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Fractional neutral stochastic differential equations driven by α-stable process

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Abstract

In this paper, we are concerned with a class of fractional neutral stochastic partial differential equations driven by α-stable process. By the stochastic analysis technique, the properties of operator semigroup and combining the Banach fixed-point theorem, we prove the existence and uniqueness of the mild solutions to this kind of equations driven by α-stable process. In the end, an example is given to demonstrate the theory of our work.

Keywords: Fractional neutral SDEs, α-stable process, existence and uniqueness.
2010 MSC: 60H15, 60G15, 60H05.

1. Introduction

In recent years, fractional calculus and fractional differential equations have attracted the attention of many researchers due to their important applications to problems in mathematical physics, chemistry, biology and engineering. Many results on existence and stability of solutions to various type of fractional differential equations have been obtained. For more details on this topic, one can refer to [2, 9, 16, 29].

The deterministic models often fluctuate due to noise. Systems are often subjected to random perturbations. Stochastic differential equations have been investigated by many authors due to playing a very important role in formulation and analysis of many phenomena in economic and finance, physics, mechanics, electric and control engineering, see, for example, Da Prato and Zabczyk [5], Liu [15], Luo and Liu [17], Jahanipur [8], and references therein. Subsequently, with the help of semigroup theory and fractional calculus technique, some authors have also considered fractional stochastic differential equations driven by Brownian motion. One can refer to the literatures [4, 10–13, 21, 22].

On the other hand, many researchers show the widespread interest in the topic of stochastic differential equations driven by α-stable processes owing to the fact that the α-stable noise exhibits the heavy tails, which have plenty of applications to problems in mathematical physics, chemistry, biology and engineering. For example, Priola and Zabczyk [20] gave a proper starting point on the investigation of structural properties of SPDEs driven by an additive cylindrical stable noise, Dong et al. [6] studied the invariant measures of stochastic 2D Navier-Stokes equation driven by α-stable processes, Xu [25] studied the ergodicity of the stochastic real Ginzburg-Landau equation driven by α-stable noise, Zhang [27]

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established the Bismut-Elworthy-Li derivative formula for stochastic differential equations driven by α-stable noise, and Bao and Yuan [3] discussed strong convergence of exponential integrator scheme based on spatial and time discretization for neutral stochastic partial differential equations driven by α-stable processes. However, to the best of our knowledge, there are no results on fractional stochastic differential equations driven by α-stable processes due to the fact that these process only has finite p-th moment for $p \in (0, \alpha)$ and the usual stochastic evolution does not admit a stochastic differential, a fact which leads to some powerful tools such as Itô's formula or Burkholder-Davis-Gundy's inequality in stochastic calculus being unavailable. To close the gap, we will make the attempt to investigate the property of dynamics for fractional stochastic differential equations driven by α-stable processes in this paper.

To this end, in this paper we will focus on the following neutral fractional stochastic partial differential equations with delay driven by α-stable noise

$$\begin{cases} {}^{C}D_t^q[x(t) - g(t, x(t-\tau))] = [Ax(t) + f(t, x(t-\tau))]dt + \sigma(t)dZ(t), & t \in [0, T], \\ x_0(\cdot) = \xi \in D([-\tau, 0], H), & t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where the fractional derivative ${}^{C}D^q$, $q \in (1/2, 1]$, is understood in the Caputo sense, A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t)\}_{t\geqslant 0}$ in a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, $D([-\tau, 0], H)$ is the space of all càdlág functions paths from $[-\tau, 0]$ into H, and $g, f : H \to H$, $\sigma : [0, +\infty) \to \mathbb{R}^+$ are given functions to be specified later. The aim of this paper is to investigate the existence and uniqueness of the mild solution to (1.1) by using the stochastic analysis techniques, the properties of operator semigroup and combining the fixed point theorem.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we devote to investigating the existence and uniqueness of the mild solutions to (1.1). In Section 4, we give an example to illustrate the efficiency of the obtained result.

2. Preliminary

In this section we collect some notions, conceptions and lemmas on α-stable process and recall some basic results which will be used throughout the whole of this paper.

