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Convergence and some control conditions of hybrid steepest-descent methods for systems of variational inequalities and hierarchical variational inequalities

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Abstract

The purpose of this paper is to find a solution of a general system of variational inequalities (for short, GSVI), which is also a unique solution of a hierarchical variational inequality (for short, HVI) for an infinite family of nonexpansive mappings in Banach spaces. We introduce general implicit and explicit iterative algorithms, which are based on the hybrid steepest-descent method and the Mann iteration method. Under some appropriate conditions, we prove the strong convergence of the sequences generated by the proposed iterative algorithms to a solution of the GSVI, which is also a unique solution of the HVI.

Keywords: System of variational inequalities, nonexpansive mapping, fixed point, hybrid steepest-descent method, global convergence.
2010 MSC: 49J30, 47H09, 47J20, 49M05.

1. Introduction and formulations

Let X be a real Banach space with its topological dual X^*, and C be a nonempty closed convex subset of X. Let $T : C \to X$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers. A mapping $T : C \to X$ is called L-Lipschitz continuous if there exists a constant $L \geqslant 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|, \quad \forall x, y \in C.$$

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In particular, if $L = 1$, then T is called a nonexpansive mapping, if $L \in [0, 1)$, then T is called a contraction.

The normalized dual mapping $J : X \to 2^{X^*}$ is defined as

$$J(x) := \{\varphi \in X^* : \langle x, \varphi \rangle = \|x\|^2 = \|\varphi\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Let X be a smooth Banach space. Let $A, B : C \to X$ be two nonlinear mappings and λ, μ be two positive real numbers. The general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\left\{ \begin{array}{ll} \langle \lambda A y^* + x^* - y^*, j(x - x^*) \rangle \geqslant 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, j(x - y^*) \rangle \geqslant 0, & \forall x \in C, \end{array} \right\}. \tag{1.1}$$

The equivalence between the GSVI (1.1) and the fixed point problem of some nonexpansive mapping defined on a Banach space is established in Yao et al. [31]. The authors [31] introduced and analyzed implicit and explicit iterative algorithms for solving the GSVI (1.1) by using this equivalence, and proved the strong convergence of the sequences generated by the proposed algorithms. Subsequently, Ceng et al. [4] proposed and analyzed an implicit algorithm of Mann's type and another explicit algorithm of Mann's type for solving GSVI (1.1).

Special case 1. Find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C.$$

This problem is a fundamental problem in the variational analysis; in particular, in the optimization theory and mechanics; see e.g., [6–8, 13–16, 18, 20, 22, 30, 32–34] and the references therein. A large number of algorithms for solving this problem are essentially projection algorithms that employ projections onto the feasible set C of the VI, or onto some related set, so as to iteratively reach a solution. In particular, Korpelevich [17] proposed an algorithm for solving the VI in Euclidean space. known as the extragradient method. This method further has been improved by several researchers; see e.g., [10, 24] and the references therein.

Special case 2. Find a point $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*) \rangle \geqslant 0, \quad \forall x \in C, \tag{1.2}$$

where C is a nonempty closed convex subset of a Banach space.

Aoyama et al. [1] proposed an iterative scheme to find the approximate solution of (1.2) and proved the weak convergence of the sequences generated by the proposed scheme. For several related results, please refer to [2, 5, 35].

The purpose of this paper is to find a solution of a general system of variational inequalities (GSVI), which is also a unique solution of a hierarchical variational inequality (HVI) for an infinite family of nonexpansive mappings in a real strictly convex and 2-uniformly smooth Banach space. We introduce general implicit and explicit iterative algorithms, which are based on the hybrid steepest-descent method and the Mann iteration method. Under some mild conditions, we prove the strong convergence of the sequences generated by the proposed iterative algorithms to a solution of the GSVI, which is also a unique solution of the HVI. Furthermore, we also present a weak convergence theorem for the proposed explicit iterative algorithm involving an infinite family of nonexpansive mappings in a real Hilbert space. Our results improve and extend the corresponding results announced by some others, e.g., Ceng et al. [4] and Buong and Phuong [2].

2. Preliminaries and algorithms

Let X be a real Banach space with the dual space X^*. For simplicity, the norms of X and X^* are denoted by the symbol $\|\cdot\|$. Let X be a nonempty closed convex subset of a real Banach space X. We write $x_n \rightharpoonup x$ (respectively, $x_n \to x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x.

Let $U := \{x \in X : \|x\| = 1\}$. A Banach space X is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|\frac{x+y}{2}\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon$. It is known that a uniformly convex Banach space is reflexive and strictly convex. Also, it is known that if a Banach space X is reflexive, then X is strictly convex if and only if X^* is smooth as well as X is smooth if and only if X^* is strictly convex.

Here we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in X, \quad \|x\| = 1, \quad \|y\| = \tau\}.$$

Lemma 2.1 ([27]). *Let q be a given real number with $1 < q \leqslant 2$ and let X be a q-uniformly smooth Banach space. Then*

$$\|x + y\|^q \leqslant \|x\|^q + q\langle y, J_q(x)\rangle + 2\|\kappa y\|^q, \quad \forall x, y \in X,$$

where κ is the q-uniformly smooth constant of X and J_q is the generalized duality mapping from X into 2^{X^*} defined by

$$J_q(x) = \{\varphi \in X^* : \langle x, \varphi\rangle = \|x\|^q, \ \|\varphi\| = \|x\|^{q-1}\}, \quad \forall x \in X.$$

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geqslant 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Lemma 2.2 ([21]). *Let C be a nonempty closed convex subset of a smooth Banach space X and D be a nonempty subset of C and Π be a retraction of C onto D. Then the following are equivalent*

(i) *Π is sunny and nonexpansive;*
(ii) *$\|\Pi(x) - \Pi(y)\|^2 \leqslant \langle x - y, j(\Pi(x) - \Pi(y))\rangle, \ \forall x, y \in C$;*
(iii) *$\langle x - \Pi(x), j(y - \Pi(x))\rangle \leqslant 0, \ \forall x \in C, y \in D$.*

Lemma 2.3 ([31]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the GSVI (1.1) if and only if $x^* \in GSVI(C, A, B)$ where $GSVI(C, A, B)$ is the set of fixed points of the mapping $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$ and $y^* = \Pi_C(x^* - \mu Bx^*)$.*

Proposition 2.4 ([31]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Then,*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leqslant \|x - y\|^2 + 2\lambda(\kappa^2\lambda - \alpha)\|Ax - Ay\|^2,$$

and

$$\|(I - \mu B)x - (I - \mu B)y\|^2 \leqslant \|x - y\|^2 + 2\mu(\kappa^2\mu - \beta)\|Bx - By\|^2.$$

In particular, if $0 \leqslant \lambda \leqslant \frac{\alpha}{\kappa^2}$ and $0 \leqslant \mu \leqslant \frac{\beta}{\kappa^2}$, then $I - \lambda A$ and $I - \mu B$ are nonexpansive.

Lemma 2.5 ([31]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let the mapping $G : C \to C$ be defined as*

$$G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B).$$

If $0 \leqslant \lambda \leqslant \frac{\alpha}{\kappa^2}$ and $0 \leqslant \mu \leqslant \frac{\beta}{\kappa^2}$, then $G : C \to C$ is nonexpansive.

Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $\lambda \in (0, \frac{\alpha}{\kappa^2})$ and $\mu \in (0, \frac{\beta}{\kappa^2})$ where κ is the 2-uniformly smooth constant of X (see Lemma 2.2). Very recently, in order to solve GSVI (1.1), Ceng et al. [4] introduced an implicit algorithm of Mann's type.

Algorithm 2.6 ([4, Algorithm 3.6])**.** For each $t \in (0, 1)$, choose a number $\theta_t \in (0, 1)$ arbitrarily. The net $\{x_t\}$ is generated by the implicit method

$$x_t = t\Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_t + (1 - t)\Pi_C(I - \theta_t F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_t, \quad \forall t \in (0, 1),$$

where x_t is a unique fixed point of the contraction

$$W_t = t\Pi_C(I - \lambda A)\Pi_C(I - \mu B) + (1 - t)\Pi_C(I - \theta_t F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B).$$

It was proven in [4] that the net $\{x_t\}$ converges in norm, as $t \to 0^+$, to the unique solution $x^* \in \mathrm{GSVI}(C, A, B)$ to the following VI:

$$\langle F(x^*), j(x - x^*)\rangle \geqslant 0, \quad \forall x \in \mathrm{GSVI}(C, A, B), \tag{2.1}$$

provided $\lim_{t\to 0^+} \theta_t = 0$. In the meantime, the authors also proposed another explicit algorithm of Mann's type.

Algorithm 2.7 ([4, Algorithm 3.8])**.** For arbitrarily given $x_0 \in C$, let the sequence $\{x_k\}$ be generated iteratively by

$$\begin{aligned} x_{k+1} = \;& \beta_k x_k + \gamma_k \Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_k \\ & + (1 - \beta_k - \gamma_k)\Pi_C(I - \lambda_k F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_k, \end{aligned}$$

where $\{\lambda_k\}, \{\beta_k\}$ and $\{\gamma_k\}$ are three sequences in $[0, 1]$ such that $\beta_k + \gamma_k \leqslant 1$, for all $k \geqslant 0$.

A mapping F with domain $D(F)$ and range $R(F)$ in X is called

(a) accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \geqslant 0,$$

where J is the normalized duality mapping.

(b) δ-strongly accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \geqslant \delta\|x - y\|^2, \quad \text{for some } \delta \in (0, 1).$$

(c) α-inverse-strongly accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \geqslant \alpha\|Fx - Fy\|^2, \quad \text{for some } \alpha \in (0, 1).$$

(d) ζ-strictly pseudocontractive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \leqslant \|x - y\|^2 - \zeta\|x - y - (Fx - Fy)\|^2, \quad \text{for some } \zeta \in (0, 1).$$

It is easy to see that (2.1) can be rewritten as

$$\langle (I - F)x - (I - F)y, j(x - y)\rangle \geqslant \zeta\|(I - F)x - (I - F)y\|^2,$$

where I denotes the identity mapping of X. Clearly, if F is ζ-strictly pseudocontractive with $\zeta = 0$, then it is said to be pseudocontractive. It is not hard to find that every nonexpansive mapping is pseudocontractive.

Let C be a nonempty closed convex subset of a smooth Banach space X and $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C. Then we set $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$. In 2013, Buong and Phuong [2] considered the following HVI with $C = X$: find $x^* \in \mathcal{F}$ such that

$$\langle F(x^*), j(x - x^*)\rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{2.2}$$

In the case where $X = H$, a Hilbert space, we have $J = I$, and hence problem (2.2) reduces to the HVI: find $x^* \in \mathcal{F}$ such that

$$\langle F(x^*), x - x^*\rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{2.3}$$

Assume that $\mathcal{F} = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ is the set of common fixed points of a family of N nonexpansive mappings T_i on H, and F is an L-Lipschitz continuous and η-strongly monotone mapping, i.e.,

$$\|Fx - Fy\| \leqslant L\|x - y\|,$$

and

$$\langle Fx - Fy, x - y\rangle \geqslant \eta\|x - y\|^2$$

for all $x, y \in H$. Zeng and Yao [35] introduced the following implicit iteration: for an arbitrarily initial point $x_0 \in H$, the sequence $\{x_k\}_{k=1}^{\infty}$ is generated as follows:

$$x_k = \beta_k x_{k-1} + (1 - \beta_k)[T_{[k]}x_k - \lambda_k \mu F(T_{[k]}x_k)], \quad \forall k \geqslant 1, \tag{2.4}$$

where $T_{[n]} = T_{n \bmod N}$, for integer $n \geqslant 1$, with the mod function taking values in the set $\{1, 2, ..., N\}$. They proved the following result.

