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Fixed point theorems in dislocated quasi-metric spaces

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Abstract

In this paper, we discuss the existence and uniqueness of a fixed point in a dislocated quasi-metric space. Several fixed point theorems for distinct type of contractive conditions are presented that generalize, extend, and unify a number of related results reported in the literature. Illustrative examples are provided.

Keywords: Dislocated quasi-metric space, fixed point, contraction mapping, self-mapping, Cauchy sequence.
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1. Introduction

The notion of fixed point theory was presented by Poincaré in 1886. Fixed point theory is one of the most crucial and dynamic research subject of nonlinear analysis. In the area of fixed point theory, the first important and remarkable result was presented by Banach [4] for a contraction mapping in a complete metric space. Since then, a number of generalizations have been made by many researchers in their works. For instance, Dass and Gupta [5] presented the generalized form of well-known Banach contraction principle in a metric space for some rational type contractive conditions. The idea of metric space has also been generalized in different directions. Some of well-known and important generalizations of metric spaces are dislocated metric space, quasi-metric space, dislocated quasi-metric space, generalized quasi-metric space, b-metric space, cone metric space, cone b-metric space, etc.

Abramsky and Jung [3] discussed some facts about dislocated metrics under the special name of metric domains in the context of domain theory. Hitzler and Seda [6] presented the concept of dislocated metric spaces and generalized the well-known Banach contraction mapping principle in complete dislocated metric spaces. Dislocated metric space has a key role in logic programming and electronics engineering. Zeyada et al. [15] established the notion of dislocated quasi-metric spaces by generalizing the results of

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Hitzler and Seda [6]. Aage and Salunke [1, 2], Isufati [7], Jha and Panthi [8], Jha et al. [9], Panthi et al. [10], Patel and Patel [11], Sarwar et al. [13], Shrivastava et al. [14], Zoto et al. [16], and Zoto et al. [17] gave some fixed point results in dislocated metric and dislocated quasi-metric spaces. Piri [12] obtained some Suzuki type fixed point theorems in complete cone b-metric spaces over a solid vector space.

In the present paper, we prove some fixed point results in the setting of dislocated quasi-metric spaces for single and a pair of continuous self-mappings which generalize, improve, and fuse the results reported in the cited papers. Throughout, $\mathbb{R}^+$ stands for the set of all nonnegative real numbers.

2. Preliminaries

We need the following auxiliary definitions and results.

Definition 2.1 ([15]). Let $\mathcal{X}$ be a nonempty set and $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a distance function. We give the following conditions:

(c_1) $d(\xi, \xi) = 0$ for all $\xi \in \mathcal{X}$;

(c_2) $d(\xi, \eta) = d(\eta, \xi) = 0$ yields $\xi = \eta$;

(c_3) $d(\xi, \eta) = d(\eta, \xi)$ for all $\xi, \eta \in \mathcal{X}$;

(c_4) $d(\xi, \eta) \leqslant d(\xi, \zeta) + d(\zeta, \eta)$ for all $\xi, \eta, \zeta \in \mathcal{X}$.

If d satisfies conditions (c_1)-(c_4), then d is called a metric on $\mathcal{X}$, if d satisfies conditions (c_2)-(c_4), then d is said to be a dislocated metric (d-metric) on $\mathcal{X}$, and if d satisfies conditions (c_2) and (c_4), then d is called a dislocated quasi-metric (dq-metric) on $\mathcal{X}$. $\mathcal{X}$ together with dq-metric d, i.e., $(\mathcal{X}, d)$ is termed a dislocated quasi-metric space (dq-metric space).

It is clear that every metric on $\mathcal{X}$ is also a d-metric on $\mathcal{X}$, but the converse is not true. The following example shows this fact.

Example 2.2 ([13]). Let $\mathcal{X} = \mathbb{R}^+$ and define the distance function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ by

$$d(\xi, \eta) = \max\{\xi, \eta\}.$$

Definition 2.3 ([15]). A sequence $\{\xi_n\}$ in a dq-metric space $(\mathcal{X}, d)$ is called Cauchy sequence if for any $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $d(\xi_m, \xi_n) < \epsilon$ or $d(\xi_n, \xi_m) < \epsilon$ for all $m, n \geqslant n_0$.

In the above definition, if we replace $d(\xi_m, \xi_n) < \epsilon$ or $d(\xi_n, \xi_m) < \epsilon$ with

$$\max\{d(\xi_m, \xi_n), d(\xi_n, \xi_m)\} < \epsilon,$$

then $\{\xi_n\}$ is called "bi" Cauchy sequence.