Let $Z(t)$ be a cylindrical α-stable process, $\alpha \in (0, 2)$, defined by

$$Z(t) := \sum_{m=1}^{\infty} \beta_m Z_m(t) e_m. \tag{2.1}$$

Here $\{e_m\}_{m\geqslant 1}$ is an orthonormal basis of H, $\{Z_m(t)\}_{m\geqslant 1}$ are independent, real-valued, normalized, symmetric α-stable Lévy processes defined on stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, and $\{\beta_m\}_{m\geqslant 1}$ is a sequence of positive numbers. Recall that a stochastic process $\{Z_{\alpha,\beta}(t) : t \geqslant 0\}$ is called an α-stable Lévy process if

(i) $Z_{\alpha,\beta}(0) = 0$ a.s.;
(ii) $Z_{\alpha,\beta}$ has independent increments;
(iii) $Z_{\alpha,\beta}(t) - Z_{\alpha,\beta}(s) \sim \eta$ for any $0 \leqslant s < t < \infty$,

where η stands for an α-stable random variable, which is uniquely determined by its characteristic function involving four parameters: $\alpha \in (0, 2]$, the index of stability; $\beta \in [-1, 1]$, the skewness parameter; $\sigma \in (0, \infty)$, the scale parameter; $\mu \in (-\infty, \infty)$ the shift, which has the form

$$\phi_\eta(u) = \mathbb{E}\exp(iu\eta) = \exp\{-\sigma^\alpha |u|^\alpha (1 - i\beta \mathrm{sgn}(u)\Phi) + i\mu u\}, \quad u \in \mathbb{R},$$

where $\Phi = \tan(\pi\alpha/2)$ for $\alpha \neq 1$ and $\Phi = -(2/\pi)\log|u|$ for $\alpha = 1$. We call η is strictly α-stable whenever $\mu = 0$, and if, in addition, $\beta = 0$, η is said to be symmetric α-stable. For a real-valued normalized (standard) symmetric α-stable Lévy process $z(t)$, $\alpha \in (0, 2)$, it has the characteristic function

$$\mathbb{E}\exp(iuz(t)) = e^{-t|u|^\alpha}, \quad u \in \mathbb{R},$$

and the Lévy measure $\lambda_\alpha(dx) := \frac{c_\alpha}{|x|^{1+\alpha}}$, $x \in \mathbb{R} - 0$, where c_α is some constant. For more details of α-stable processes, we can refer to [1] and [24].

Let A be the infinitesimal generator of an analytic semigroup $(S(t))_{t\geqslant 0}$ and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^k$ for $0 < k \leqslant 1$, as a closed linear operator on its domain $D(-A)^k$. Furthermore, the subspace $D(-A)^k$ is dense in H, and the expression

$$\|h\|_k = \|(-A)^k h\|$$

defines a norm in $D(-A)^k$. If H_k represents the space $D(-A)^k$ endowed with the norm $\|\cdot\|_k$, then the following properties are well-known (cf. [19, Theorem 6.13 p.74]).

Lemma 2.1. *Suppose that the previous conditions are satisfied.*

(1) *Let* $0 < k \leqslant 1$. *Then* H_k *is a Banach space.*
(2) *If* $0 < k \leqslant l$ *then the injection* $H_l \hookrightarrow H_k$ *is continuous.*
(3) *For every* $0 < k \leqslant 1$ *there exists* $M_k > 0$ *such that*

$$\|(-A)^k S(t)\| \leqslant M_k t^{-k} e^{-\lambda t}, \quad t > 0,\ \lambda > 0.$$

Now, we recall some notations and preliminary results about fractional calculus and some special functions.

Definition 2.2. The Riemann-Liouville fractional integral of the order $q > 0$ of $f : [0, T] \to H$ is defined by

$$J_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\,ds,$$

where $\Gamma(\cdot)$ is the standard Gamma function.

Definition 2.3. The Riemann-Liouville fractional derivative of the order $q \in (0, 1]$ of $f : [0, T] \to H$ is defined by

$$D_t^q f(t) = \frac{d}{dt} J_t^{1-q} f(t).$$

Definition 2.4. The Caputo fractional derivative of the order $q \in (0, 1]$ of $f : [0, T] \to H$ is defined by

$${}^C D_t^q f(t) = D_t^q (f(t) - f(0)).$$

The Laplace transform of Caputo fractional derivative is given by

$$L\{{}^C D_t^q u(t)\} = \lambda^q \hat{u}(\lambda) - \lambda^{q-1} u(0),$$

where $\hat{u}(\lambda)$ is the Laplace transform of u defined by

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\,dt, \quad \Re\lambda > \omega,$$

where $\Re\lambda$ stands for the real part of the complex number λ.

Definition 2.5. The Mittag-Leffler function is defined by

$$E_{q,p}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(qn+p)}, \ p, q > 0, z \in \mathbb{C}. \tag{2.2}$$

When $p = 1$, set $E_q(z) = E_{q,1}(z)$.