Theorem 2.8 ([35, Theorem 2.1])**.** *Let H be a real Hilbert space and let $F : H \to H$ be a mapping such that, for some positive constants L and η, F is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^{N}$ be N nonexpansive mappings on H such that $\mathcal{F} : \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/L^2)$, $x_0 \in H$, $\{\lambda_k\}_{k=1}^{\infty} \subset [0, 1)$ and $\{\beta_k\}_{k=1}^{\infty} \subset (0, 1)$ satisfying the conditions $\sum_{k=1}^{\infty} \lambda_k < \infty$, and let $a \leqslant \beta_k \leqslant b, k \geqslant 1$, for some $a, b \in (0, 1)$. Then the sequence $\{x_k\}_{k=0}^{\infty}$, defined by (2.4), converges weakly to $x^* \in \mathcal{F}$, solving (2.3).*

It is well-known that if $\sum_{k=1}^{\infty} \lambda_k < \infty$, then $\lambda_k \to 0$, as $k \to \infty$, and the inversion is not right. Recently, in order to obtain the strong convergence and decrease the strictness of the condition on λ_k, the following implicit iteration method was proposed:

$$x_t = T^t x_t, \quad T^t := T_0^t T_N^t \cdots T_1^t, \quad t \in (0, 1), \tag{2.5}$$

where $\{T_i^t\}_{i=0}^{N}$ are defined by

$$T_i^t x := (1 - \beta_t^i)x + \beta_t^i T_i x, \quad i = 1, ..., N, \quad T_0^t y := (I - \lambda_t \mu F)y, \quad x, y \in H,$$

and proved that the net $\{x_t\}$, defined by (2.5), converges strongly to an element x^* in (2.3) under the conditions on μ, β_t^i that are similar to Theorem 2.8, and $\lambda_t \to 0$ as $t \to 0^+$. When $N = 1$, X is a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and T is a continuous pseudocontractive mapping, Ceng et al. [3] proved the following result.

Theorem 2.9 ([3, Theorem 4.3])**.** *Let F be a δ-strongly accretive and ζ-strictly pseudocontractive mapping with $\delta + \zeta > 1$ and let T be a continuous and pseudocontractive mapping on X, which is a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, such that $\mathcal{F} := \mathrm{Fix}(T) \neq \emptyset$. For each $t \in (0, 1)$, choose a number $\mu_t \in (0, 1)$ arbitrarily and let $\{z_t\}$ be defined by*

$$z_t = t(I - \mu_t F)z_t + (1 - t)Tz_t. \tag{2.6}$$

Then, as $t \to 0^+$, $\{z_t\}$ converges strongly to $x^* \in \mathcal{F}$, solving (2.2).

To find a common fixed point of an infinite family $\{T_i\}_{i=1}^{\infty}$ of nonexpansive mappings on a nonempty, closed and convex subset C in H, Takahashi introduced a W-mapping, generated by $T_k, T_{k-1}, \cdots, T_1$ and real numbers $\alpha_k, \alpha_{k-1}, \cdots, \alpha_1$ as follows:

$$\begin{cases} U_{k,k+1} = I, \\ U_{k,k} = \alpha_k T_k U_{k,k+1} + (1-\alpha_k)I, \\ U_{k,k-1} = \alpha_{k-1} T_{k-1} U_{k,k} + (1-\alpha_{k-1})I, \\ \quad \vdots \\ U_{k,2} = \alpha_2 T_2 U_{k,3} + (1-\alpha_2)I, \\ W_k = U_{k,1} = \alpha_1 T_1 U_{k,2} + (1-\alpha_1)I, \end{cases}$$

and, based on a contractive mapping f on C, Kikkawa and Takahashi [11] proved strong convergence of a sequence $\{x_k\}_{k=1}^{\infty}$, defined by the following implicit iterative scheme: $x_k = \gamma_k f(x_k) + (1-\gamma_k)W_k x_k$ with $0 < \alpha_1 \leqslant 1$ and $0 < \alpha_i \leqslant b < 1$, for $i = 2, 3, \cdots$. Next, in [12], when C is a nonempty, closed and convex subset of a uniformly convex Banach space X with a uniformly Gâteaux differentiable norm, they considered the following strongly convergent implicit method:

$$S_k x = (1 - \frac{1}{k})Ux + \frac{1}{k}f(x), \quad \text{and} \quad Ux = \lim_{k\to\infty} W_k x = \lim_{k\to\infty} U_{k,1}x. \tag{2.7}$$

Note that the method (2.7) contains the limit mapping U, and hence, it is quite difficult to realize.

In [2], motivated by methods (2.5) and (2.6), by introducing a mapping V_k, defined by

$$V_k = V_k^1, \quad V_k^i = T^i T^{i+1} \cdots T^k, \quad T^i = (1-\alpha_i)I + \alpha_i T_i, \quad i = 1, 2, \cdots, k, \tag{2.8}$$

where

$$\alpha_i \in (0, 1) \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_i < \infty, \tag{2.9}$$

Buong and Phuong considered two implicit methods. In both methods, the iteration sequence $\{x_k\}_{k=1}^{\infty}$ is defined, respectively, by

$$x_k = V_k(I - \lambda_k F)x_k, \quad \forall k \geqslant 1, \tag{2.10}$$

and

$$x_k = \gamma_k (I - \lambda_k F)x_k + (I - \gamma_k)V_k x_k, \quad \forall k \geqslant 1, \tag{2.11}$$

where λ_k and γ_k are the positive parameters, satisfying some additional conditions. The authors [2] proved the strong convergence theorems for the methods (2.10) and (2.11).

We will make use of the following well-known results.

Lemma 2.10. *Let X be a real Banach space. Then for all* $x, y \in X$

(i) $\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle$ *for all* $j(x+y) \in J(x+y)$;
(ii) $\|x+y\|^2 \geqslant \|x\|^2 + 2\langle y, j(x)\rangle$ *for all* $j(x) \in J(x)$.

Lemma 2.11 ([29, Theorem 4.1]). *Let X be a uniformly smooth Banach space, C be a nonempty closed convex subset of* X, $T: C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *and* $f: C \to C$ *be a fixed contractive mapping. Let* $\{x_t\}$ *be defined by* $x_t = tf(x_t) + (1-t)Tx_t$. *Then as* $t \to 0$, $\{x_t\}$ *converges strongly to a unique solution* $x^* \in \mathrm{Fix}(T)$ *to the following VI:*

$$\langle (I-f)(x^*), j(x - x^*)\rangle \geqslant 0, \quad \forall x \in \mathrm{Fix}(T).$$

Let LIM be a continuous linear functional on l^∞ and $s = (a_1, a_2, ...) \in l^\infty$. We write $\mathrm{LIM}_k a_k$ instead of LIM(s). LIM is called a Banach limit if LIM satisfies $\|\mathrm{LIM}\| = \mathrm{LIM}_k 1 = 1$ and $\mathrm{LIM}_k a_{k+1} = \mathrm{LIM}_k a_k$ for all $(a_1, a_2, \cdots) \in l^\infty$. If LIM is a Banach limit, then there hold the following:

(i) for all $k \geqslant 1$, $a_k \leqslant c_k$ implies $\text{LIM}_k a_k \leqslant \text{LIM}_k c_k$;
(ii) $\text{LIM}_k a_{k+m} = \text{LIM}_k a_k$ for any fixed positive integer m;
(iii) $\liminf_{k\to\infty} a_k \leqslant \text{LIM}_k a_k \leqslant \limsup_{k\to\infty} a_k$, for all $(a_1, a_2, \cdots) \in l^\infty$.

Lemma 2.12 ([36]). *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_k\} \in l^\infty$ satisfy the condition $\text{LIM}_k a_k \leqslant a$ for all Banach limit LIM. If $\limsup_{k\to\infty}(a_{k+m} - a_k) \leqslant 0$, then $\limsup_{k\to\infty} a_k \leqslant a$.*

In particular, if $m = 1$ in Lemma 2.12, then we immediately obtain the following corollary.

Corollary 2.13 ([23]). *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_k\} \in l^\infty$ satisfy the condition $\text{LIM}_k a_k \leqslant a$ for all Banach limit LIM. If $\limsup_{k\to\infty}(a_{k+1} - a_k) \leqslant 0$, then $\limsup_{k\to\infty} a_k \leqslant a$.*

Lemma 2.14 ([3]). *Let X be a real smooth Banach space and $F : X \to X$ be a mapping.*

(a) *If F is ζ-strictly pseudocontractive, then F is Lipschitz continuous with constant $1 + \frac{1}{\zeta}$.*
(b) *If F is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$, then $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\zeta}} \in (0, 1)$.*
(c) *If F is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$, then for any fixed number $\lambda \in (0, 1)$, $I - \lambda F$ is contractive with constant $1 - \lambda(1 - \sqrt{\frac{1-\delta}{\zeta}}) \in (0, 1)$.*

Recall that X satisfies Opial's property provided, for each sequence $\{x_k\}$ in X, the condition $x_k \rightharpoonup x$ implies

$$\limsup_{k\to\infty} \|x_k - x\| < \limsup_{k\to\infty} \|x_k - y\|, \quad \forall y \in X, \quad y \neq x.$$

It is known in [19] that each l^p $(1 \leqslant p < \infty)$ enjoys this property, while L^p does not unless $p = 2$. It is known that any separable Banach space can be equivalently renormed so that it satisfies Opial's property. We denote by $\omega_w(x_k)$ the weak ω-limit set of $\{x_k\}$, i.e.,

$$\omega_w(x_k) = \{\bar{x} \in X : x_{k_i} \rightharpoonup \bar{x} \quad \text{for some subsequence } \{x_{k_i}\} \text{ of } \{x_k\}\}.$$

Finally, recall that in a Hilbert space H, there holds the following equality

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

We also use the following elementary lemmas.

Lemma 2.15 ([26]). *Let $\{a_k\}$ and $\{b_k\}$ be sequences of nonnegative real numbers such that $\sum_{k=1}^\infty b_k < \infty$ and $a_{k+1} \leqslant a_k + b_k$ for all $k \geqslant 1$. Then $\lim_{k\to\infty} a_k$ exists.*

Lemma 2.16 ([9, Demiclosedness Principle]). *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_k\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_k\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

3. Iterative algorithms and convergence criteria

In this section, we introduce general implicit and explicit iterative algorithms, which are based on the hybrid steepest-descent method and the Mann iteration method. Under some suitable conditions, we prove the strong convergence of the sequences generated by the proposed iterative algorithms to a solution of a general system of variational inequalities (GSVI), which is also a unique solution of a hierarchical variational inequality (HVI), in a strictly convex and 2-uniformly smooth Banach space X. Furthermore, we also establish a weak convergence theorem for the proposed explicit iterative algorithm involving an infinite family of nonexpansive mappings in a Hilbert space.

The following lemmas and proposition will be used to prove our main results in the sequel.

Lemma 3.1 ([2, Lemma 3.1]). *Let* C *be a nonempty closed convex subset of a strictly convex Banach space* X *and let* $\{T_i\}_{i=1}^k$, $k \geqslant 1$, *be* k *nonexpansive self-mappings on* C *such that the set of common fixed points* $\mathcal{F} := \bigcap_{i=1}^k \mathrm{Fix}(T_i) \neq \emptyset$. *Let* a, b *and* α_i, $i = 1, 2, \cdots, k$, *be real numbers such that* $0 < a \leqslant \alpha_i \leqslant b < 1$, *and let* V_k *be a mapping, defined by* (2.8) *for all* $k \geqslant 1$. *Then,* $\mathrm{Fix}(V_k) = \mathcal{F}$.

Lemma 3.2 ([2, Lemma 3.2]). *Let* C *be a nonempty closed convex subset of a Banach space* X *and let* $\{T_i\}_{i=1}^\infty$ *be an infinite family of nonexpansive self-mappings on* C *such that the set of common fixed points* $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$. *Let* V_k *be a mapping, defined by* (2.8), *and let* α_i *satisfy* (2.9). *Then, for each* $x \in C$ *and* $i \geqslant 1$, $\lim_{k\to\infty} V_k^i x$ *exists.*

Remark 3.3.