Definition 2.4 ([15]). A sequence $\{\xi_n\}$ in a dq-metric space $(\mathcal{X}, d)$ is said to be dislocated quasi-convergent (dq-convergent) to ξ if

$$\lim_{n\to\infty} d(\xi_n, \xi) = \lim_{n\to\infty} d(\xi, \xi_n) = 0.$$

In this case, ξ is called a dq-limit of sequence $\{\xi_n\}$ and we write $\xi_n \to \xi$ as $n \to \infty$.

Definition 2.5 ([15]). Let $(\mathcal{X}, d_1)$ and (Y, d_2) be two dq-metric spaces and let $\mathcal{T} : \mathcal{X} \to Y$ be a function. $\mathcal{T}$ is said to be continuous if for each sequence $\{\xi_n\}$ which is d_1q-convergent to $\xi_0 \in \mathcal{X}$, the sequence $\{\mathcal{T}\xi_n\}$ is d_2q-convergent to $\mathcal{T}\xi_0$ in Y.

Definition 2.6 ([15]). A dq-metric space $(\mathcal{X}, d)$ is called complete if every Cauchy sequence in it is dq-convergent.

Definition 2.7 ([15]). Let $(\mathcal{X}, d)$ be a dq-metric space. A self-mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called contraction if there exists a $\lambda \in [0, 1)$ such that $d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant \lambda d(\xi, \eta)$ for all $\xi, \eta \in \mathcal{X}$.

Lemma 2.8 ([15]). *Every subsequence of a* dq*-convergent sequence to a point* ξ_0 *is* dq*-convergent to* ξ_0.

Lemma 2.9 ([15]). *Let* $(\mathcal{X}, d)$ *be a* dq*-metric space. If* $g : \mathcal{X} \to \mathcal{X}$ *is a contraction function, then* $\{g^n(\xi_0)\}$ *is a Cauchy sequence for each* $\xi_0 \in \mathcal{X}$.

Lemma 2.10 ([15]). dq*-limits in a* dq*-metric space are unique.*

In what follows, we present some fixed point theorems in dq-metric spaces which consist of contractive type conditions and rational contractive conditions.

Theorem 2.11 ([15]). *Let* $(\mathcal{X}, d)$ *be a complete* dq*-metric space and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a continuous contraction function. Then* $\mathcal{T}$ *has a unique fixed point.*

Theorem 2.12 ([2]). *Let* $(\mathcal{X}, d)$ *be a complete* dq*-metric space and* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a continuous self-mapping satisfying*

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant \lambda[d(\xi, \mathcal{T}\xi) + d(\eta, \mathcal{T}\eta)]$$

for all $\xi, \eta \in \mathcal{X}$, *where* $0 \leqslant \lambda < 1/2$. *Then* $\mathcal{T}$ *has a unique fixed point.*

Theorem 2.13 ([7]). *Let* $(\mathcal{X}, d)$ *be a complete* dq*-metric space and* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a continuous self-mapping satisfying*

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \mathcal{T}\eta) + a_2^* d(\eta, \mathcal{T}\xi) + a_3^* d(\xi, \eta)$$

for all $\xi, \eta \in \mathcal{X}$, *where* $a_1^*, a_2^*, a_3^* \in \mathbb{R}^+$ *which may depend on both* ξ *and* η *and satisfy* $\sup\{2a_1^* + 2a_2^* + a_3^*\} < 1$. *Then* $\mathcal{T}$ *has a unique fixed point.*

Theorem 2.14 ([14]). *Let* $\mathcal{T}$ *be a continuous self-mapping defined on a complete* dq*-metric space* $(\mathcal{X}, d)$. *If*

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \eta) + a_2^* \frac{d(\xi, \mathcal{T}\xi) d(\eta, \mathcal{T}\eta)}{d(\xi, \eta)}$$

for all $\xi, \eta \in \mathcal{X}$ *satisfying* $d(\xi, \eta) \neq 0$, *where* $a_1^*, a_2^* \in \mathbb{R}^+$ *and* $0 \leqslant a_1^* + a_2^* < 1$, *then* $\mathcal{T}$ *has a unique fixed point.*

Theorem 2.15 ([14]). *Let* $(\mathcal{X}, d)$ *be a complete* dq*-metric space and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a continuous self-mapping. If*