Definition 2.6. The Mainardi's function is defined by

$$M_q(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-qn+1-q)}, \ 0 < q < 1, z \in \mathbb{C}.$$

The Laplace transform of Mainardi's function $M_q(r)$ is (see [18]):

$$\int_0^\infty e^{-\lambda r} M_q(r)dr = E_q(-\lambda). \tag{2.3}$$

By (2.2) and (2.3), it is clear that

$$\int_0^\infty M_q(r)dr = 1,\ 0 < q < 1.$$

On the other hand, $M_q(z)$ satisfies the following equality (see [18])

$$\int_0^\infty \frac{q}{r^{q+1}} M_q(1/r^q)e^{-\lambda r}dr = e^{-\lambda^q}$$

and the equality (see [18])

$$\int_0^\infty r^\delta M_q(r)dr = \frac{\Gamma(\delta+1)}{\Gamma(q\delta+1)},\ \delta > -1,\ 0 < q < 1.$$

Throughout this paper we impose the following assumptions.

(H1) The operator $(A, D(A))$ is a self-adjoint operator on the separable Hilbert space H admitting a discrete spectrum

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m \leqslant \cdots \leqslant \lim_{m\to\infty} \lambda_m = \infty$$

with corresponding eigenbasis $\{e_m\}_{m\geqslant 1}$ of H and generating an analytic semigroup $S(t) = e^{tA}$, $t \geqslant 0$, $0 \in \rho(A)$, such that $\|e^{tA}\| \leqslant Me^{-\lambda_1 t}$.

(H2) There exists a positive constant K_1 such that for all $x, y \in H$ and $t \geqslant 0$,

$$\|f(t,x) - f(t,y)\| \leqslant K_1\|x - y\|, \qquad \|f(t,x)\| \leqslant K_1(1 + \|x\|).$$

(H3) There exists $k \in (0, 1]$ and a positive constant K_2 such that for all $x, y \in H$ and $t \geqslant 0$,

$$\|(-A)^k g(t,x) - (-A)^k g(t,y)\| \leqslant K_2\|x - y\|, \quad g(t,0) = 0, \quad 5M^p\|(-A)^{-k}\|^p K_2^p < 1.$$

(H4) There exists a constant $\lambda > \frac{1}{1+\alpha q-\alpha}$ such that the function $\sigma : [0, +\infty) \to \mathbb{R}^+$ satisfies

$$\int_0^T \sigma^{\alpha\lambda}(s)ds < \infty.$$

3. Existence and uniqueness

In this section, we shall prove the existence and uniqueness of the mild solution to equation (1.1). For $0 < q < 1$, set $T_q(t)x = \int_0^\infty M_q(r)S(t^q r)xdr$ and $S_q(t)x = \int_0^\infty qrM_q(r)S(t^q r)xdr$, $t \geqslant 0$, $x \in X$. It is known that $u(t) = T_q(t)u_0 + \int_0^t (t-s)^{q-1}S_q(t-s)f(s,u(s))ds$, and is the mild solution to the deterministic fractional equation

$$\begin{cases} {}^C D_t^q u(t) = A(t)u(t) + f(t,u(t))dt,\ t \in [0,b], \\ u(0) = u_0, \end{cases}$$

see, for example, [14, 28]. Motivated by this result and noting Definitions 2.4 and 2.5, we present the following definition of mild solutions to (1.1).

Definition 3.1. An $\mathcal{F}_t$-adapted càdlág stochastic process $x(t)$, $t \in [-\tau, T]$ is called the mild solution for (1.1) if

(i) $x_0 = \xi \in D([-\tau, 0]; H)$;
(ii) for arbitrary $t \in [0, T]$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) =& T_q(t)[\xi(0) + g(0, \xi(-\tau))] + g(t, x(t-\tau)) + \int_0^t (t-s)^{q-1} A S_q(t-s) g(s, x(s-\tau)) ds \\ &+ \int_0^t (t-s)^{q-1} S_q(t-s) f(s, x(s-\tau)) ds + \int_0^t (t-s)^{q-1} S_q(t-s) \sigma(s) dZ(s). \end{aligned}$$

The following properties of $T_q(t)$ and $S_q(t)$ appeared in [29] are useful.