(i) We can define the mappings

$$V_\infty^i x := \lim_{k\to\infty} V_k^i x \quad \text{and} \quad Vx := V_\infty^1 x = \lim_{k\to\infty} V_k x, \quad \forall x \in C.$$

(ii) It can be readily seen from the proof of Lemma 3.2 that if D is a nonempty and bounded subset of C, then the following holds:

$$\lim_{k\to\infty} \sup_{x\in D} \|V_k^i x - V_\infty^i x\| = 0, \quad \forall i \geqslant 1.$$

In particular, whenever $i = 1$, we have

$$\lim_{k\to\infty} \sup_{x\in D} \|V_k x - Vx\| = 0.$$

Lemma 3.4 ([2, Lemma 3.3]). *Let* C *be a nonempty closed convex subset of a strictly convex Banach space* X *and let* $\{T_i\}_{i=1}^\infty$ *be an infinite family of nonexpansive self-mappings on* C *such that the set of common fixed points* $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$. *Let* α_i *satisfy the first condition in* (2.9). *Then,* $\mathrm{Fix}(V) = \mathcal{F}$.

Inspired by Lemma 3.4, we present the following.

Proposition 3.5. *Let* C *be a nonempty closed convex subset of a strictly convex and* 2*-uniformly smooth Banach space* X. *Let* Π_C *be a sunny nonexpansive retraction from* X *onto* C. *Let the mappings* $A, B : X \to X$ *be* α*-inverse-strongly accretive and* β*-inverse-strongly accretive, respectively. Let the mapping* $G : X \to C \subset X$ *be defined as* $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$ *where* $0 < \lambda \leqslant \frac{\alpha}{\kappa^2}$ *and* $0 < \mu \leqslant \frac{\beta}{\kappa^2}$. *Let* $\{T_i\}_{i=1}^\infty$ *be an infinite family of nonexpansive self-mappings on* C *such that* $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$. *Let* α_i *satisfy the first condition in* (2.9). *Then,* $\mathrm{Fix}(V \circ G) = \mathcal{F}$.

Proof. Let $p \in \mathcal{F}$. Then it is obvious that $Gp = p$ and $V_k^i p = p$ for all integers $i, k \geqslant 1$ with $k \geqslant i$. So, we have $V_\infty^i Gp = p$ for all integers $i \geqslant 1$. In particular, we have $(V \circ G)p = V_\infty^1 Gp$ and hence $\mathcal{F} \subset \mathrm{Fix}(V \circ G)$. Next, we prove that $\mathrm{Fix}(V \circ G) \subset \mathcal{F}$. Now, let $x \in \mathrm{Fix}(V \circ G)$ and $y \in \mathcal{F}$. Then,

$$\begin{aligned} \|V_k Gx - V_k Gy\| &= \|V_k^1 Gx - V_k^1 Gy\| = \|(1-\alpha_1)(V_k^2 Gx - V_k^2 Gy) + \alpha_1(T_1 V_k^2 Gx - T_1 V_k^2 Gy)\| \\ &\leqslant (1-\alpha_1)\|V_k^2 Gx - V_k^2 Gy\| + \alpha_1\|V_k^2 Gx - V_k^2 Gy\| = \|V_k^2 Gx - V_k^2 Gy\| \\ &\leqslant \|V_k^{i+1} Gx - V_k^{i+1} Gy\| \leqslant \|V_k^k Gx - V_k^k Gy\| \leqslant \|Gx - Gy\| \leqslant \|x - y\|, \end{aligned}$$

which together with $\|(V \circ G)x - (V \circ G)y\| = \|x - y\|$ implies that

$$\|V_\infty^i Gx - V_\infty^i Gy\| = \|V_\infty^{i+1} Gx - V_\infty^{i+1} Gy\| = \|Gx - y\|.$$

Therefore, we have

$$\|(1-\alpha_i)(V_\infty^{i+1} Gx - V_\infty^{i+1} Gy) + \alpha_i(T_i V_\infty^{i+1} Gx - T_i V_\infty^{i+1} Gy)\| = \|V_\infty^{i+1} Gx - V_\infty^{i+1} Gy\| = \|Gx - y\|.$$

Since X is strictly convex, $0 < \alpha_i < 1$, and $y \in \mathcal{F}$, we have $Gx - y = T_i V_\infty^{i+1} Gx - T_i V_\infty^{i+1} Gy = T_i V_\infty^{i+1} Gx - y$ and $Gx - y = V_\infty^{i+1} Gx - V_\infty^{i+1} Gy = V_\infty^{i+1} Gx - y$, and hence, $Gx = T_i V_\infty^{i+1} Gx$ and $Gx = V_\infty^{i+1} Gx$ for every $i \geqslant 1$. Consequently, for every $i \geqslant 1$, we have $Gx = T_i Gx$. In particular, when $i = 1$, we have that $Gx = T_1 V_\infty^2 Gx$ and $Gx = V_\infty^2 Gx$. So, it follows that

$$x = (V \circ G)x = (1-\alpha_1)V_\infty^2 Gx + \alpha_1 T_1 V_\infty^2 Gx = Gx,$$

which together with $Gx = T_i Gx$, for all $i \geqslant 1$, implies that for every $i \geqslant 1$, we have $x = T_i x$. It means that $x \in \mathcal{F}$. □

Lemma 3.6 ([25]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\alpha_k\}$ be a sequence in $[0, 1]$ such that $0 < \liminf_{k\to\infty} \alpha_k \leqslant \limsup_{k\to\infty} \alpha_k < 1$. Suppose that $x_{k+1} = \alpha_k x_k + (1-\alpha_k)z_k$, for all $k \geqslant 1$, and $\limsup_{k\to\infty}(\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|) \leqslant 0$. Then $\lim_{k\to\infty} \|z_k - x_k\| = 0$.*

Lemma 3.7 ([28]). *Assume that $\{a_k\}$ is a sequence of nonnegative real numbers such that*

$$a_{k+1} \leqslant (1-\gamma_k)a_k + \gamma_k \delta_k, \quad \forall k \geqslant 1,$$

*where $\{\gamma_k\}$ is a sequence in $[0, 1]$ and $\{\delta_k\}$ is a sequence in **R** such that*

(i) $\sum_{k=1}^\infty \gamma_k = \infty$;
(ii) $\limsup_{k\to\infty} \delta_k \leqslant 0$ *or* $\sum_{k=1}^\infty |\gamma_k \delta_k| < \infty$.

Then, $\lim_{k\to\infty} a_k = 0$.

Now, we are in a position to prove the following main results.

Theorem 3.8. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : X \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $F : X \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $\lambda \in (0, \frac{\alpha}{\kappa^2})$ and $\mu \in (0, \frac{\beta}{\kappa^2})$ where κ is the 2-uniformly smooth constant of X. Let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$. Let $\{V_k\}_{k=1}^\infty$ be defined by (2.8) and (2.9). Let $\{x_k\}_{k=1}^\infty$ be generated in the implicit manner*

$$\begin{cases} y_k = \beta_k x_k + (1-\beta_k)V_k G x_k, \\ x_k = \gamma_k (I - \lambda_k F)x_k + (1-\gamma_k)y_k, \quad \forall k \geqslant 1, \end{cases} \tag{3.1}$$

where $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$, and $\{\lambda_k\}_{k=1}^\infty \subset (0, 1]$, $\{\gamma_k\}_{k=1}^\infty \subset (0, 1)$ and $\{\beta_k\}_{k=1}^\infty \subset [0, 1)$ such that

(C1) $0 < \gamma_k \leqslant \sqrt{\frac{1-\delta}{\zeta}}$;
(C2) $\limsup_{k\to\infty} \beta_k < 1$.

Then,

$$x_k \to x^* \quad \Leftrightarrow \quad \gamma_k \lambda_k F(x_k) \to 0,$$

where $x^* \in \mathcal{F}$ is a unique solution of the VI:

$$\langle F(x^*), j(x - x^*)\rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{3.2}$$

Proof. Since $\limsup_{k\to\infty} \beta_k < 1$, we may assume, without loss of generality, that $\{\beta_k\}_{k=1}^\infty \subset [0, b] \subset [0, 1)$. Let the mapping $G : X \to C \subset X$ be defined as $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$ where $0 < \lambda < \frac{\alpha}{\kappa^2}$ and $0 < \mu < \frac{\beta}{\kappa^2}$. In terms of Lemma 2.5 we know that G is a nonexpansive mapping on X. Then, the implicit iterative scheme (3.1) can be rewritten as

$$x_k = \gamma_k (I - \lambda_k F)x_k + (1-\gamma_k)(\beta_k x_k + (1-\beta_k)V_k G x_k), \quad \forall k \geqslant 1. \tag{3.3}$$

Consider the mapping

$$U_k x = \gamma_k (I - \lambda_k F)x + (1-\gamma_k)(\beta_k x + (1-\beta_k)V_k G x), \quad \forall x \in X.$$

Utilizing Lemmas 2.5 and 2.14 (c), we obtain that for all $x, y \in X$,

$$\begin{aligned}
\|U_k x - U_k y\| &\leqslant \gamma_k \|(I-\lambda_k F)x - (I-\lambda_k F)y\| + (1-\gamma_k)\|\beta_k x + (1-\beta_k)V_k G x - \beta_k y - (1-\beta_k)V_k G y\| \\
&\leqslant \gamma_k \|(I-\lambda_k F)x - (I-\lambda_k F)y\| + (1-\gamma_k)[\beta_k\|x-y\| + (1-\beta_k)\|V_k G x - V_k G y\|] \\
&\leqslant \gamma_k (1-\lambda_k \tau)\|x-y\| + (1-\gamma_k)[\beta_k \|x-y\| + (1-\beta_k)\|x-y\|] \\
&= \gamma_k(1-\lambda_k\tau)\|x-y\| + (1-\gamma_k)\|x-y\| \\
&= (1-\gamma_k\lambda_k\tau)\|x-y\|,
\end{aligned}$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}} \in (0,1)$. This shows that $U_k : X \to X$ is a contraction. By the Banach contraction principle, the fixed point equation (3.3) has a unique solution $x_k \in X$ for each $k \geqslant 1$. Thus, the sequence $\{x_k\}_{k=1}^{\infty}$ is well-defined.

Again from Lemmas 2.5 and 2.14 (c), it follows that for each $k \geqslant 1$,

$$\begin{aligned}
\|x_k - p\|^2 &\leqslant \gamma_k \|(I-\lambda_k F)x_k - p\|^2 + (1-\gamma_k)\|(\beta_k x_k + (1-\beta_k)V_k G x_k) - p\|^2 \\
&= \gamma_k \|(I-\lambda_k F)x_k - (I-\lambda_k F)p - \lambda_k F(p)\|^2 + (1-\gamma_k)\|(\beta_k x_k + (1-\beta_k)V_k G x_k) - p\|^2 \\
&\leqslant \gamma_k [\|(I-\lambda_k F)x_k - (I-\lambda_k F)p\| + \lambda_k \|F(p)\|]^2 + (1-\gamma_k)[\beta_k \|x_k - p\|^2 \\
&\quad + (1-\beta_k)\|V_k G x_k - p\|^2] \\
&\leqslant \gamma_k [(1-\lambda_k\tau)\|x_k - p\| + \lambda_k \|F(p)\|]^2 + (1-\gamma_k)[\beta_k\|x_k-p\|^2 + (1-\beta_k)\|x_k - p\|^2] \\
&= \gamma_k [(1-\lambda_k\tau)\|x_k - p\| + \lambda_k \tau \cdot \tau^{-1}\|F(p)\|]^2 + (1-\gamma_k)\|x_k - p\|^2 \\
&\leqslant \gamma_k (1-\lambda_k \tau)\|x_k - p\|^2 + \gamma_k\lambda_k \cdot \tau^{-1}\|F(p)\|^2 + (1-\gamma_k)\|x_k - p\|^2 \\
&= (1-\gamma_k\lambda_k\tau)\|x_k-p\|^2 + \gamma_k\lambda_k \cdot \tau^{-1}\|F(p)\|^2.
\end{aligned}$$

Therefore, $\|x_k - p\| \leqslant \|F(p)\|/\tau$, which implies the boundedness of $\{x_k\}_{k=1}^{\infty}$. So, the sequences $\{Gx_k\}_{k=1}^{\infty}$, $\{V_k G x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ and $\{F(x_k)\}_{k=1}^{\infty}$, where $y_k = \beta_k x_k + (1-\beta_k)V_k G x_k$, are also bounded.