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \eta) + a_2^* \frac{d(\xi, \mathcal{T}\xi) d(\eta, \mathcal{T}\eta)}{d(\xi, \eta)} + a_3^*[d(\xi, \mathcal{T}\xi) + d(\eta, \mathcal{T}\eta)] + a_4^*[d(\xi, \mathcal{T}\eta) + d(\eta, \mathcal{T}\xi)]$$

for all $\xi, \eta \in \mathcal{X}$ *satisfying* $d(\xi, \eta) \neq 0$, *where* $a_1^*, a_2^*, a_3^*, a_4^* \in \mathbb{R}^+$ *and* $0 \leqslant a_1^* + a_2^* + 2a_3^* + 2a_4^* < 1$, *then* $\mathcal{T}$ *has a unique fixed point.*

Theorem 2.16 ([16]). *Let* $(\mathcal{X}, d)$ *be a complete* dq*-metric space and* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a continuous self-mapping. If*

$$\begin{aligned} d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \eta) + a_2^* \frac{d(\xi, \mathcal{T}\xi) d(\eta, \mathcal{T}\eta)}{d(\xi, \eta)} + a_3^*[d(\xi, \mathcal{T}\xi) + d(\eta, \mathcal{T}\eta)] \\ + a_4^*[d(\xi, \mathcal{T}\eta) + d(\eta, \mathcal{T}\xi)] + a_5^*[d(\xi, \mathcal{T}\xi) + d(\xi, \eta)] \end{aligned}$$

for all $\xi, \eta \in \mathcal{X}$ *satisfying* $d(\xi, \eta) \neq 0$, *where* $a_1^*, a_2^*, a_3^*, a_4^*, a_5^* \in \mathbb{R}^+$ *and* $0 \leqslant a_1^* + a_2^* + 2a_3^* + 2a_4^* + 2a_5^* < 1$, *then* $\mathcal{T}$ *has a unique fixed point.*

Theorem 2.17 ([10]). *Let* $(\mathcal{X}, d)$ *be a complete* dq*-metric space and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a continuous self-mapping. Assume that*

$$\begin{aligned} d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \eta) + a_2^* \frac{d(\xi, \mathcal{T}\xi) d(\eta, \mathcal{T}\eta)}{d(\xi, \eta)} + a_3^*[d(\xi, \mathcal{T}\xi) + d(\eta, \mathcal{T}\eta)] + a_4^*[d(\xi, \mathcal{T}\eta) + d(\eta, \mathcal{T}\xi)] \\ + a_5^*[d(\xi, \mathcal{T}\xi) + d(\xi, \eta)] + a_6^*[d(\eta, \mathcal{T}\eta) + d(\xi, \eta)] \end{aligned}$$

for all $\xi, \eta \in \mathcal{X}$ *satisfying* $d(\xi, \eta) \neq 0$, *where* $a_1^*, a_2^*, a_3^*, a_4^*, a_5^*, a_6^* \in \mathbb{R}^+$ *and*

$$0 \leqslant a_1^* + a_2^* + 2a_3^* + 4a_4^* + 2a_5^* + 2a_6^* < 1.$$

Then $\mathcal{T}$ has a unique fixed point.

Theorem 2.18 ([1]). *Let $(\mathcal{X}, d)$ be a complete* dq*-metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous self-mapping. If*

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \eta) + a_2^* d(\xi, \mathcal{T}\xi) + a_3^* d(\eta, \mathcal{T}\eta)$$

for all $\xi, \eta \in \mathcal{X}$, where $a_1^*, a_2^*, a_3^* \in \mathbb{R}^+$ and $0 \leqslant a_1^* + a_2^* + a_3^* < 1$, then $\mathcal{T}$ has a unique fixed point.

Theorem 2.19 ([1]). *Let $(\mathcal{X}, d)$ be a complete* dq*-metric space and let $S, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be two continuous self-mappings. If*

$$d(S\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \eta) + a_2^* d(\xi, S\xi) + a_3^* d(\eta, \mathcal{T}\eta)$$

for all $\xi, \eta \in \mathcal{X}$, where $a_1^*, a_2^*, a_3^* \in \mathbb{R}^+$ and $0 \leqslant a_1^* + a_2^* + a_3^* < 1$, then S and $\mathcal{T}$ have a unique common fixed point.