Lemma 3.2. *Under the assumption* (H1),

(i) *for any fixed* $t \geqslant 0$, $T_q(t)$ *and* $S_q(t)$ *are linear and bounded operators such that for any* $x \in H$, $\|T_\alpha(t)x\| \leqslant M\|x\|$, $\|S_\alpha(t)x\| \leqslant \frac{Mq}{\Gamma(1+q)}\|x\|$;
(ii) $T_q(t)$ *and* $S_q(t)$ *are strongly continuous;*
(iii) *for any* $x \in H$, $\beta \in (0, 1)$, *and* $k \in (0, 1]$, *we have*

$$AS_q(t)x = A^{1-\beta} S_q(t) A^\beta x \quad \textit{and} \quad \|A^k S_q(t)\| \leqslant \frac{qM_k}{t^{qk}} \frac{\Gamma(2-k)}{\Gamma(1+q(1-k))}.$$

Lemma 3.3. *Let (H1) hold, then for any* $t \geqslant 0$ *and* $p > 0$

$$\mathbb{E}\Big\| \int_0^t (t-s)^{q-1} S_q(t-s) \sigma(s) dZ(s) \Big\|^p \leqslant C_{p,\alpha} \Big(\sum_{k=1}^\infty \beta_k^\alpha \int_0^t (t-s)^{\alpha q - \alpha} S_q^\alpha(t-s) \sigma^\alpha(s) ds \Big)^{p/\alpha},$$

where the constant $C_{p,\alpha} > 0$ *depends on* p *and* α.

Proof. By virtue of (2.1), we can easily calculate that

$$\int_0^t (t-s)^{q-1} S_q(t-s) \sigma(s) dZ(s) = \sum_{k=1}^\infty \Lambda_k(t) e_k,$$

where $\Lambda_k(t) := \int_0^t \beta_k (t-s)^{q-1} S_q(t-s) \sigma(s) dZ_k(s)$. Let $\{r_k\}_{k \geqslant 1}$ be a Rademacher sequence defined on a new probability space $(\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_{t \geqslant 0}, \mathbb{P}')$, i.e., $r_k : \Omega' \to \{1, -1\}$ are i.i.d. with $\mathbb{P}'(r_k = 1) = \mathbb{P}'(r_k = -1) = 1/2$.

The following Khintchine's inequality holds, for arbitrary real numbers $c_1, \cdots, c_n$, for any $p > 0$

$$\Big(\sum_{k \geqslant 1} c_k^2 \Big)^{1/2} \leqslant M_1(p) \Big(\mathbb{E}' \Big| \sum_{k \geqslant 1} r_k c_k \Big|^p \Big)^{1/p} \tag{3.1}$$

(see, for instance, [7]), where the constant C_p depends only on p (for $p = 1$, we have $c_1 = \sqrt{2}$) and $\mathbb{E}'$ indicates the expectation with respect to $\mathbb{P}'$.

Then, by (3.1), we can write

$$\Big(\sum_{k=1}^\infty \Big(\int_0^t (t-s)^{q-1} S_q(t-s) \sigma(s) dZ_k(s) \Big)^2 \Big)^{1/2} \leqslant M_1(p) \Big(\mathbb{E}' \Big| \sum_{k \geqslant 1} r_k \Lambda_k(t) \Big|^p \Big)^{1/p}.$$

Using the Fubini theorem and the property of the α-stable process $Z(t)$ (see [20]), we have

$$\begin{aligned} &\mathbb{E}\Big\| \int_0^t (t-s)^{q-1} S_q(t-s) \sigma(s) dZ(s) \Big\|^p \\ \leqslant& (M_1(p))^p \mathbb{E}\Big[\mathbb{E}' \Big| \sum_{k \geqslant 1} r_k \Lambda_k(t) \Big|^p \Big] \\ =& (M_1(p))^p \mathbb{E}'\Big[\mathbb{E} \Big| \sum_{k \geqslant 1} r_k \Lambda_k(t) \Big|^p \Big] \\ =& (M_1(p))^p \mathbb{E}'\Big[\mathbb{E} \Big| \sum_{k \geqslant 1} r_k \int_0^t \beta_k (t-s)^{q-1} S_q(t-s) \sigma(s) dZ_k(s) \Big|^p \Big]. \end{aligned} \tag{3.2}$$

Note that $|r_k| = 1$ for any $k \geqslant 1$. Then by using properties of α-stable processes again, it is not difficult to get for any $t \geqslant 0$, $\lambda \in \mathbb{R}$ that

$$\mathbb{E}\exp\left[i\lambda\sum_{k=1}^{\infty} r_k\Lambda_k(t)\right] = \exp\left[-|\lambda|^{\alpha}\sum_{k=1}^{\infty}\beta_k^{\alpha}\int_0^t (t-s)^{\alpha q-\alpha}S_q^{\alpha}(t-s)\sigma^{\alpha}(s)ds\right].$$