Suppose that $\gamma_k \lambda_k F(x_k) \to 0$ as $k \to \infty$. From (3.1) we observe that

$$0 = x_k - x_k = -\gamma_k\lambda_k F(x_k) + (1-\gamma_k)(y_k - x_k),$$

and

$$y_k - x_k = (1-\beta_k)(V_k G x_k - x_k).$$

Then from $\|\gamma_k\lambda_k F(x_k)\| \to 0$ and condition (C1) it follows that as $k \to \infty$,

$$\tau\|y_k - x_k\| \leqslant (1-\gamma_k)\|y_k - x_k\| = \|\gamma_k\lambda_k F(x_k)\| \to 0,$$

and

$$(1-b)\|V_k G x_k - x_k\| \leqslant (1-\beta_k)\|V_k G x_k - x_k\| = \|y_k - x_k\| \to 0,$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}} \in (0,1)$. That is,

$$\lim_{k\to\infty} \|x_k - y_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|x_k - V_k G x_k\| = 0. \tag{3.4}$$

Let us show $\mathrm{LIM}_k \|x_k - VGz_n\|^2 \leqslant \mathrm{LIM}_k\|x_k - z_n\|^2$ for any Banach limit LIM, where for each $n \geqslant 1$, z_n is a unique element in X such that $z_n = \frac{1}{n}(I-F)z_n + (1-\frac{1}{n})VGz_n$.

Indeed, in terms of Lemma 2.14 (b) we know that $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\zeta}} \in (0,1)$. Utilizing Lemma 2.11 and Proposition 3.5, we conclude that $\{z_n\}$ converges strongly to a unique solution

$x^* \in \mathrm{Fix}(VG) = \mathcal{F}$ to the following VI:

$$\langle (I - (I - F))x^*, j(x - x^*) \rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{3.5}$$

Since the VI (3.5) is equivalent to the VI (3.2), we know that $\{z_n\}$ converges strongly to a unique solution $x^* \in \mathcal{F}$ to the VI (3.2). Moreover, since V_k is a nonexpansive mapping for each $k \geqslant 1$, V is a nonexpansive mapping on C. Note that $x_k = \gamma_k(I - \lambda_k F)x_k + (1-\gamma_k)y_k$, where $y_k = \beta_k x_k + (1-\beta_k)V_k G x_k$. Also, observe that for each $k, n \geqslant 1$

$$\begin{aligned}
\|x_k - VGz_n\| &= \|\gamma_k[(I - \lambda_k F)x_k - VGz_n] + (1-\gamma_k)(y_k - VGz_n)\| \\
&\leqslant \gamma_k \|(I - \lambda_k F)x_k - VGz_n\| + (1-\gamma_k)[\|y_k - V_k G z_n\| + \|V_k G z_n - VGz_n\|] \\
&\leqslant \gamma_k(\|x_k - VGz_n\| + \|\lambda_k F(x_k)\|) + (1-\gamma_k)[\|y_k - V_k G z_n\| + \|V_k G z_n - VGz_n\|] \\
&\leqslant \gamma_k(\|x_k - VGz_n\| + \|\lambda_k F(x_k)\|) + (1-\gamma_k)[\beta_k\|x_k - V_k G z_n\| + (1-\beta_k)\|V_k G x_k - V_k G z_n\| \\
&\quad + \|V_k G z_n - VGz_n\|] \\
&\leqslant \gamma_k(\|x_k - VGz_n\| + \|\lambda_k F(x_k)\|) + (1-\gamma_k)[\beta_k(\|x_k - V_k G x_k\| + \|V_k G x_k - V_k G z_n\|) \\
&\quad + (1-\beta_k)\|V_k G x_k - V_k G z_n\| + \|V_k G z_n - VGz_n\|] \\
&\leqslant \gamma_k\|x_k - VGz_n\| + \|\gamma_k\lambda_k F(x_k)\| + (1-\gamma_k)[\|x_k - V_k G x_k\| + \|x_k - z_n\| \\
&\quad + \|V_k G z_n - VGz_n\|],
\end{aligned}$$

which together with condition (C1), yields

$$\begin{aligned}
\|x_k - VGz_n\| &\leqslant \frac{\|\gamma_k\lambda_k F(x_k)\|}{1-\gamma_k} + \|x_k - V_k G x_k\| + \|x_k - z_n\| + \|V_k G z_n - VGz_n\| \\
&\leqslant \frac{1}{\tau}\|\gamma_k\lambda_k F(x_k)\| + \|x_k - V_k G x_k\| + \|x_k - z_n\| + \|V_k G z_n - VGz_n\|.
\end{aligned} \tag{3.6}$$

Furthermore, from Remark 3.3 (ii), we deduce that if D is a nonempty and bounded subset of C, then, for $\varepsilon > 0$, there exists $m_0 > i$ such that for all $k > m_0$

$$\sup_{x\in D} \|V_k^i x - V_\infty^i x\| \leqslant \varepsilon. \tag{3.7}$$

Taking $D = \{Gz_n : n \geqslant 1\}$, $\{Gx_k : k \geqslant 1\}$, respectively, and setting $i = 1$, from (3.7) we have

$$\|V_k G z_n - VGz_n\| \leqslant \sup_{x\in D}\|V_k x - Vx\| \leqslant \varepsilon \quad \text{and} \quad \|V_k G x_k - VGx_k\| \leqslant \sup_{x\in D}\|V_k x - Vx\| \leqslant \varepsilon,$$

which immediately imply that

$$\lim_{k\to\infty}\|V_k G x_k - VGx_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|V_k G z_n - VGz_n\| = 0, \quad \forall n \geqslant 1. \tag{3.8}$$

Since $\|\gamma_k\lambda_k F(x_k)\| \to 0$ as $k \to \infty$, from (3.4), (3.6) and (3.8) we obtain

$$\mathrm{LIM}_k\|x_k - VGz_n\|^2 \leqslant \mathrm{LIM}_k\|x_k - z_n\|^2. \tag{3.9}$$

Let us show $\mathrm{LIM}_k\langle F(x^*), j(x^* - x_k)\rangle \leqslant 0$. Indeed, since $z_n = \frac{1}{n}(I - F)z_n + (1 - \frac{1}{n})VGz_n$, we have

$$x_k - z_n = \frac{1}{n}(x_k - (I - F)z_n) + (1 - \frac{1}{n})(x_k - VGz_n),$$

that is,

$$(1 - \frac{1}{n})(x_k - VGz_n) = x_k - z_n - \frac{1}{n}(x_k - (I - F)z_n). \tag{3.10}$$

From Lemma 2.10 (ii) and (3.10) it follows that

$$\begin{aligned}(1-\frac{1}{n})^2\|x_k - VGz_n\|^2 &\geqslant \|x_k - z_n\|^2 - \frac{2}{n}\langle x_k - z_n + z_n - (I-F)z_n, j(x_k - z_n)\rangle \\ &= (1-\frac{2}{n})\|x_k - z_n\|^2 + \frac{2}{n}\langle F(z_n), j(z_n - x_k)\rangle.\end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11), we have

$$(1-\frac{1}{n})^2 \mathrm{LIM}_k\|x_k - z_n\|^2 \geqslant (1-\frac{2}{n})\mathrm{LIM}_k\|x_k - z_n\|^2 + \frac{2}{n}\mathrm{LIM}_k\langle F(z_n), j(z_n - x_k)\rangle,$$

and hence

$$\frac{1}{n^2}\mathrm{LIM}_k\|x_k - z_n\|^2 \geqslant \frac{2}{n}\mathrm{LIM}_k\langle F(z_n), j(z_n - x_k)\rangle.$$

This implies that $\frac{1}{2n}\mathrm{LIM}_k\|x_k - z_n\|^2 \geqslant \mathrm{LIM}_k\langle F(z_n), j(z_n - x_k)\rangle$. Since $z_n \to x^* \in \mathcal{F}$ as $n \to \infty$, by the uniform Fréchet differentiability of the norm of X we have

$$\mathrm{LIM}_k\langle F(x^*), j(x^* - x_k)\rangle \leqslant 0. \tag{3.12}$$

Let us show $\mathrm{LIM}_k\|x_k - x^*\|^2 = 0$. Indeed, since $x_k = \gamma_k(I-\lambda_k F)x_k + (1-\gamma_k)y_k$, where

$$y_k = \beta_k x_k + (1-\beta_k)V_k G x_k,$$

we have

$$\begin{aligned}x_k - (I-\lambda_k F)x_k &= (1-\gamma_k)[y_k - (I-\lambda_k F)x_k] \\ &= (1-\gamma_k)[y_k - V_kGx_k + V_kGx_k - x_k + x_k - (I-\lambda_k F)x_k] \\ &= (1-\gamma_k)[\beta_k(x_k - V_kGx_k) + V_kGx_k - x_k + x_k - (I-\lambda_k F)x_k] \\ &= (1-\gamma_k)[-(1-\beta_k)(x_k - V_kGx_k) + x_k - (I-\lambda_k F)x_k],\end{aligned}$$

which hence implies that

$$\lambda_k F(x_k) = x_k - (I-\lambda_k F)x_k = -\frac{(1-\gamma_k)(1-\beta_k)}{\gamma_k}(I - V_kG)x_k.$$

Consequently, for $x^* \in \mathcal{F}$ we conclude that

$$\langle F(x_k), j(x_k - x^*)\rangle = -\frac{(1-\gamma_k)(1-\beta_k)}{\gamma_k\lambda_k}\langle (I-V_kG)x_k - (I-V_kG)x^*, j(x_k - x^*)\rangle \leqslant 0. \tag{3.13}$$

On the other hand, utilizing Lemma 2.14 (b) we get

$$\begin{aligned}\langle F(x_k), j(x_k - x^*)\rangle &= \langle (I-(I-F))x_k, j(x_k - x^*)\rangle \\ &= \|x_k - x^*\|^2 + \langle (I-(I-F))x^*, j(x_k - x^*)\rangle \\ &\quad + \langle (I-F)x^* - (I-F)x_k, j(x_k - x^*)\rangle \\ &\geqslant (1-\sqrt{\frac{1-\delta}{\zeta}})\|x_k - x^*\|^2 + \langle F(x^*), j(x_k - x^*)\rangle \\ &= \tau\|x_k - x^*\|^2 + \langle F(x^*), j(x_k - x^*)\rangle.\end{aligned} \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\|x_k - x^*\|^2 \leqslant \frac{1}{\tau}\langle F(x^*), j(x^* - x_k)\rangle.$$

From (3.12) we conclude that $\mathrm{LIM}_k\|x_k - x^*\|^2 \leqslant 0$, that is,

$$\mathrm{LIM}_k\|x_k - x^*\|^2 = 0.$$

Let us show $\lim_{k\to\infty}\|x_k - x^*\| = 0$. Indeed, from $\mathrm{LIM}_k\|x_k - x^*\|^2 = 0$ it follows that there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ which converges strongly to $x^* \in \mathcal{F}$. Noting that

$$\|x_k - VGx_k\| \leqslant \|x_k - V_kGx_k\| + \|V_kGx_k - VGx_k\|,$$

we deduce from (3.4) and (3.8) that

$$\lim_{k\to\infty}\|x_k - VGx_k\| = 0.$$

Now assume that there exists another subsequence $\{x_{m_i}\}$ of $\{x_k\}$ such that $x_{m_i} \to \hat{x} \in \mathrm{Fix}(V\circ G) = \mathcal{F}$ (because $\|x_k - VGx_k\| \to 0$ as $k \to \infty$). Then we have that $\|F(x_{m_i}) - F(\hat{x})\| \to 0$ as $i \to \infty$. We claim that $\hat{x}$ is a solution in $\mathcal{F}$ to the VI (3.2). As a matter of fact, since for any $p \in \mathcal{F}$ the sequences $\{x_{m_i} - p\}$ and $\{F(x_{m_i})\}$ are bounded and j is norm to norm uniformly continuous on bounded subsets of X, we obtain that as $i \to \infty$

$$|\langle F(x_{m_i}), j(x_{m_i} - p)\rangle - \langle F(\hat{x}), j(\hat{x} - p)\rangle| \leqslant \|F(x_{m_i}) - F(\hat{x})\|\|x_{m_i} - p\| + |\langle F(\hat{x}), j(x_{m_i} - p) - j(\hat{x} - p)\rangle| \to 0.$$

In addition, repeating the same arguments as those of (3.13), we obtain that for any $p \in \mathcal{F}$

$$\langle F(x_k), j(x_k - p)\rangle \leqslant 0,$$

which immediately yields

$$\langle F(\hat{x}), j(\hat{x} - p)\rangle = \lim_{i\to\infty}\langle F(x_{m_i}), j(x_{m_i} - p)\rangle \leqslant 0.$$

That is, $\hat{x} \in \mathcal{F}$ is a solution of the VI (3.2) and hence $\hat{x} = x^*$ by uniqueness. Therefore, each cluster point of $\{x_k\}$ equals x^*, and so $\{x_k\}$ converges strongly to x^*, which is the unique solution of the VI (3.2) in $\mathcal{F}$.

