s\in(0,1)$ and $q\geqslant 1$, then we have the following inequality*
$$\left|A^{\frac{s}{q}+1}(a,b)-L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right| \leqslant \left(\frac{s}{q}+1\right)\frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\begin{array}{c}\left(\frac{1}{(s+2)2^s}a^s+\frac{s+3}{(s+1)(s+2)2^s}b^s\right)^{\frac{1}{q}}\\ +\left(\frac{s+3}{(s+1)(s+2)2^s}a^s+\frac{1}{(s+2)2^s}b^s\right)^{\frac{1}{q}}\end{array}\right].$$

Proof. The proof follows by (4) of Corollary 2.5, applied for the function $f:[0,1]\to[0,1]$, $f(x)=\frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$ where $s\in(0,1)$ and $q\geqslant 1$ (it is clear that $|f'(x)|^q=x^s$ is s-convex mapping). □

Proposition 3.3. *Let $0<a<b$, $s\in(0,1)$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality*
$$\left|A^{\frac{s}{q}+1}(a,b)-L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right| \leqslant \left(\frac{s}{q}+1\right)\frac{(b-a)}{4}\frac{1}{2^{\frac{s}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{1}{(s+1)}a^s+\frac{2^{s+1}-1}{(s+1)}b^s\right)^{\frac{1}{q}}\\ +\left(\frac{2^{s+1}-1}{(s+1)}a^s+\frac{1}{(s+1)}b^s\right)^{\frac{1}{q}}\end{array}\right].$$

Proof. The proof follows by (4) of Corollary 2.8, applied for the function $f:[0,1]\to[0,1]$, $f(x)=\frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$ where $s\in(0,1)$ and $q\geqslant 1$ (it is clear that $|f'(x)|^q=x^s$ is s-convex mapping). □

Proposition 3.4. *Let $0<a<b$, $s\in(0,1)$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality*
$$\left|A^{\frac{s}{q}+1}(a,b)-L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a,b)\right| \leqslant \left(\frac{s}{q}+1\right)\frac{(b-a)}{2^{\frac{1}{p}}}\left[\begin{array}{c}\left(\frac{1}{2^{q+s+1}(q+s+1)}a^s+B_{\frac{1}{2}}(q+1,s+1)\,b^s\right)^{\frac{1}{q}}\\ +\left(B_{\frac{1}{2}}(q+1,s+1)\,a^s+\frac{1}{2^{q+s+1}(q+s+1)}b^s\right)^{\frac{1}{q}}\end{array}\right].$$

Proof. The proof follows by (4) of Corollary 2.11, applied for the function $f:[0,1]\to[0,1]$, $f(x)=\frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$, where $s\in(0,1)$ and $q\geqslant 1$ (it is clear that $|f'(x)|^q=x^s$ is s-convex mapping). □

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