Recall the fact (see page 18, [23]) that if Y is a symmetric random variable satisfying

$$\mathbb{E}[e^{i\lambda Y}] = e^{-\sigma^{\alpha}|\lambda|^{\alpha}}, \qquad \sigma \in \mathbb{R}$$

for some $\alpha \in (0,2)$ and any $\lambda \in \mathbb{R}$, then for all $p \in (0,\alpha)$,

$$\mathbb{E}|Y|^p = M_2(\alpha,p)\sigma^p, \tag{3.3}$$

where $M_2(\alpha,p) > 0$ is a constant depending only on α and p. Applying this result to (3.2) and (3.3), we then obtain that

$$\mathbb{E}\left\|\int_0^t (t-s)^{q-1}S_q(t-s)\sigma(s)dZ(s)\right\|^p \leqslant C_{p,\alpha}\Big(\sum_{k=1}^{\infty}\beta_k^{\alpha}\int_0^t (t-s)^{\alpha q-\alpha}S_q^{\alpha}(t-s)\sigma^{\alpha}(s)ds\Big)^{p/\alpha},$$

where the constant $C_{p,\alpha} = (M_1(p))^p M_2(\alpha,p) > 0$ depends on p and α. The proof is thus complete. □

Theorem 3.4. *Assume that* (H1)-(H4) *hold, and*

$$C := \sum_{k=1}^{\infty}\beta_k^{\alpha} < \infty \tag{3.4}$$

holds for $\alpha \in (1,2)$, $q \in (1/2,1)$. *Then* (1.1) *has a unique mild solution on* $[-\tau, T]$.

Proof. Fix $T > 0$ and denote by S_T the Banach space of all càdlàg H-valued processes $x(t) \in D([-\tau,T];H)$ with initial data $x(t) = \xi(t)$ for $t \in [-\tau, 0]$ equipped with the supremum norm

$$\|x(t)\|_{S_T} = \sup_{-\tau\leqslant t\leqslant T}\Big(\mathbb{E}\|x(t)\|^p\Big)^{1/p}.$$

Define the operator Ψ on S_T by $\Psi(x)(t) = \xi(t)$ for $t \in [-\tau, 0]$ and for $t \in [0,T]$

$$\begin{aligned}\Psi(x)(t) =& T_q(t)[\xi(0) + g(0,\xi(-\tau))] + g(t,x(t-\tau)) + \int_0^t (t-s)^{q-1}AS_q(t-s)g(s,x(s-\tau))ds\\ &+ \int_0^t (t-s)^{q-1}S_q(t-s)f(s,x(s-\tau))ds + \int_0^t (t-s)^{q-1}S_q(t-s)\sigma(s)dZ(s).\end{aligned}$$

Then it is clear that to prove the existence of mild solutions to (1.1) is equivalent to find a fixed point for the operator Ψ. Next we will show by using the Banach fixed point theorem that Ψ has a unique fixed point. We divide the subsequent proof into two steps.

Step 1. We show that $\Psi(x)(t) \subset S_T$ for $t \in [-\tau, T]$. It is trivial for the case $t \in [-\tau, 0]$. For $t \in [0,T]$, and for any fixed $x \in S_T$, using the essential inequality: $\Big(\sum_{i=1}^n a_i\Big)^p \leqslant C_r\Big(\sum_{i=1}^n a_i^p\Big)$, here $C_r = 1$ when $p \leqslant 1$, $C_r = n^{p-1}$ when $p > 1$, we have

$$\begin{aligned}\mathbb{E}\|\Psi(x)(t)\|^p \leqslant& 5^{p-1}\mathbb{E}\|T_q(t)[\xi(0)+g(0,\xi(-\tau))]\|^p + 5^{p-1}\mathbb{E}\|g(t,x(t-\tau))\|^p\\ &+ 5^{p-1}\mathbb{E}\left\|\int_0^t AS_q(t-s)(t-s)^{q-1}g(s,x(s-\tau))ds\right\|^p\\ &+ 5^{p-1}\mathbb{E}\left\|\int_0^t S_q(t-s)(t-s)^{q-1}f(s,x(s-\tau))ds\right\|^p\\ &+ 5^{p-1}\mathbb{E}\left\|\int_0^t S_q(t-s)(t-s)^{q-1}\sigma(s)dZ(s)\right\|^p := \sum_{i=1}^5 I_i.\end{aligned} \tag{3.5}$$

It follows from (H1), (H3), and Lemma 3.2 that

$$\begin{aligned} I_1 =&5^{p-1}\mathbb{E}\|(-A)^{-k}T_q(t)(-A)^k[\xi(0)+g(0,\xi(-\tau))]\|^p \\ \leqslant&10^{p-1}M^p\mathbb{E}\|\xi(0)\|^p+10^{p-1}M^pK_2^p\|(-A)^{-k}\|^p\mathbb{E}\|\xi(-\tau)\|^p \\ \leqslant&10^{p-1}M^p(1+K_2^p\|(-A)^{-k}\|^p)\sup_{-\tau\leqslant s\leqslant 0}\mathbb{E}\|\xi(s)\|^p. \end{aligned} \tag{3.6}$$