Theorem 2.20 ([11]). *Let $(\mathcal{X}, d)$ be a complete* dq*-metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous self-mapping satisfying*

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \eta) + a_2^* d(\xi, \mathcal{T}\xi) + a_3^* d(\eta, \mathcal{T}\eta) + a_4^*[d(\xi, \mathcal{T}\xi) + d(\eta, \mathcal{T}\eta)] + a_5^*[d(\xi, \mathcal{T}\eta) + d(\eta, \mathcal{T}\xi)]$$

for all $\xi, \eta \in \mathcal{X}$, where $a_1^*, a_2^*, a_3^*, a_4^*, a_5^* \in \mathbb{R}^+$ and $0 \leqslant a_1^* + a_2^* + a_3^* + 2a_4^* + 2a_5^* < 1$. Then $\mathcal{T}$ has a unique fixed point.

Theorem 2.21 ([13]). *Let $(\mathcal{X}, d)$ be a complete* dq*-metric space and let $S, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be two continuous self-mappings satisfying*

$$d(S\xi, \mathcal{T}\eta) \leqslant a_1^* d(\xi, \eta) + a_2^* d(\xi, S\xi) + a_3^* d(\eta, \mathcal{T}\eta) + a_4^*[d(\xi, S\xi) + d(\eta, \mathcal{T}\eta)] + a_5^*[d(\xi, \mathcal{T}\eta) + d(\eta, S\xi)]$$

for all $\xi, \eta \in \mathcal{X}$, where $a_1^*, a_2^*, a_3^*, a_4^*, a_5^* \in \mathbb{R}^+$ and $0 \leqslant a_1^* + a_2^* + a_3^* + 2a_4^* + 4a_5^* < 1$. Then S and $\mathcal{T}$ have a unique common fixed point.

3. Main results

In this section, we prove some fixed point theorems in complete dq-metric spaces.

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a complete* dq*-metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous self-mapping. If*

$$\begin{aligned} d(\mathcal{T}\xi, \mathcal{T}\eta) \leqslant{}& a_1 d(\xi, \eta) + a_2 \frac{d(\xi, \mathcal{T}\xi) d(\eta, \mathcal{T}\eta)}{d(\xi, \eta)} + a_3[d(\xi, \mathcal{T}\xi) + d(\eta, \mathcal{T}\eta)] + a_4[d(\xi, \mathcal{T}\eta) + d(\eta, \mathcal{T}\xi)] \\ &+ a_5[d(\xi, \mathcal{T}\xi) + d(\xi, \eta)] + a_6[d(\eta, \mathcal{T}\eta) + d(\xi, \eta)] + a_7[d(\xi, \mathcal{T}\eta) + d(\xi, \eta)] \end{aligned} \tag{3.1}$$

for all $\xi, \eta \in \mathcal{X}$ satisfying $d(\xi, \eta) \neq 0$, where $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{R}^+$ and

$$0 \leqslant a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 < 1,$$

then $\mathcal{T}$ has a unique fixed point.

Proof. Let $\xi_0 \in \mathcal{X}$ and define a sequence $\{\xi_n\}$ by

$$\mathcal{T}\xi_n = \xi_{n+1} \quad \text{for } n = 0, 1, 2, \ldots,$$

where $d(\xi_{n-1}, \xi_n) \neq 0$. Set $\xi = \xi_{n-1}$ and $\eta = \xi_n$. It follows from (3.1) and