Conversely, assume that $x_k \to x^*$ as $k \to \infty$, where $x^* \in \mathcal{F}$ is a unique solution of the VI (3.2). Then from (3.1) it follows that

$$\begin{aligned}
\|y_k - x^*\| &= \|\beta_k(x_k - x^*) + (1-\beta_k)(V_kGx_k - x^*)\| \\
&\leqslant \beta_k\|x_k - x^*\| + (1-\beta_k)\|V_kGx_k - x^*\| \\
&\leqslant \beta_k\|x_k - x^*\| + (1-\beta_k)\|x_k - x^*\| \\
&= \|x_k - x^*\| \to 0 \quad \text{as} \quad k \to \infty,
\end{aligned}$$

that is, $y_k \to x^*$ as $k \to \infty$. Again from (3.1) we deduce that

$$0 = x_k - x_k = -\gamma_k\lambda_kF(x_k) + (1-\gamma_k)(y_k - x_k),$$

which immediately yields

$$\|\gamma_k\lambda_kF(x_k)\| = (1-\gamma_k)\|y_k - x_k\| \leqslant \|y_k - x^*\| + \|x_k - x^*\|.$$

Since $x_k \to x^*$ and $y_k \to x^*$ as $k \to \infty$, we obtain that $\gamma_k\lambda_kF(x_k) \to x^*$ as $k \to \infty$. This completes the proof. □

Theorem 3.9. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : X \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $F : X \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\sqrt{\frac{1-\delta}{\zeta}} < \frac{1}{2}$. Assume that $\lambda \in (0, \frac{\alpha}{\kappa^2})$ and $\mu \in (0, \frac{\beta}{\kappa^2})$ where κ is the 2-uniformly smooth constant of X. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that*

$\mathcal{F} := \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \cap \text{GSVI}(C, A, B) \neq \emptyset$. *Let* $\{V_k\}_{k=1}^{\infty}$ *be defined by* (2.8) *and* (2.9)*. For an arbitrary* $x_1 \in X$*, let* $\{x_k\}_{k=1}^{\infty}$ *be generated by*

$$\left\{\begin{array}{l} y_k = \beta_k x_k + (1-\beta_k) V_k G x_k, \\ x_{k+1} = \gamma_k (I - \lambda_k F) x_k + (1-\gamma_k) y_k, \quad \forall k \geqslant 1, \end{array}\right\}, \tag{3.15}$$

where $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$*, and* $\{\lambda_k\}_{k=1}^{\infty} \subset (0, 1]$, $\{\gamma_k\}_{k=1}^{\infty} \subset (0, 1)$ *and* $\{\beta_k\}_{k=1}^{\infty} \subset [0, 1]$ *such that*

(C1) $\sum_{k=1}^{\infty} \gamma_k = \infty$ *and* $0 < \gamma_k \leqslant \min\{1-\tau, \frac{1}{2\tau}\}$ *with* $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}}$;

(C2) $\lim_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0$ *and* $\liminf_{k\to\infty} \lambda_k > \frac{1}{2\tau}$;

(C3) $0 < \liminf_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \beta_k < 1$;

(C4) $\lim_{k\to\infty}\left(\frac{\gamma_{k+1}}{1-(1-\gamma_{k+1})\beta_{k+1}} - \frac{\gamma_k}{1-(1-\gamma_k)\beta_k}\right) = 0.$

Then,

$$x_k \to x^* \quad \Leftrightarrow \quad \gamma_k F(x_k) \to 0,$$

where $x^* \in \mathcal{F}$ *is a unique solution of the VI:*

$$\langle F(x^*), j(x - x^*)\rangle \geqslant 0, \quad \forall x \in \mathcal{F}. \tag{3.16}$$

Proof. First of all, it is not difficult to find that

$$\sqrt{\frac{1-\delta}{\zeta}} < \frac{1}{2} \quad \Leftrightarrow \ 2\Big(1 - \sqrt{\frac{1-\delta}{\zeta}}\Big) > 1 \quad \Leftrightarrow \quad 2\tau > 1.$$

Since $0 < \liminf_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \beta_k < 1$ and $\liminf_{k\to\infty} \lambda_k > \frac{1}{2\tau}$, we may assume, without loss of generality, that $\{\beta_k\}_{k=1}^{\infty} \subset [a, b] \subset (0, 1)$ and $\{\lambda_k\} \subset (\frac{1}{2\tau}, 1]$. Let the mapping $G : X \to C \subset X$ be defined as $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$ where $0 < \lambda < \frac{\alpha}{\kappa^2}$ and $0 < \mu < \frac{\beta}{\kappa^2}$. In terms of Lemma 2.5 we know that G is a nonexpansive mapping on X. Take a fixed $p \in \mathcal{F}$ arbitrarily. Then, by Remark 3.3 and Lemma 2.14 (c), we obtain that for each $k \geqslant 1$,

$$\begin{aligned}
\|x_{k+1} - p\| &\leqslant \gamma_k \|(I - \lambda_k F)x_k - p\| + (1-\gamma_k)\|(\beta_k x_k + (1-\beta_k)V_k G x_k) - p\| \\
&= \gamma_k \|(I - \lambda_k F)x_k - (I - \lambda_k F)p - \lambda_k F(p)\| + (1-\gamma_k)\|(\beta_k x_k + (1-\beta_k)V_k G x_k) - p\| \\
&\leqslant \gamma_k[\|(I - \lambda_k F)x_k - (I - \lambda_k F)p\| + \lambda_k\|F(p)\|] + (1-\gamma_k)[\beta_k\|x_k - p\| \\
&\quad + (1-\beta_k)\|V_k G x_k - p\|] \\
&\leqslant \gamma_k[(1 - \lambda_k\tau)\|x_k - p\| + \lambda_k\|F(p)\|] + (1-\gamma_k)[\beta_k\|x_k - p\| + (1-\beta_k)\|x_k - p\|] \\
&= \gamma_k\Big[(1 - \lambda_k\tau)\|x_k - p\| + \lambda_k\tau \cdot \frac{\|F(p)\|}{\tau}\Big] + (1-\gamma_k)\|x_k - p\| \\
&\leqslant \gamma_k \max\Big\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\Big\} + (1-\gamma_k)\|x_k - p\| \\
&\leqslant \max\Big\{\|x_k - p\|, \frac{\|F(p)\|}{\tau}\Big\},
\end{aligned}$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}} \in (\frac{1}{2}, 1)$. By induction, we have

$$\|x_k - p\| \leqslant \max\Big\{\|x_0 - p\|, \frac{\|F(p)\|}{\tau}\Big\}, \quad \forall k \geqslant 1,$$

which hence implies the boundedness of $\{x_k\}_{k=1}^{\infty}$. So, the sequences $\{Gx_k\}_{k=1}^{\infty}$, $\{V_k G x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ and $\{F(x_k)\}_{k=1}^{\infty}$, where $y_k = \beta_k x_k + (1-\beta_k)V_k G x_k$, are also bounded.

Suppose that $\gamma_k F(x_k) \to 0$ as $k \to \infty$. We claim that $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0$. Indeed, put

$$\rho_k = (1-\gamma_k)\beta_k, \quad \forall k \geqslant 1.$$

Then it follows from (C1) and (C3) that

$$\beta_k \geqslant \rho_k = (1-\gamma_k)\beta_k \geqslant \tau\beta_k, \quad \forall k \geqslant 1,$$

and hence

$$0 < \liminf_{k\to\infty} \rho_k \leqslant \limsup_{k\to\infty} \rho_k < 1. \tag{3.17}$$

Define

$$x_{k+1} = \rho_k x_k + (1-\rho_k) w_k.$$

Observe that

$$\begin{aligned}
w_{k+1} - w_k &= \frac{x_{k+2} - \rho_{k+1}x_{k+1}}{1-\rho_{k+1}} - \frac{x_{k+1} - \rho_k x_k}{1-\rho_k} \\
&= \frac{\gamma_{k+1}(I-\lambda_{k+1}F)x_{k+1} + (1-\gamma_{k+1})y_{k+1} - \rho_{k+1}x_{k+1}}{1-\rho_{k+1}} \\
&\quad - \frac{\gamma_k(I-\lambda_k F)x_k + (1-\gamma_k)y_k - \rho_k x_k}{1-\rho_k} \\
&= \Big(\frac{\gamma_{k+1}(I-\lambda_{k+1}F)x_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k(I-\lambda_k F)x_k}{1-\rho_k}\Big) - \frac{(1-\gamma_k)[\beta_k x_k + (1-\beta_k)V_k G x_k] - \rho_k x_k}{1-\rho_k} \\
&\quad + \frac{(1-\gamma_{k+1})[\beta_{k+1}x_{k+1} + (1-\beta_{k+1})V_{k+1}Gx_{k+1}] - \rho_{k+1}x_{k+1}}{1-\rho_{k+1}} \\
&= \Big(\frac{\gamma_{k+1}(I-\lambda_{k+1}F)x_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k(I-\lambda_k F)x_k}{1-\rho_k}\Big) \\
&\quad + \frac{(1-\gamma_{k+1})(1-\beta_{k+1})V_{k+1}Gx_{k+1}}{1-\rho_{k+1}} - \frac{(1-\gamma_k)(1-\beta_k)V_k G x_k}{1-\rho_k} \\
&= \Big(\frac{\gamma_{k+1}(I-\lambda_{k+1}F)x_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k(I-\lambda_k F)x_k}{1-\rho_k}\Big) + (V_{k+1}Gx_{k+1} - V_k G x_k) \\
&\quad - \frac{\gamma_{k+1}}{1-\rho_{k+1}} V_{k+1}Gx_{k+1} + \frac{\gamma_k}{1-\rho_k} V_k G x_k \\
&= \Big(\frac{\gamma_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k}{1-\rho_k}\Big)(I-\lambda_{k+1}F)x_{k+1} + ((I-\lambda_{k+1}F)x_{k+1} - (I-\lambda_k F)x_k)\frac{\gamma_k}{1-\rho_k} \\
&\quad + (V_{k+1}Gx_{k+1} - V_k G x_k) - \Big(\frac{\gamma_{k+1}}{1-\rho_{k+1}} \\
&\quad - \frac{\gamma_k}{1-\rho_k}\Big)V_{k+1}Gx_{k+1} - (V_{k+1}Gx_{k+1} - V_k G x_k)\frac{\gamma_k}{1-\rho_k} \\
&= \Big(\frac{\gamma_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k}{1-\rho_k}\Big)[(I-\lambda_{k+1}F)x_{k+1} - V_{k+1}Gx_{k+1}] \\
&\quad + ((I-\lambda_{k+1}F)x_{k+1} - (I-\lambda_k F)x_k)\frac{\gamma_k}{1-\rho_k} \\
&\quad + \frac{1-\rho_k-\gamma_k}{1-\rho_k}(V_{k+1}Gx_{k+1} - V_k G x_k),
\end{aligned}$$