For the second term, I_2, using the assumption (H3) again, we have

$$I_2 =5^{p-1}\mathbb{E}\|(-A)^{-k}(-A)^k g(t,x(t-\tau))]\|^p \leqslant 5^{p-1}M^pK_2^p\|(-A)^{-k}\|^p\mathbb{E}\|x(t-\tau)\|^p. \tag{3.7}$$

From (H3), the Hölder's inequality and Lemma 3.2, we can obtain

$$\begin{aligned} I_3 &= 5^{p-1}\mathbb{E}\Big\|\int_0^t(-A)^{1-k}S_q(t-s)(t-s)^{q-1}(-A)^k g(s,x(s-\tau))ds\Big\|^p \\ &\leqslant 5^{p-1}\Big|\frac{K_2qM_{1-k}\Gamma(2-k)}{\Gamma(1+q(1-k))}\Big|^p\mathbb{E}\Big(\int_0^t(t-s)^{qk-1}\|x(s-\tau)\|ds\Big)^p \\ &\leqslant 5^{p-1}\Big|\frac{K_2qM_{1-k}\Gamma(2-k)}{\Gamma(1+q(1-k))}\Big|^p\Big(\int_0^t(t-s)^{qk-1}ds\Big)^{p-1}\int_0^t(t-s)^{qk-1}\mathbb{E}\|x(s-\tau)\|^p ds \\ &\leqslant 5^{p-1}\Big|\frac{K_2qM_{1-k}\Gamma(2-k)t^{qk}}{\Gamma(1+q(1-k))qk}\Big|^p\sup_{-\tau\leqslant s\leqslant t}\mathbb{E}\|x(s)\|^p. \end{aligned} \tag{3.8}$$

By using Lemma 3.2, (H2), and the Hölder's inequality, we have

$$\begin{aligned} I_4 =&5^{p-1}\mathbb{E}\Big\|\int_0^t S_q(t-s)(t-s)^{q-1}f(s,x(s-\tau))ds\Big\|^p \\ \leqslant&5^{p-1}\Big|\frac{K_1qM}{\Gamma(1+q)}\Big|^p\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big(\int_0^t(t-s)^{q-1}(1+\|x(s-\tau)\|)ds\Big)^p \\ \leqslant&10^{p-1}\Big|\frac{K_1Mt}{\Gamma(1+q)}\Big|^p+10^{p-1}\Big|\frac{K_1Mq}{\Gamma(1+q)}\Big|^p\Big(\int_0^t(t-s)^{q-1}ds\Big)^{p-1}\int_0^t(t-s)^{q-1}\mathbb{E}\|x(s-\tau)\|^p ds \\ \leqslant&10^{p-1}\Big|\frac{K_1Mt}{\Gamma(1+q)}\Big|^p+10^{p-1}\Big|\frac{K_1Mt}{\Gamma(1+q)}\Big|^p\sup_{-\tau\leqslant s\leqslant t}\mathbb{E}\|x(s)\|^p. \end{aligned} \tag{3.9}$$

By Lemma 3.3, we know that

$$\begin{aligned} I_5 \leqslant&C_{p,\alpha}\Big(\sum_{k=1}^{\infty}\beta_k^{\alpha}\int_0^t(S_q(t-s))^{\alpha}(t-s)^{\alpha(q-1)}\sigma^{\alpha}(s)ds\Big)^{p/\alpha} \\ \leqslant&C_{p,\alpha}\Big|\frac{Mq}{\Gamma(1+q)}\Big|^p\Big(\sum_{k=1}^{\infty}\beta_k^{\alpha}\int_0^t(t-s)^{\alpha(q-1)}\sigma^{\alpha}(s)ds\Big)^{p/\alpha}. \end{aligned}$$

Notice that $\alpha\in(1,2)$ and $q\in(1/2,1]$, then $\lambda>\frac{1}{1+\alpha q-\alpha}>1$. So we can obtain by the Hölder's inequality that

$$\begin{aligned} I_5 \leqslant&C_{p,\alpha}\Big|\frac{Mq}{\Gamma(1+q)}\Big|^p\Big[\sum_{k=1}^{\infty}\beta_k^{\alpha}\Big(\int_0^t(t-s)^{\frac{\lambda\alpha(q-1)}{\lambda-1}}ds\Big)^{\frac{\lambda-1}{\lambda}}\cdot\Big(\int_0^t\sigma^{\alpha\lambda}(s)ds\Big)^{1/\lambda}\Big]^{p/\alpha} \\ \leqslant&C_{p,\alpha}\Big|\frac{Mq}{\Gamma(1+q)}\Big|^pT^{\frac{(\alpha q-\alpha+1)\lambda-1}{\lambda}}\Big(\int_0^t\sigma^{\alpha\lambda}(s)ds\Big)^{\frac{p}{\alpha\lambda}}\Big(\sum_{k=1}^{\infty}\beta_k^{\alpha}\Big)^{p/\alpha} \\ <&\infty. \end{aligned} \tag{3.10}$$

