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Uniqueness result for the cantilever beam equation with fully nonlinear term

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Abstract

In this paper, the uniqueness of solution for the cantilever beam equation with fully nonlinear term is obtained by using the method of order reduction and the theory of linear operators. A simple comparison is given to show that the obtained results provide the same results with weaker conditions.

Keywords: Fully fourth-order boundary value problem, uniqueness theorem, order reduction, Banach's contraction mapping principle.

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1. Introduction

In this paper we establish some new results on the uniqueness of solution for the fully fourth order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{1.1}$$

Boundary value problem (1.1) models the deflection of the elastic beam fixed at left and freed at right. In mechanics, this problem is called cantilever beam equation. Because of the wide applications in mechanics, it has been studied extensively in recent years, see, for instance, [1–3, 6, 8–13] and references therein. However, most of the known results in this area concentrate on the existence and multiplicity of solutions or positive solutions of boundary value problem (1.1). To our knowledge, there are few papers investigating the uniqueness results of boundary value problem (1.1). For some recent works on the uniqueness result for boundary value problem, we refer the reader to [4, 5, 7, 14] and the references therein. The objective of the present paper is to fill this gap.

By the method of order reduction and the theory of linear operators, we give some new result on the uniqueness of solution for the fully fourth order boundary value problem (1.1) under the assumption

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that the nonlinearity is a Lipschitz continuous function. Then the obtained results is compared to those ones obtained when the Banach's contraction mapping principle is applied. The nonlinearity is shown to satisfy weaker conditions. The interesting point is that the Lipschitz constant is related to the relevant linear operators.

2. Main results

Let $C[0,1]$ denote the Banach space of real-valued continuous function with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Firstly, we use the method of order reduction to transform (1.1) to a nonlinear integral equation. To do this, we let

$$(T_3v)(t) = \int_t^1 v(s)ds, \tag{2.1}$$

$$(T_2v)(t) = \int_0^t (T_3v)(s)ds = \int_0^1 G_2(t,s)v(s)ds, \tag{2.2}$$

$$(T_1v)(t) = \int_0^t (T_2v)(s)ds = \int_0^1 G_1(t,s)v(s)ds, \tag{2.3}$$

where

$$G_2(t,s) = \begin{cases} t, & 0 \leqslant t \leqslant s \leqslant 1, \\ s, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \qquad G_1(t,s) = \begin{cases} \dfrac{t^2}{2}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{2ts-s^2}{2}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

From the above formulas, it follows that

$$(T_1v)'''(t) = (T_2v)''(t) = (T_3v)'(t) = -v(t), \quad t \in [0,1].$$

Thus by the above transformations T_i $(i = 1,2,3)$, BVP (1.1) can be converted into a terminal value problem

$$-v'(t) = f(t,(T_1v)(t),(T_2v)(t),(T_3v)(t),-v(t)), \quad v(1) = 0,$$

which is rewritten to the equivalent nonlinear integral equation

$$v(t) = \int_t^1 f(s,(T_1v)(s),(T_2v)(s),(T_3v)(s),-v(s))ds.$$

Define an operator $A : C[0,1] \to C[0,1]$ by

$$(Av)(t) = \int_t^1 f(s,(T_1v)(s),(T_2v)(s),(T_3v)(s),-v(s))ds, \quad v \in C[0,1].$$

Then the existence of solution of BVP (1.1) is equivalent to the existence of fixed point of A on $C[0,1]$.

Take $u_0(t) = 1-t$. By (2.3), we get

$$(T_1u_0)(t) = \int_0^t \frac{2ts-s^2}{2}(1-s)ds + \int_t^1 \frac{t^2}{2}(1-s)ds = \frac{t^2}{4} - \frac{t^3}{6} + \frac{t^4}{24},$$

and

$$\int_t^1 (T_1u_0)(s)ds = \frac{1-t^3}{12} - \frac{1-t^4}{24} + \frac{1-t^5}{120} = \frac{1-t}{120}(6+6t+6t^2-4t^3+t^4).$$

After simple computation, we conclude that

$$\int_t^1 (T_1u_0)(s)ds \leqslant \frac{1}{8}u_0(t). \tag{2.4}$$

Analogously, from (2.1) and (2.2) we have

$$\begin{aligned}
(T_2u_0)(t) &= \int_0^t s(1-s)ds + \int_t^1 t(1-s)ds = \frac{t}{2} - \frac{t^2}{2} + \frac{t^3}{6},\\
\int_t^1 (T_2u_0)(s)ds &= \frac{1-t^2}{4} - \frac{1-t^3}{6} + \frac{1-t^4}{24} \leqslant \frac{u_0(t)}{6},
\end{aligned} \tag{2.5}$$

and

$$\int_t^1 (T_3u_0)(s)ds = \int_t^1 \frac{(1-s)^2}{2}ds = \frac{(1-t)^3}{6} \leqslant \frac{u_0(t)}{6}. \tag{2.6}$$

By use of (2.4), (2.5), and (2.6), we present the main result of this paper.