and hence

$$\begin{aligned}
\|w_{k+1} - w_k\| &\leqslant \Big|\frac{\gamma_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k}{1-\rho_k}\Big| \|(I-\lambda_{k+1}F)x_{k+1} - V_{k+1}Gx_{k+1}\| \\
&\quad + \|(I-\lambda_{k+1}F)x_{k+1} - (I-\lambda_k F)x_k\| \frac{\gamma_k}{1-\rho_k} + \frac{1-\rho_k-\gamma_k}{1-\rho_k}\|V_{k+1}Gx_{k+1} - V_k G x_k\| \\
&\leqslant \Big|\frac{\gamma_{k+1}}{1-\rho_{k+1}} - \frac{\gamma_k}{1-\rho_k}\Big| \|(I-\lambda_{k+1}F)x_{k+1} - V_{k+1}Gx_{k+1}\| \\
&\quad + (\|(I-\lambda_{k+1}F)x_{k+1} - (I-\lambda_k F)x_{k+1}\|
\end{aligned}$$

$$
\begin{aligned}
&+\|(I-\lambda_k F)x_{k+1}-(I-\lambda_k F)x_k\|)\frac{\gamma_k}{1-\rho_k}+\frac{1-\rho_k-\gamma_k}{1-\rho_k}(\|V_{k+1}Gx_{k+1}-V_{k+1}Gx_k\| \\
&+\|V_{k+1}Gx_k-V_kGx_k\|) \\
\leqslant &|\frac{\gamma_{k+1}}{1-\rho_{k+1}}-\frac{\gamma_k}{1-\rho_k}|(\|x_{k+1}\|+\|F(x_{k+1})\|+\|V_{k+1}Gx_{k+1}\|)+(|\lambda_{k+1}-\lambda_k|\|F(x_{k+1})\| \\
&+\|x_{k+1}-x_k\|)\frac{\gamma_k}{1-\rho_k}+\frac{1-\rho_k-\gamma_k}{1-\rho_k}(\|x_{k+1}-x_k\|+\alpha_{k+1}\|T_{k+1}Gx_k-Gx_k\|) \qquad (3.18)\\
= &|\frac{\gamma_{k+1}}{1-\rho_{k+1}}-\frac{\gamma_k}{1-\rho_k}|(\|x_{k+1}\|+\|F(x_{k+1})\|+\|V_{k+1}Gx_{k+1}\|)+\|x_{k+1}-x_k\| \\
&+\frac{\gamma_k}{1-\rho_k}|\lambda_{k+1}-\lambda_k|\|F(x_{k+1})\|+\frac{1-\rho_k-\gamma_k}{1-\rho_k}\alpha_{k+1}\|T_{k+1}Gx_k-Gx_k\| \\
\leqslant &\|x_{k+1}-x_k\|+|\frac{\gamma_{k+1}}{1-\rho_{k+1}}-\frac{\gamma_k}{1-\rho_k}|(\|x_{k+1}\|+\|F(x_{k+1})\|+\|V_{k+1}Gx_{k+1}\|) \\
&+|\lambda_{k+1}-\lambda_k|\|F(x_{k+1})\|+\alpha_{k+1}\|T_{k+1}Gx_k-Gx_k\|.
\end{aligned}
$$

Thus, from (3.18), $\lim_{k\to\infty}\alpha_k=0$, and conditions (C2), (C4), it follows that (noticing the boundedness of $\{x_k\}$ and the nonexpansivity of T_k)

$$\limsup_{k\to\infty}(\|w_{k+1}-w_k\|-\|x_{k+1}-x_k\|)\leqslant 0.$$

Since $0<\liminf_{k\to\infty}\rho_k\leqslant\limsup_{k\to\infty}\rho_k<1$ (due to (3.17)), by Lemma 3.6 we get $\lim_{k\to\infty}\|w_k-x_k\|=0$. Consequently,

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=\lim_{k\to\infty}(1-\rho_k)\|w_k-x_k\|=0. \qquad (3.19)$$

Furthermore, from (3.15) we observe that

$$x_{k+1}-x_k=-\gamma_k\lambda_kF(x_k)+(1-\gamma_k)(y_k-x_k),$$

and

$$y_k-x_k=(1-\beta_k)(V_kGx_k-x_k).$$

Then from $\|\gamma_kF(x_k)\|\to 0$ and condition (C1) it follows that as $k\to\infty$,

$$
\begin{aligned}
\tau\|y_k-x_k\| &\leqslant (1-\gamma_k)\|y_k-x_k\| \\
&=\|x_{k+1}-x_k+\gamma_k\lambda_kF(x_k)\| \\
&\leqslant \|x_{k+1}-x_k\|+\|\gamma_kF(x_k)\|\to 0,
\end{aligned}
$$

and

$$(1-b)\|V_kGx_k-x_k\|\leqslant(1-\beta_k)\|V_kGx_k-x_k\|=\|y_k-x_k\|\to 0,$$

where $\tau=1-\sqrt{\frac{1-\delta}{\zeta}}\in(0,1)$. That is,

$$\lim_{k\to\infty}\|x_k-y_k\|=0 \quad \text{and} \quad \lim_{k\to\infty}\|x_k-V_kGx_k\|=0. \qquad (3.20)$$

Let us show $\mathrm{LIM}_k\|x_k-VGz_n\|^2\leqslant\mathrm{LIM}_k\|x_k-z_n\|^2$ for any Banach limit LIM, where for each $n\geqslant 1$, z_n is a unique element in X such that $z_n=\frac{1}{n}(I-F)z_n+(1-\frac{1}{n})VGz_n$.

Indeed, in terms of Lemma 2.14 (b) we know that $I-F$ is contractive with constant $\sqrt{\frac{1-\delta}{\zeta}}\in(0,1)$. Utilizing Lemma 2.11 and Proposition 3.5, we conclude that $\{z_n\}$ converges strongly to a unique solution $x^*\in\mathrm{Fix}(VG)=\mathcal{F}$ to the following VI:

$$\langle(I-(I-F))x^*,j(x-x^*)\rangle\geqslant 0,\quad \forall x\in\mathcal{F}. \qquad (3.21)$$

Since the VI (3.21) is equivalent to the VI (3.16), we know that $\{z_n\}$ converges strongly to a unique solution $x^*\in\mathcal{F}$ to the VI (3.16). Moreover, since V_k is a nonexpansive mapping for each $k\geqslant 1$, V is a nonexpansive mapping on C. Note that $x_{k+1}=\gamma_k(I-\lambda_kF)x_k+(1-\gamma_k)y_k$, where $y_k=\beta_kx_k+(1-\beta_k)V_kGx_k$. Also, observe that for each $k, n \geqslant 1$

$$\begin{aligned}\|x_k - VGz_n\| - \|x_k - x_{k+1}\| &\leqslant \|x_{k+1} - VGz_n\| \\ &= \|\gamma_k[(I-\lambda_k F)x_k - VGz_n] + (1-\gamma_k)(y_k - VGz_n)\| \\ &\leqslant \gamma_k\|(I-\lambda_k F)x_k - VGz_n\| + (1-\gamma_k)[\|y_k - V_k Gz_n\| + \|V_k Gz_n - VGz_n\|] \\ &\leqslant \gamma_k(\|x_k - VGz_n\| + \|\lambda_k F(x_k)\|) + (1-\gamma_k)[\|y_k - V_k Gz_n\| \\ &\quad + \|V_k Gz_n - VGz_n\|] \\ &\leqslant \gamma_k\|x_k - VGz_n\| + \|\gamma_k\lambda_k F(x_k)\| + (1-\gamma_k)[\|x_k - V_k Gx_k\| + \|x_k - z_n\| \\ &\quad + \|V_k Gz_n - VGz_n\|],\end{aligned}$$

which together with condition (C1), yields

$$\begin{aligned}\|x_k - VGz_n\| &\leqslant \frac{\|\gamma_k\lambda_k F(x_k)\| + \|x_k - x_{k+1}\|}{1-\gamma_k} + \|x_k - V_k Gx_k\| + \|x_k - z_n\| + \|V_k Gz_n - VGz_n\| \\ &\leqslant \frac{1}{\tau}(\|\gamma_k\lambda_k F(x_k)\| + \|x_k - x_{k+1}\|) + \|x_k - V_k Gx_k\| + \|x_k - z_n\| + \|V_k Gz_n - VGz_n\|.\end{aligned} \tag{3.22}$$

Repeating the same arguments as those of (3.8) in the proof of Theorem 3.8, we get

$$\lim_{k\to\infty}\|V_k Gx_k - VGx_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|V_k Gz_n - VGz_n\| = 0, \quad \forall n \geqslant 1. \tag{3.23}$$

Since $\|\gamma_k\lambda_k F(x_k)\| \to 0$ as $k \to \infty$, from (3.19), (3.20), (3.22) and (3.23) we obtain

$$\mathrm{LIM}_k\|x_k - VGz_n\|^2 \leqslant \mathrm{LIM}_k\|x_k - z_n\|^2. \tag{3.24}$$

In addition, repeating the same arguments as those of (3.12) in the proof of Theorem 3.8 and utilizing (3.24), we obtain

$$\mathrm{LIM}_k\langle F(x^*), j(x^* - x_k)\rangle \leqslant 0. \tag{3.25}$$

Let us show $\limsup_{k\to\infty}\langle F(x^*), j(x^* - x_k)\rangle \leqslant 0$. To this end, put

$$a_k := \langle F(x^*), j(x^* - x_k)\rangle, \quad \forall k \geqslant 1.$$

Then, from (3.25) we get $\mathrm{LIM}_k a_k \leqslant 0$ for any Banach limit LIM. Since $\|x_{k+1} - x_k\| \to 0$ (due to (3.19)) and j is uniformly continuous on bounded subsets of X, we know that

$$\limsup_{k\to\infty}(a_{k+1} - a_k) = \limsup_{k\to\infty}\langle F(x^*), j(x^* - x_{k+1}) - j(x^* - x_k)\rangle = 0.$$

Then, by Lemma 2.12, we obtain $\limsup_{k\to\infty} a_k \leqslant 0$, that is,

$$\limsup_{k\to\infty}\langle F(x^*), j(x^* - x_k)\rangle = \limsup_{k\to\infty} a_k \leqslant 0. \tag{3.26}$$

Next, let us show $\lim_{k\to\infty}\|x_k - x^*\| = 0$. Indeed, observe that

$$\begin{aligned}x_{k+1} - x^* &= \gamma_k[(I-\lambda_k F)x_k - x^*] + (1-\gamma_k)(y_k - x^*) \\ &= \gamma_k[(I-\lambda_k F)x_k - x^*] + (1-\gamma_k)(1-\beta_k)(V_k Gx_k - x^*) + (1-\gamma_k)\beta_k(x_k - x^*).\end{aligned}$$

Then, utilizing Lemma 2.10 (i), $1-\gamma_k \geqslant \tau$ (due to (C1)) and $\lambda_k \geqslant \frac{1}{2\tau}$ (due to (C2)) we get