Thus, we derive from (3.5)-(3.10) that, for some constants c_1, c_2, and c_3,

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\|\Psi(x)(t)\|^p \leqslant c_1 + c_2\sup_{-\tau\leqslant\theta\leqslant 0}\mathbb{E}\|\xi(\theta)\|^p + c_3\sup_{0\leqslant t\leqslant T}\mathbb{E}\|x(t)\|^p.$$

Hence, $\Psi(x)(t) \subset S_T$.

Step 2. We shall show that the mapping Ψ is contractive. Let $x, y \in S_T$. For any fixed $t \in [0, T]$, we have

$$\begin{aligned}\mathbb{E}\|\Psi(x)(t)-\Psi(y)(t)\|^p \leqslant & 3^{p-1}\mathbb{E}\|g(t,x(t-\tau))-g(t,y(t-\tau))\|^p\\ &+3^{p-1}\mathbb{E}\left\|\int_0^t AS_q(t-s)(t-s)^{q-1}[g(s,x(s-\tau))-g(s,y(s-\tau))]ds\right\|^p\\ &+3^{p-1}\mathbb{E}\left\|\int_0^t S_q(t-s)(t-s)^{q-1}[f(s,x(s-\tau))-f(s,y(s-\tau))]ds\right\|^p.\end{aligned}$$

By the assumptions (H2), (H3), and the Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\|\Psi(x)(t)-\Psi(y)(t)\|^p \leqslant & 3^{p-1}M^pK_2^p\|(-A)^{-k}\|^p\mathbb{E}\|x(t-\tau)-y(t-\tau)\|^p\\ &+3^{p-1}\left|\frac{K_2qM_{1-k}\Gamma(2-k)t^{qk}}{\Gamma(1+q(1-k))qk}\right|^p\sup_{0\leqslant s\leqslant t}\mathbb{E}\|x(s)-y(s)\|^p\\ &+3^{p-1}\left|\frac{K_1Mt}{\Gamma(1+q)}\right|^p\sup_{0\leqslant s\leqslant t}\mathbb{E}\|x(s)-y(s)\|^p.\end{aligned}$$

Then,

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\|\Psi(x)(t)-\Psi(y)(t)\|^p \leqslant 3^{p-1}(M^pK_2^p\|(-A)^{-k}\|^p + C_1T^{qkp} + C_2T^p)\sup_{0\leqslant t\leqslant T}\mathbb{E}\|x(t)-y(t)\|^p,$$

where $x(t) = y(t)$ on $[-\tau, 0]$, and $C_1 > 0$, $C_2 > 0$ are two bounded constants. Hence, by the condition $5M^pK_2^p\|(-A)^{-k}\|^p < 1$, choosing sufficiently small T_1 such that

$$3^{p-1}(M^pK_2^p\|(-A)^{-k}\|^p + C_1T^{qkp} + C_2T^p) < 1,$$

we can conclude that Ψ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of (1.1) on $[0, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[0, T]$ in finitely many steps. This completes the proof. □

4. An example

In this section, an example is provided to illustrate the theory obtained.

Example 4.1. We consider the following neutral fractional stochastic partial functional differential equation driven by α-stable process:

$$^CD_t^q\left[x(t)-\int_0^\pi \varphi(-\tau,\zeta,x)u(t-\tau,\zeta)d\zeta)\right] = \left[\frac{\partial^2}{\partial x^2}u(t,x)+\phi(u(t-\tau),x)\right]dt+\sigma(t)dZ(t,x), \tag{4.1}$$

with the Dirichlet boundary condition

$$u(t,0) = u(t,\pi) = 0, \quad t \in [0,T],$$

and the initial condition

$$u(\theta,x) = \Psi(\theta,x), \quad \theta\in[-\tau,0],\ x\in(0,\pi).$$

Furthermore, let $\phi : \mathbb{R} \to \mathbb{R}$ be Lipschitzian. Assume further that $\varphi : [-\tau, 0] \times [0, \pi] \times [0, \pi] \to \mathbb{R}$ is measurable such that $\varphi(\cdot, \cdot, 0) = \varphi(\cdot, \cdot, \pi) = 0$ and

$$N := \int_0^{\pi} \int_0^{\pi} \left(\frac{\partial}{\partial x} \varphi(-\tau, \zeta, x) \right)^2 d\zeta dx < \infty. \tag{4.2}$$