$$d(\xi_n, \xi_n) \leqslant d(\xi_{n-1}, \xi_n) + d(\xi_n, \xi_{n+1})$$

that

$$\begin{aligned}
d(\xi_n, \xi_{n+1}) &= d(\mathcal{T}\xi_{n-1}, \mathcal{T}\xi_n) \\
&\leqslant a_1 d(\xi_{n-1}, \xi_n) + a_2 \frac{d(\xi_{n-1}, \mathcal{T}\xi_{n-1}) d(\xi_n, \xi_{n+1})}{d(\xi_{n-1}, \xi_n)} \\
&\quad + a_3[d(\xi_{n-1}, \mathcal{T}\xi_{n-1}) + d(\xi_n, \mathcal{T}\xi_n)] + a_4[d(\xi_{n-1}, \mathcal{T}\xi_n) + d(\xi_n, \mathcal{T}\xi_{n-1})] \\
&\quad + a_5[d(\xi_{n-1}, \mathcal{T}\xi_{n-1}) + d(\xi_{n-1}, \xi_n)] + a_6[d(\xi_n, \mathcal{T}\xi_n) + d(\xi_{n-1}, \xi_n)] \\
&\quad + a_7[d(\xi_{n-1}, \mathcal{T}\xi_n) + d(\xi_{n-1}, \xi_n)] \\
&= a_1 d(\xi_{n-1}, \xi_n) + a_2 \frac{d(\xi_{n-1}, \xi_n) d(\xi_n, \xi_{n+1})}{d(\xi_{n-1}, \xi_n)} + a_3[d(\xi_{n-1}, \xi_n) + d(\xi_n, \xi_{n+1})] \\
&\quad + a_4[d(\xi_{n-1}, \xi_{n+1}) + d(\xi_n, \xi_n)] + a_5[d(\xi_{n-1}, \xi_n) + d(\xi_{n-1}, \xi_n)] \\
&\quad + a_6[d(\xi_n, \xi_{n+1}) + d(\xi_{n-1}, \xi_n)] + a_7[d(\xi_{n-1}, \xi_{n+1}) + d(\xi_{n-1}, \xi_n)] \\
&\leqslant (a_1 + a_3 + 2a_4 + 2a_5 + a_6 + 2a_7) d(\xi_{n-1}, \xi_n) + (a_2 + a_3 + 2a_4 + a_6 + a_7) d(\xi_n, \xi_{n+1}).
\end{aligned}$$

Hence, we have

$$d(\xi_n, \xi_{n+1}) \leqslant \frac{a_1 + a_3 + 2a_4 + 2a_5 + a_6 + 2a_7}{1 - (a_2 + a_3 + 2a_4 + a_6 + a_7)} d(\xi_{n-1}, \xi_n).$$

Let

$$h = \frac{a_1 + a_3 + 2a_4 + 2a_5 + a_6 + 2a_7}{1 - (a_2 + a_3 + 2a_4 + a_6 + a_7)}.$$

Then $0 \leqslant h < 1$,

$$d(\xi_n, \xi_{n+1}) \leqslant h d(\xi_{n-1}, \xi_n) \quad \text{and} \quad d(\xi_{n-1}, \xi_n) \leqslant h d(\xi_{n-2}, \xi_{n-1}).$$

Continuing this process, we conclude that

$$d(\xi_n, \xi_{n+1}) \leqslant h^n d(\xi_0, \xi_1).$$

Now, for any m, n satisfying $m > n$, using triangle inequality, we obtain

$$\begin{aligned}
d(\xi_n, \xi_m) &\leqslant d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+2}) + \cdots + d(\xi_{m-1}, \xi_m) \\
&\leqslant h^n d(\xi_0, \xi_1) + h^{n+1} d(\xi_0, \xi_1) + \cdots + h^{m-1} d(\xi_0, \xi_1) \\
&\leqslant (h^n + h^{n+1} + h^{n+2} + \cdots) d(\xi_0, \xi_1) = \frac{h^n}{1-h} d(\xi_0, \xi_1).
\end{aligned}$$

By $h \in [0, 1)$, $h^n \to 0$ as $n \to \infty$, which shows that $\{\xi_n\}$ is a Cauchy sequence in the complete dq-metric space $(\mathcal{X}, d)$. Therefore, by virtue of the fact that $\mathcal{T}$ is continuous, there exists a point $p \in \mathcal{X}$ such that

$$\mathcal{T}p = \mathcal{T} \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \mathcal{T}\xi_n = \lim_{n\to\infty} \xi_{n+1} = p.$$

Uniqueness. Suppose to the contrary that p and p^* are two different fixed points of $\mathcal{T}$. Then $\mathcal{T}p = p$ and $\mathcal{T}p^* = p^*$. We assert that $d(p, p) = d(p^*, p^*) = 0$. If $d(p, p) > 0$ and $d(p^*, p^*) > 0$, then we derive from (3.1) that

$$\begin{aligned}
d(p, p) &= d(\mathcal{T}p, \mathcal{T}p) \\
&\leqslant a_1 d(p, p) + a_2 d(p, p) + 2a_3 d(p, p) + 2a_4 d(p, p) \\
&\quad + 2a_5 d(p, p) + 2a_6 d(p, p) + 2a_7 d(p, p) \\
&= (a_1 + a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7) d(p, p)
\end{aligned}$$

and

$$d(p^*, p^*) \leqslant (a_1 + a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7) d(p^*, p^*),$$

respectively, which are contradictions due to $0 \leqslant a_1 + a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 < 1$. Now, we consider the following three cases separately.