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\leqslant \|(1-\gamma_k)\beta_k(x_k - x^*) + (1-\gamma_k)(1-\beta_k)(V_k Gx_k - x^*)\|^2 \\ &\quad + 2\gamma_k\langle(I-\lambda_k F)x_k - x^*, j(x_{k+1} - x^*)\rangle \\ &\leqslant [(1-\gamma_k)\beta_k\|x_k - x^*\| + (1-\gamma_k)(1-\beta_k)\|x_k - x^*\|]^2\end{aligned}$$

$$
\begin{aligned}
&+2\gamma_k\langle (I-\lambda_k F)x_k-(I-\lambda_k F)x^*, j(x_{k+1}-x^*)\rangle \\
&+2\gamma_k\langle (I-\lambda_k F)x^*-x^*, j(x_{k+1}-x^*)\rangle \\
\leqslant\ & (1-\gamma_k)^2\|x_k-x^*\|^2+2\gamma_k(1-\lambda_k\tau)\|x_k-x^*\|\|x_{k+1}-x^*\| \\
&+2\gamma_k\lambda_k\langle F(x^*), j(x^*-x_{k+1})\rangle \\
\leqslant\ & (1-\gamma_k)^2\|x_k-x^*\|^2+\gamma_k(1-\lambda_k\tau)[\|x_k-x^*\|^2+\|x_{k+1}-x^*\|^2] \\
&+2\gamma_k\lambda_k\langle F(x^*), j(x^*-x_{k+1})\rangle \\
=\ & [(1-\gamma_k)^2+\gamma_k(1-\lambda_k\tau)]\|x_k-x^*\|^2+\gamma_k(1-\lambda_k\tau)\|x_{k+1}-x^*\|^2 \\
&+2\gamma_k\lambda_k\langle F(x^*), j(x^*-x_{k+1})\rangle \\
=\ & [1-\gamma_k(1-\gamma_k)-\gamma_k\lambda_k\tau]\|x_k-x^*\|^2+\gamma_k(1-\lambda_k\tau)\|x_{k+1}-x^*\|^2 \\
&+2\gamma_k\lambda_k\langle F(x^*), j(x^*-x_{k+1})\rangle \\
\leqslant\ & [1-\gamma_k\tau-\gamma_k\lambda_k\tau]\|x_k-x^*\|^2+\gamma_k\lambda_k\tau\|x_{k+1}-x^*\|^2 \\
&+2\gamma_k\lambda_k\langle F(x^*), j(x^*-x_{k+1})\rangle,
\end{aligned}
$$

which immediately yields

$$
\begin{aligned}
\|x_{k+1}-x^*\|^2 &\leqslant \frac{1-\gamma_k\tau-\gamma_k\lambda_k\tau}{1-\gamma_k\lambda_k\tau}\|x_k-x^*\|^2+\frac{2\gamma_k\lambda_k}{1-\gamma_k\lambda_k\tau}\langle F(x^*), j(x^*-x_{k+1})\rangle \\
&=(1-\frac{\gamma_k\tau}{1-\gamma_k\lambda_k\tau})\|x_k-x^*\|^2+\frac{\gamma_k\tau}{1-\gamma_k\lambda_k\tau}\cdot\frac{2\lambda_k}{\tau}\langle F(x^*), j(x^*-x_{k+1})\rangle.
\end{aligned} \tag{3.27}
$$

Now, note that when $0<\gamma_k\leqslant\frac{1}{2\tau}$, one has

$$
\gamma_k\tau+\gamma_k\lambda_k\tau\leqslant 2\gamma_k\tau\leqslant 1,
$$

which yields $\frac{\gamma_k\tau}{1-\gamma_k\lambda_k\tau}\leqslant 1$. Since $\sum_{k=1}^{\infty}\frac{\gamma_k\tau}{1-\gamma_k\lambda_k\tau}\geqslant\sum_{k=1}^{\infty}\gamma_k\tau=\infty$ and $\limsup_{k\to\infty}\frac{2\lambda_k}{\tau}\langle F(x^*), j(x^*-x_{k+1})\rangle\leqslant 0$ (due to (3.26)), applying Lemma 3.7 to (3.27) we infer that

$$
\lim_{k\to\infty}\|x_k-x^*\|=0.
$$

Conversely, if $x_k\to x^*\in\mathcal{F}$ as $k\to\infty$, then from (3.15) it follows that

$$
\begin{aligned}
\|y_k-x^*\| &= \|\beta_k(x_k-x^*)+(1-\beta_k)(V_kGx_k-x^*)\| \\
&\leqslant \beta_k\|x_k-x^*\|+(1-\beta_k)\|V_kGx_k-x^*\| \\
&\leqslant \beta_k\|x_k-x^*\|+(1-\beta_k)\|x_k-x^*\| \\
&= \|x_k-x^*\|\to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

that is, $y_k\to x^*$. Again from (3.15) we obtain that

$$
\begin{aligned}
\frac{1}{2\tau}\|\gamma_kF(x_k)\| &\leqslant \|\gamma_k[(I-\lambda_kF)x_k-x_k]\| \\
&= \|x_{k+1}-x_k-(1-\gamma_k)(y_k-x_k)\| \\
&\leqslant \|x_{k+1}-x_k\|+(1-\gamma_k)\|y_k-x_k\| \\
&\leqslant \|x_{k+1}-x^*\|+\|x_k-x^*\|+(1-\gamma_k)(\|y_k-x^*\|+\|x_k-x^*\|) \\
&\leqslant \|x_{k+1}-x^*\|+2\|x_k-x^*\|+\|y_k-x^*\|.
\end{aligned}
$$

Since $x_k\to x^*$ and $y_k\to x^*$, we get $\gamma_kF(x_k)\to 0$. This completes the proof. □

Corollary 3.10. *Let* C *be a nonempty closed convex subset of a strictly convex and* 2*-uniformly smooth Banach space* X*. Let* Π_C *be a sunny nonexpansive retraction from* X *onto* C*. Let the mappings* $A, B : X\to X$ *be* α*-inverse-strongly accretive and* β*-inverse-strongly accretive, respectively. Let* $F : X\to X$ *be* δ*-strongly accretive* *and ζ-strictly pseudocontractive with $\sqrt{\frac{1-\delta}{\zeta}} < \frac{1}{2}$. Assume that $\lambda \in (0, \frac{\alpha}{\kappa^2})$ and $\mu \in (0, \frac{\beta}{\kappa^2})$ where κ is the 2-uniformly smooth constant of X. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$. Let $\{V_k\}_{k=1}^{\infty}$ be defined by (2.8) and (2.9). For an arbitrary $x_1 \in X$, let $\{x_k\}_{k=1}^{\infty}$ be generated by (3.15), where $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$, and $\{\lambda_k\}_{k=1}^{\infty} \subset (0, 1]$, $\{\gamma_k\}_{k=1}^{\infty} \subset (0, 1)$ and $\{\beta_k\}_{k=1}^{\infty} \subset [0, 1]$ such that*

(C1) $\sum_{k=1}^{\infty} \gamma_k = \infty$ *and* $0 < \gamma_k \leqslant \min\{1 - \tau, \frac{1}{2\tau}\}$ *with* $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}}$;

(C2) $\lim_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0$ *and* $\liminf_{k\to\infty} \lambda_k > \frac{1}{2\tau}$;

(C3) $0 < \liminf_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \beta_k < 1$;

(C4) $\lim_{k\to\infty} |\beta_{k+1} - \beta_k| = 0$ *and* $\lim_{k\to\infty} |\gamma_{k+1} - \gamma_k| = 0$.

Then,

$$x_k \to x^* \quad \Leftrightarrow \quad \gamma_k F(x_k) \to 0,$$

where $x^* \in \mathcal{F}$ is a unique solution of the VI (3.16).

Proof. Observe that

$$\begin{aligned}
&\frac{\gamma_{k+1}}{1-(1-\gamma_{k+1})\beta_{k+1}} - \frac{\gamma_k}{1-(1-\gamma_k)\beta_k} \\
&= \frac{\gamma_{k+1}(1-(1-\gamma_k)\beta_k) - \gamma_k(1-(1-\gamma_{k+1})\beta_{k+1})}{(1-(1-\gamma_{k+1})\beta_{k+1})(1-(1-\gamma_k)\beta_k)} \\
&= \frac{(\gamma_{k+1}-\gamma_k) - \gamma_{k+1}\beta_k + \gamma_k\beta_{k+1} + \gamma_{k+1}\gamma_k\beta_k - \gamma_k\gamma_{k+1}\beta_{k+1}}{(1-(1-\gamma_{k+1})\beta_{k+1})(1-(1-\gamma_k)\beta_k)} \\
&= \frac{(\gamma_{k+1}-\gamma_k) - \gamma_{k+1}(\beta_k - \beta_{k+1}) - \beta_{k+1}(\gamma_{k+1}-\gamma_k) + \gamma_k\gamma_{k+1}(\beta_k - \beta_{k+1})}{(1-(1-\gamma_{k+1})\beta_{k+1})(1-(1-\gamma_k)\beta_k)} \\
&= \frac{(\gamma_{k+1}-\gamma_k)(1-\beta_{k+1}) - \gamma_{k+1}(\beta_k - \beta_{k+1})(1-\gamma_k)}{(1-(1-\gamma_{k+1})\beta_{k+1})(1-(1-\gamma_k)\beta_k)}.
\end{aligned}$$

Since $\lim_{k\to\infty} |\gamma_{k+1} - \gamma_k| = 0$ and $\lim_{k\to\infty} |\beta_{k+1} - \beta_k| = 0$, we conclude that

$$\lim_{k\to\infty} \left(\frac{\gamma_{k+1}}{1-(1-\gamma_{k+1})\beta_{k+1}} - \frac{\gamma_k}{1-(1-\gamma_k)\beta_k}\right) = 0.$$

Consequently, all conditions of Theorem 3.9 are satisfied. So, utilizing Theorem 3.9 we obtain the desired result. □

Theorem 3.11. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : X \to X$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $F : X \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $\lambda \in (0, \frac{\alpha}{\kappa^2})$ and $\mu \in (0, \frac{\beta}{\kappa^2})$ where κ is the 2-uniformly smooth constant of X. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$. Let $\{V_k\}_{k=1}^{\infty}$ be defined by (2.8) and (2.9). For an arbitrary $x_1 \in X$, let $\{x_k\}_{k=1}^{\infty}$ be generated by (3.15), where $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$, and $\{\lambda_k\}_{k=1}^{\infty} \subset (0, 1]$, $\{\gamma_k\}_{k=1}^{\infty} \subset (0, 1)$ and $\{\beta_k\}_{k=1}^{\infty} \subset [0, 1]$ such that*

(C1) $\sum_{k=1}^{\infty} \gamma_k = \infty$ *and* $\lim_{k\to\infty} \gamma_k = 0$;

(C2) $\lim_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0$ *and* $\liminf_{k\to\infty} \lambda_k > 0$;

(C3) $0 < \liminf_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \beta_k < 1$.

Then $\{x_k\}_{k=1}^{\infty}$ converges strongly to a unique solution $x^* \in \mathcal{F}$ to the VI (3.16).

Proof. Since $\lim_{k\to\infty} \gamma_k = 0$ and $0 < \liminf_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \beta_k < 1$, we may assume, without lossof generality, that $0 < \gamma_k \leqslant \sqrt{\frac{1-\delta}{\zeta}}$ and $\{\beta_k\} \subset [a, b] \subset (0, 1)$. In this case, it is easy to see that

$$\lim_{k\to\infty} \left(\frac{\gamma_{k+1}}{1-(1-\gamma_{k+1})\beta_{k+1}} - \frac{\gamma_k}{1-(1-\gamma_k)\beta_k}\right) = 0.$$

Repeating the same arguments as in the proof of Theorem 3.9 we know that $\{x_k\}_{k=1}^{\infty}$ is bounded. So, the sequences $\{Gx_k\}_{k=1}^{\infty}$, $\{V_kGx_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ and $\{F(x_k)\}_{k=1}^{\infty}$, where $y_k = \beta_k x_k + (1-\beta_k)V_kGx_k$, are also bounded.