Let $H = L^2(0, \pi)$ and A be given by

$$A := \frac{\partial^2}{\partial x^2}, \quad D(A) := H^2(0, \pi) \cap H_0^1(0, \pi),$$

where $H^k(0, \pi), k = 1, 2$, represents the classical Sobolev spaces, and $H_0^1(0, \pi)$ is the subspace of $H^1(0, \pi)$ of all functions vanishing at 0 and π. Note that A is a self-adjoint negative operator in H and $Ae_k = -k^2 e_k$ with $e_k(\xi) = (2/\pi)^{1/2} \sin m\xi$, for $m \in \mathbb{N}$ and $\xi \in [0, \pi]$. We can easily know that A is the infinitesimal generator of an analytic semigroup $S(t) = e^{tA}$, $t \geqslant 0$, in H and the operator $(A, D(A))$ is a self-adjoint operator on the separable Hilbert space H admitting a discrete spectrum

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m \leqslant \cdots \leqslant \lim_{m \to \infty} \lambda_m = \infty$$

with $\lambda_k = k^2$ and $\|S(t)\| \leqslant e^{-t}$. This implies that (H1) holds. Furthermore, we know that

$$(-A)^{-\frac{1}{2}} \zeta = \sum_{n=1}^{\infty} \frac{1}{n} \langle \zeta, e_n \rangle e_n, \quad \zeta \in H, \tag{4.3}$$

and

$$(-A)^{\frac{1}{2}} \zeta = \sum_{n=1}^{\infty} n \langle \zeta, e_n \rangle e_n, \quad \zeta \in D((-A)^{\frac{1}{2}}), \tag{4.4}$$

which in particular yields $\|(-A)^{-\frac{1}{2}}\| = 1$.

Let $Z(t, \xi) := \sum_{k=1}^{\infty} \beta_k Z_k(t) e_k(\xi)$, where $\rho < 2q - 1 - \frac{1}{\alpha}$ and $\{Z_k(t)\}_{k \geqslant 1}$ is a cylindrical α-stable process on H with $\{Z_k(t)\}_{k \in \mathbb{N}}$ being i.i.d. one dimensional symmetric α-stable process sequence with $1 < \alpha < 2$. Moreover, there exist some C_1, $C_2 > 0$ such that for $k \in \mathbb{N}$,

$$C_1 \lambda_k^{-\beta} \leqslant |\beta_k| \leqslant C_2 \lambda_k^{-\beta} \quad \text{for some} \quad \beta > 0.$$

For $t \in [0, T]$ and $x \in [0, \pi]$, let

$$x(t)(x) := u(t, x), \quad g(t, x(t-\tau))(x) := \int_0^{\pi} \varphi(-\tau, \zeta, x) u(t-\tau, \zeta) d\zeta$$

and

$$f(t, x(t-\tau))(x) := \phi(u(t-\tau), x).$$

Then (4.1) can be rewritten in the form (1.1). Then, using $\varphi(\cdot, \cdot, 0) = \varphi(\cdot, \cdot, \pi) = 0$, combining (4.2)-(4.4) with the Hölder's inequality, we obtain $\|(-A)^{\frac{1}{2}}(g(t, x(t-\tau)) - g(t, y(t-\tau))\|^2 \leqslant N\|x(t-\tau) - y(t-\tau)\|^2$ (see [26]). Hence, (H2) and (H3) hold.

On the other hand, by simple calculation we have

$$\sum_{k=1}^{\infty} \beta_k^{\alpha} \leqslant \sum_{k=1}^{\infty} C_2^{\alpha} \lambda_k^{-\alpha\beta} = \sum_{k=1}^{\infty} C_2^{\alpha} k^{-2\alpha\beta}.$$

Thus, we can see that (3.4) holds if $\beta > \frac{1}{2\alpha}$. Consequently, by Theorem 3.4, there exists a unique mild solution to (4.1) provided that $\beta > \frac{1}{2\alpha}$ and σ satisfies the assumption (H4).

5. Conclusion

In this paper, a class of fractional neutral stochastic partial differential equations driven by α-stable process are discussed. By estimating the pth moment of α-stable noise and using the Banach fixed-point theorem, the existence and uniqueness of the mild solutions to this kind of equations driven by α-stable process are obtained.

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