Case 1. Assume first that $d(p, p^*) > 0$ and $d(p^*, p) = 0$. An application of (3.1) yields

$$\begin{aligned} d(p, p^*) &= d(\mathcal{T}p, \mathcal{T}p^*) \\ &\leqslant a_1 d(p, p^*) + a_2 \frac{d(p, p)d(p^*, p^*)}{d(p, p^*)} + a_3[d(p, p) + d(p^*, p^*)] + a_4[d(p, p^*) + d(p^*, p)] \\ &\quad + a_5[d(p, p) + d(p, p^*)] + a_6[d(p^*, p^*) + d(p, p^*)] + a_7[d(p, p^*) + d(p, p^*)] \\ &= (a_1 + a_4 + a_5 + a_6 + 2a_7)d(p, p^*) + a_4 d(p^*, p), \end{aligned} \tag{3.2}$$

which is a contradiction due to $d(p^*, p) = 0$ and $0 \leqslant a_1 + a_4 + a_5 + a_6 + 2a_7 < 1$.

Case 2. Suppose that $d(p, p^*) = 0$ and $d(p^*, p) > 0$. By virtue of (3.1),

$$d(p^*, p) \leqslant (a_1 + a_4 + a_5 + a_6 + 2a_7)d(p^*, p) + a_4 d(p, p^*), \tag{3.3}$$

which is a contradiction due to $d(p, p^*) = 0$ and $0 \leqslant a_1 + a_4 + a_5 + a_6 + 2a_7 < 1$.

Case 3. Assume now that $d(p, p^*) > 0$ and $d(p^*, p) > 0$. It follows from (3.1) that (3.2) and (3.3) hold. Combining (3.2) and (3.3), we are led to

$$|d(p, p^*) - d(p^*, p)| \leqslant (a_1 + a_5 + a_6 + 2a_7)|d(p, p^*) - d(p^*, p)|,$$

which implies that $d(p, p^*) = d(p^*, p)$ due to $0 \leqslant a_1 + a_5 + a_6 + 2a_7 < 1$. Using (3.2), we conclude that

$$d(p, p^*) \leqslant (a_1 + 2a_4 + a_5 + a_6 + 2a_7)d(p, p^*),$$

which is a contradiction due to $0 \leqslant a_1 + 2a_4 + a_5 + a_6 + 2a_7 < 1$.

Hence, $d(p, p^*) = d(p^*, p) = 0$. This fact yields $p = p^*$. The proof is complete. □

Example 3.2. Let $\mathcal{X} = [0, 1]$. Define a complete dq-metric by $d(\xi, \eta) = |\eta|$ for all $\xi, \eta \in \mathcal{X}$ and define a continuous self-mapping $\mathcal{T}$ by $\mathcal{T}\eta = \eta/4$ for all $\eta \in \mathcal{X}$. Set $a_1 = 1/8$, $a_2 = 1/12$, $a_3 = 1/20$, $a_4 = 1/24$, $a_5 = 1/30$, $a_6 = 1/34$, and $a_7 = 1/30$. Then $\mathcal{T}$ satisfies all assumptions of Theorem 3.1 and $\eta = 0$ is the unique fixed point of $\mathcal{T}$ in $\mathcal{X}$.

Remark 3.3. Theorem 3.1 includes a number of results in [2, 7, 10, 14–16]; see the following details.

(1) If $a_7 = 0$, then Theorem 3.1 reduces to Theorem 2.17.

(2) If $a_6 = a_7 = 0$, then Theorem 3.1 includes some assumptions in Theorem 2.16.

(3) If $a_5 = a_6 = a_7 = 0$, then Theorem 3.1 includes some conditions in Theorem 2.15.

(4) If $a_2 = a_3 = a_5 = a_6 = a_7 = 0$, then Theorem 3.1 involves some assumptions in Theorem 2.13.

(5) If $a_3 = a_4 = a_5 = a_6 = a_7 = 0$, then Theorem 3.1 reduces to Theorem 2.14.

(6) If $a_1 = a_2 = a_4 = a_5 = a_6 = a_7 = 0$, then Theorem 3.1 reduces to Theorem 2.12.

(7) If $a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$, then Theorem 3.1 reduces to Theorem 2.11.