Repeating the same arguments as those of (3.19) and (3.20) in the proof of Theorem 3.9 we know that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0, \quad \lim_{k\to\infty} \|x_k - y_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|x_k - V_kGx_k\| = 0.$$

Repeating the same arguments as those of (3.26) in the proof of Theorem 3.9 we know that

$$\limsup_{k\to\infty} \langle F(x^*), j(x^* - x_k)\rangle \leqslant 0. \tag{3.28}$$

Next, let us show $\lim_{k\to\infty} \|x_k - x^*\| = 0$. Indeed, observe that

$$\begin{aligned} x_{k+1} - x^* &= \gamma_k[(I-\lambda_k F)x_k - x^*] + (1-\gamma_k)(y_k - x^*) \\ &= \gamma_k[(I-\lambda_k F)x_k - x^*] + (1-\gamma_k)(1-\beta_k)(V_kGx_k - x^*) + (1-\gamma_k)\beta_k(x_k - x^*). \end{aligned}$$

Then, utilizing Lemma 2.10 (i), we get

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\leqslant \|(1-\gamma_k)\beta_k(x_k - x^*) + (1-\gamma_k)(1-\beta_k)(V_kGx_k - x^*)\|^2 \\ &\quad + 2\gamma_k\langle (I-\lambda_k F)x_k - x^*, j(x_{k+1} - x^*)\rangle \\ &\leqslant [(1-\gamma_k)\beta_k\|x_k - x^*\| + (1-\gamma_k)(1-\beta_k)\|x_k - x^*\|]^2 \\ &\quad + 2\gamma_k\langle (I-\lambda_k F)x_k - (I-\lambda_k F)x^*, j(x_{k+1} - x^*)\rangle \\ &\quad + 2\gamma_k\langle (I-\lambda_k F)x^* - x^*, j(x_{k+1} - x^*)\rangle \\ &\leqslant (1-\gamma_k)^2\|x_k - x^*\|^2 + 2\gamma_k(1-\lambda_k\tau)\|x_k - x^*\|\|x_{k+1} - x^*\| \\ &\quad + 2\gamma_k\lambda_k\langle F(x^*), j(x^* - x_{k+1})\rangle \\ &\leqslant (1-\gamma_k)^2\|x_k - x^*\|^2 + \gamma_k(1-\lambda_k\tau)[\|x_k - x^*\|^2 + \|x_{k+1} - x^*\|^2] \\ &\quad + 2\gamma_k\lambda_k\langle F(x^*), j(x^* - x_{k+1})\rangle \\ &= [(1-\gamma_k)^2 + \gamma_k(1-\lambda_k\tau)]\|x_k - x^*\|^2 + \gamma_k(1-\lambda_k\tau)\|x_{k+1} - x^*\|^2 \\ &\quad + 2\gamma_k\lambda_k\langle F(x^*), j(x^* - x_{k+1})\rangle, \end{aligned}$$

which immediately yields

$$\|x_{k+1} - x^*\|^2 \leqslant \frac{(1-\gamma_k)^2 + \gamma_k(1-\lambda_k\tau)}{1-\gamma_k(1-\lambda_k\tau)}\|x_k - x^*\|^2 + \frac{2\gamma_k\lambda_k}{1-\gamma_k(1-\lambda_k\tau)}\langle F(x^*), j(x^* - x_{k+1})\rangle. \tag{3.29}$$

Observe that for all $k \geqslant 1$

$$\begin{aligned} \frac{(1-\gamma_k)^2 + \gamma_k(1-\lambda_k\tau)}{1-\gamma_k(1-\lambda_k\tau)} &= \frac{1-(1-\lambda_k\tau)\gamma_k - 2\gamma_k[1-(1-\lambda_k\tau)] + \gamma_k^2}{1-\gamma_k(1-\lambda_k\tau)} \\ &= 1 - \frac{2\gamma_k[1-(1-\lambda_k\tau)]}{1-\gamma_k(1-\lambda_k\tau)} + \frac{\gamma_k^2}{1-\gamma_k(1-\lambda_k\tau)} \\ &\leqslant 1 - 2\gamma_k[1-(1-\lambda_k\tau)] + \frac{\gamma_k^2}{1-\gamma_k(1-\lambda_k\tau)} \\ &= 1 - 2\gamma_k\lambda_k\tau + \frac{\gamma_k^2}{1-\gamma_k(1-\lambda_k\tau)}. \end{aligned}$$

Then it follows from (3.29) that

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\leqslant (1-2\gamma_k\lambda_k\tau)\|x_k-x^*\|^2+\frac{2\gamma_k}{1-\gamma_k(1-\lambda_k\tau)}[\frac{\gamma_k}{2}\|x_k-x^*\|^2+\lambda_k\langle F(x^*),j(x^*-x_{k+1})\rangle] \\ &= (1-2\gamma_k\lambda_k\tau)\|x_k-x^*\|^2+2\gamma_k\lambda_k\tau\cdot\frac{1}{\tau-\tau\gamma_k(1-\lambda_k\tau)}[\frac{\gamma_k}{2\lambda_k}\|x_k-x^*\|^2 \\ &\quad +\langle F(x^*),j(x^*-x_{k+1})\rangle].\end{aligned}\tag{3.30}$$

Since $\lim_{k\to\infty}\gamma_k=0$ and $\liminf_{k\to\infty}\lambda_k>0$, we deduce from (3.28) that

$$\limsup_{k\to\infty}\frac{1}{\tau-\tau\gamma_k(1-\lambda_k\tau)}[\frac{\gamma_k}{2\lambda_k}\|x_k-x^*\|^2+\langle F(x^*),j(x^*-x_{k+1})\rangle]\leqslant 0.$$

Noticing $\sum_{k=1}^{\infty}\gamma_k=\infty$, we get $\sum_{k=1}^{\infty}2\gamma_k\lambda_k\tau=\infty$. Therefore, according to Lemma 3.7 we conclude from (3.30) that $\lim_{k\to\infty}\|x_k-x^*\|=0$. This completes the proof. □

Next we give a weak convergence theorem for hybrid steepest-descent method (3.15) involving an infinite family $\{T_i\}_{i=1}^{\infty}$ of nonexpansive self-mappings in a Hilbert space H.

Theorem 3.12. *Let C be a nonempty closed convex subset of a Hilbert space H. Let P_C be the metric projection from H onto C. Let the mappings $A,B:X\to X$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $F:X\to X$ be δ-strongly monotone and ζ-strictly pseudocontractive (in the Browder-Petryshyn sense) with $\delta+\zeta>1$. Assume that $\lambda\in(0,2\alpha)$ and $\mu\in(0,2\beta)$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F}:=\bigcap_{i=1}^{\infty}\mathrm{Fix}(T_i)\cap \mathrm{GSVI}(C,A,B)\neq\emptyset$. Let $\{V_k\}_{k=1}^{\infty}$ be defined by (2.8) and (2.9). For an arbitrary $x_1\in H$, let $\{x_k\}_{k=1}^{\infty}$ be generated by*

$$\begin{cases} y_k=\beta_k x_k+(1-\beta_k)V_kGx_k,\\ x_{k+1}=\gamma_k(I-\lambda_kF)x_k+(1-\gamma_k)y_k, \quad \forall k\geqslant 1,\end{cases}$$

where $G:=P_C(I-\lambda A)P_C(I-\mu B)$, and $\{\lambda_k\}_{k=1}^{\infty}\subset(0,1]$, $\{\gamma_k\}_{k=1}^{\infty}\subset(0,1)$ and $\{\beta_k\}_{k=1}^{\infty}\subset[0,1]$ such that

(C1) $\sum_{k=1}^{\infty}\gamma_k<\infty$;

(C2) $0<\liminf_{k\to\infty}\beta_k\leqslant\limsup_{k\to\infty}\beta_k<1$.

Then $\{x_k\}_{k=1}^{\infty}$ converges weakly to an element $x^*\in\mathcal{F}$.

Proof. Take a fixed $p\in\mathcal{F}$ arbitrarily. Repeating the same arguments as in the proof of Theorem 3.9 we know that $\{x_k\}_{k=1}^{\infty}$ is bounded. So, the sequences $\{Gx_k\}_{k=1}^{\infty}$, $\{V_kGx_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ and $\{F(x_k)\}_{k=1}^{\infty}$, where $y_k=\beta_kx_k+(1-\beta_k)V_kGx_k$, are also bounded.

Observe that

$$\begin{aligned}\|x_{k+1}-p\|^2 &\leqslant (1-\gamma_k)\|y_k-p\|^2+\gamma_k\|(I-\lambda_kF)x_k-p\|^2\\ &\leqslant \|y_k-p\|^2+\gamma_k\|(I-\lambda_kF)x_k-p\|^2\\ &= \|\beta_k(x_k-p)+(1-\beta_k)(V_kGx_k-p)\|^2+\gamma_k\|(I-\lambda_kF)x_k-p\|^2\\ &= \beta_k\|x_k-p\|^2+(1-\beta_k)\|V_kGx_k-p\|^2-\beta_k(1-\beta_k)\|x_k-V_kGx_k\|^2\\ &\quad +\gamma_k\|(I-\lambda_kF)x_k-p\|^2\\ &\leqslant \beta_k\|x_k-p\|^2+(1-\beta_k)\|x_k-p\|^2-\beta_k(1-\beta_k)\|x_k-V_kGx_k\|^2\\ &\quad +\gamma_k\|(I-\lambda_kF)x_k-p\|^2\\ &\leqslant \|x_k-p\|^2-\beta_k(1-\beta_k)\|x_k-V_kGx_k\|^2+\gamma_k[\|x_k\|+\|F(x_k)\|+\|p\|]^2\\ &\leqslant \|x_k-p\|^2+\gamma_k[\|x_k\|+\|F(x_k)\|+\|p\|]^2.\end{aligned}\tag{3.31}$$

Since $\sum_{k=1}^{\infty} \gamma_k < \infty$ and $\{x_k\}$ and $\{F(x_k)\}$ are bounded, we get $\sum_{k=1}^{\infty} \gamma_k[\|x_k\| + \|F(x_k)\| + \|p\|]^2 < \infty$. Utilizing Lemma 2.15, we deduce that $\lim_{k\to\infty} \|x_k - p\|$ exists. Furthermore, it follows from (3.31) that for all $k \geqslant 1$

$$\beta_k(1-\beta_k)\|x_k - V_k G x_k\|^2 \leqslant \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + \gamma_k[\|x_k\| + \|F(x_k)\| + \|p\|]^2. \tag{3.32}$$

Since $\gamma_k \to 0$ and $0 < \liminf_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \beta_k < 1$, it follows from (3.32) that

$$\lim_{k\to\infty} \|x_k - V_k G x_k\| = 0. \tag{3.33}$$

Repeating the same arguments as in the proof of Theorem 3.9 we know that

$$\lim_{k\to\infty} \|V_k G x_k - V G x_k\| = 0. \tag{3.34}$$

Note that

$$\|x_k - VGx_k\| \leqslant \|x_k - V_k G x_k\| + \|V_k G x_k - VGx_k\|.$$

Combining (3.33) and (3.34) we get

$$\lim_{k\to\infty} \|x_k - VGx_k\| = 0.$$

Now, let us show that $\omega_w(x_k) \subset \mathcal{F}$. Indeed, let $\bar{x} \in \omega_w(x_k)$. Then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup \bar{x}$. Since $(I - VG)x_k \to 0$, by Lemma 2.16 we know that $\bar{x} \in \mathrm{Fix}(V \circ G) = \mathcal{F}$ (due to Proposition 3.5).

Finally, let us show that $\omega_w(x_k)$ is a singleton. Indeed, let $\{x_{m_i}\}$ be another subsequence of $\{x_k\}$ such that $x_{m_i} \rightharpoonup \hat{x}$. Then $\hat{x}$ is also an element in $\mathcal{F}$. If $\bar{x} \neq \hat{x}$, by Opial's property of H, we reach the following contradiction:

$$\begin{aligned}
\lim_{k\to\infty} \|x_k - \bar{x}\| &= \lim_{i\to\infty} \|x_{k_i} - \bar{x}\| \\
&< \lim_{i\to\infty} \|x_{k_i} - \hat{x}\| = \lim_{i\to\infty} \|x_{m_i} - \hat{x}\| \\
&< \lim_{i\to\infty} \|x_{m_i} - \bar{x}\| \\
&= \lim_{k\to\infty} \|x_k - \bar{x}\|.
\end{aligned}$$

This shows that $\omega_w(x_k)$ is a singleton. Consequently, $\{x_k\}$ converges weakly to an element in $\mathcal{F}$. □

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