Theorem 3.4. *Let* $(\mathcal{X}, d)$ *be a complete* dq*-metric space and let* $S, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be two continuous self-mappings. If*

$$\begin{aligned} d(S\xi, \mathcal{T}\eta) \leqslant\ & a_1 d(\xi, \eta) + a_2 d(\xi, S\xi) + a_3 d(\eta, \mathcal{T}\eta) + a_4[d(\xi, S\xi) + d(\eta, \mathcal{T}\eta)] \\ & + a_5[d(\xi, \mathcal{T}\eta) + d(\eta, S\xi)] + a_6[d(\xi, S\xi) + d(\xi, \eta)] \end{aligned} \tag{3.4}$$

for all $\xi, \eta \in \mathcal{X}$*, where* $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}^+$ *and* $0 \leqslant a_1 + a_2 + a_3 + 2a_4 + 4a_5 + 2a_6 < 1$*, then* S *and* $\mathcal{T}$ *have a unique common fixed point.*

Proof. Let $\xi_0 \in \mathcal{X}$ and define a sequence $\{\xi_n\}$ by

$$\xi_1 = S\xi_0, \cdots, \xi_{2n+1} = S\xi_{2n}$$

and

$$\xi_2 = \mathcal{T}\xi_1, \cdots, \xi_{2n} = \mathcal{T}\xi_{2n-1}$$

for all $n \in \mathbb{N}$. We claim that $\{\xi_n\}$ is a Cauchy sequence in $\mathcal{X}$. Note that

$$d(\xi_{2n+1}, \xi_{2n+2}) = d(S\xi_{2n}, \mathcal{T}\xi_{2n+1}).$$

By virtue of (3.4),

$$\begin{aligned} d(\xi_{2n+1}, \xi_{2n+2}) \leqslant\ & a_1 d(\xi_{2n}, \xi_{2n+1}) + a_2 d(\xi_{2n}, S\xi_{2n}) + a_3 d(\xi_{2n+1}, \mathcal{T}\xi_{2n+1}) \\ & + a_4[d(\xi_{2n}, S\xi_{2n}) + d(\xi_{2n+1}, \mathcal{T}\xi_{2n+1})] + a_5[d(\xi_{2n}, \mathcal{T}\xi_{2n+1}) + d(\xi_{2n+1}, S\xi_{2n})] \\ & + a_6[d(\xi_{2n}, S\xi_{2n}) + d(\xi_{2n}, \xi_{2n+1})] \\ =\ & a_1 d(\xi_{2n}, \xi_{2n+1}) + a_2 d(\xi_{2n}, \xi_{2n+1}) + a_3 d(\xi_{2n+1}, \xi_{2n+2}) \\ & + a_4[d(\xi_{2n}, \xi_{2n+1}) + d(\xi_{2n+1}, \xi_{2n+2})] + a_5[d(\xi_{2n}, \xi_{2n+2}) + d(\xi_{2n+1}, \xi_{2n+1})] \\ & + a_6[d(\xi_{2n}, \xi_{2n+1}) + d(\xi_{2n}, \xi_{2n+1})] \\ \leqslant\ & (a_1 + a_2 + a_4 + 2a_5 + 2a_6) d(\xi_{2n}, \xi_{2n+1}) + (a_3 + a_4 + 2a_5) d(\xi_{2n+1}, \xi_{2n+2}), \end{aligned}$$

which implies that

$$d(\xi_{2n+1}, \xi_{2n+2}) \leqslant \frac{a_1 + a_2 + a_4 + 2a_5 + 2a_6}{1 - (a_3 + a_4 + 2a_5)} d(\xi_{2n}, \xi_{2n+1}).$$

Let

$$h = \frac{a_1 + a_2 + a_4 + 2a_5 + 2a_6}{1 - (a_3 + a_4 + 2a_5)}.$$

Then $0 \leqslant h < 1$,

$$d(\xi_{2n+1}, \xi_{2n+2}) \leqslant h d(\xi_{2n}, \xi_{2n+1}) \quad \text{and} \quad d(\xi_{2n}, \xi_{2n+1}) \leqslant h d(\xi_{2n-1}, \xi_{2n}),$$

and thus

$$d(\xi_{2n+1}, \xi_{2n+2}) \leqslant h^2 d(\xi_{2n-1}, \xi_{2n}).$$

Continuing this process, we get

$$d(\xi_n, \xi_{n+1}) \leqslant h^n d(\xi_0, \xi_1).$$

Since $h \in [0, 1)$, $h^n \to 0$ as $n \to \infty$, which shows that $\{\xi_n\}$ is a Cauchy sequence in the complete dq-metric space $(\mathcal{X}, d)$. Therefore, there exists a $p^* \in \mathcal{X}$ such that $\xi_n \to p^*$ as $n \to \infty$. Furthermore, the subsequences $\{\xi_{2n}\}$ and $\{\xi_{2n+1}\}$ converge to p^*. Since $\mathcal{T}$ is a continuous mapping, we deduce that

$$\lim_{n\to\infty} \xi_{2n+1} = p^* \Rightarrow \lim_{n\to\infty} \mathcal{T}\xi_{2n+1} = \mathcal{T}p^* \Rightarrow \lim_{n\to\infty} \xi_{2n+2} = \mathcal{T}p^*.$$

Hence, $\mathcal{T}p^* = p^*$. Similarly, taking into account that S is continuous, we prove that $Sp^* = p^*$. Therefore, p^* is a common fixed point of S and $\mathcal{T}$.

Uniqueness. Suppose to the contrary that S and $\mathcal{T}$ have two different common fixed points p^* and q^*. By virtue of (3.4), we can obtain $d(p^*, p^*) = d(q^*, q^*) = 0$. It follows from (3.4) that

$$\begin{aligned} d(p^*, q^*) &= d(Sp^*, \mathcal{T}q^*) \\ &\leqslant a_1 d(p^*, q^*) + a_2 d(p^*, Sp^*) + a_3 d(q^*, \mathcal{T}q^*) + a_4[d(p^*, Sp^*) + d(q^*, \mathcal{T}q^*)] \\ &\quad + a_5[d(p^*, \mathcal{T}q^*) + d(q^*, Sp^*)] + a_6[d(p^*, Sp^*) + d(p^*, q^*)] \\ &= (a_1 + a_5 + a_6) d(p^*, q^*) + a_5 d(q^*, p^*). \end{aligned} \tag{3.5}$$

Similarly,

$$d(q^*, p^*) \leqslant (a_1 + a_5 + a_6)d(q^*, p^*) + a_5 d(p^*, q^*). \tag{3.6}$$

Subtracting (3.6) from (3.5), we have

$$|d(p^*, q^*) - d(q^*, p^*)| \leqslant (a_1 + a_6)|d(p^*, q^*) - d(q^*, p^*)|,$$

which implies that $d(p^*, q^*) = d(q^*, p^*)$ due to $0 \leqslant a_1 + a_6 < 1$. Using (3.5), we obtain

$$d(p^*, q^*) \leqslant (a_1 + 2a_5 + a_6)d(p^*, q^*),$$

which yields $d(p^*, q^*) = 0$ due to $0 \leqslant a_1 + 2a_5 + a_6 < 1$. Hence, $d(p, p^*) = d(p^*, p) = 0$. An application of this fact implies that $p = p^*$. The proof is complete. □

Example 3.5. Let $\mathcal{X} = [0, 1]$. Define a complete dq-metric by $d(\xi, \eta) = |\eta|$ for all $\xi, \eta \in \mathcal{X}$ and define two continuous self-mappings S and $\mathcal{T}$ by $S\eta = 0$ and $\mathcal{T}\eta = \eta/6$ for all $\eta \in \mathcal{X}$. Set $a_1 = 1/8$, $a_2 = 1/12$, $a_3 = 1/14$, $a_4 = 1/20$, $a_5 = 1/21$, and $a_6 = 1/24$. Then the mappings S and $\mathcal{T}$ satisfy all assumptions of Theorem 3.4 and $\eta = 0$ is the unique common fixed point of S and $\mathcal{T}$ in $\mathcal{X}$.

Remark 3.6. Theorem 3.4 includes a number of results in [1, 2, 11, 13]; see the following details.

(1) If $a_6 = 0$, then Theorem 3.4 reduces to Theorem 2.21.

(2) If $S = \mathcal{T}$ and $a_6 = 0$, then Theorem 3.4 includes some conditions in Theorem 2.20.

(3) If $a_4 = a_5 = a_6 = 0$, then Theorem 3.4 reduces to Theorem 2.19.

(4) If $S = \mathcal{T}$ and $a_4 = a_5 = a_6 = 0$, then Theorem 3.4 reduces to Theorem 2.18.

(5) If $S = \mathcal{T}$ and $a_1 = a_2 = a_3 = a_5 = a_6 = 0$, then Theorem 3.4 reduces to Theorem 2.12.

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