Theorem 2.1. *Suppose that there exist four nonnegative constants* M_i $(i = 1, 2, 3, 4)$ *with* $\frac{M_1}{8} + \frac{M_2}{6} + \frac{M_3}{6} + M_4 < 1$ *such that*

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \leqslant \sum_{i=1}^{4} M_i|x_i - y_i|, \quad x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem (1.1) *has a unique solution.*

Proof. According to the foregoing arguments, we only need to prove that A has a unique fixed point in $C[0, 1]$. Let us introduce a linear operator T on $C[0, 1]$ as

$$(Tv)(t) = \int_t^1 (M_1(T_1v)(s) + M_2(T_2v)(s) + M_3(T_3v)(s) + M_4v(s))ds. \tag{2.7}$$

As a first step, we show that for all $v \in C[0, 1]$ with $v(t) \geqslant 0$ $(t \in [0, 1])$, there exists $N = N(v)$ such that

$$(Tv)(t) \leqslant Nu_0(t), \quad t \in [0, 1]. \tag{2.8}$$

In fact, we take $N = M_1\|T_1v\| + M_2\|T_2v\| + M_3\|T_3v\| + M_4\|v\|$. Then by (2.7), we obtain that $(Tv)(t) \leqslant Nu_0(t),\ \ t \in [0, 1]$. Moreover, it follows from (2.4), (2.5), and (2.6) that

$$(Tu_0)(t) \leqslant Mu_0(t), \tag{2.9}$$

where $M = \frac{M_1}{8} + \frac{M_2}{6} + \frac{M_3}{6} + M_4 < 1$.

For any given $v_0 \in C[0, 1]$, let

$$v_n(t) = (Av_{n-1})(t), \quad w_n(t) = |v_n(t) - v_{n-1}(t)|, \quad n = 1, 2, \ldots. \tag{2.10}$$

Then for $t \in [0, 1]$, we have

$$\begin{aligned}
w_{n+1}(t) &= |v_{n+1}(t) - v_n(t)|\\
&= |(Av_n)(t) - (Av_{n-1})(t)|\\
&\leqslant \int_t^1 |f(s, (T_1v_n)(s), (T_2v_n)(s), (T_3v_n)(s), -v_n(s))\\
&\quad - f(s, (T_1v_{n-1})(s), (T_2v_{n-1})(s), (T_3v_{n-1})(s), -v_{n-1}(s))|ds\\
&\leqslant \int_t^1 (M_1T_1(|v_n - v_{n-1}|)(s) + M_2T_2(|v_n - v_{n-1}|)(s)\\
&\quad + M_3T_3(|v_n - v_{n-1}|)(s) + M_4|v_n(s) - v_{n-1}(s)|)ds = (T|v_n - v_{n-1}|)(t) = (Tw_n)(t).
\end{aligned}$$

By (2.8), (2.9), and the method of induction, there exists $N = N(w_1)$ such that

$$w_{n+1}(t) \leqslant (Tw_n)(t) \leqslant \cdots \leqslant (T^nw_1)(t) \leqslant N(T^{n-1}u_0)(t) \leqslant NM^{n-1}u_0(t), \quad t \in [0, 1].$$

Thus for all $m, n \in \mathbb{N}$ and $t \in [0,1]$,

$$\begin{aligned}|v_{n+m+1}(t) - v_n(t)| &= |v_{n+m+1}(t) - v_{n+m}(t) + \cdots + v_{n+1}(t) - v_n(t)| \\ &\leqslant w_{n+m+1}(t) + \cdots + w_{n+1}(t) \\ &\leqslant NM^{n+m-1}u_0(t) + \cdots + NM^{n-1}u_0(t) \\ &= \frac{NM^{n-1}(1-M^{m+1})}{1-M}u_0(t) < \frac{NM^{n-1}}{1-M}.\end{aligned} \tag{2.11}$$

This shows that $\{v_n\}$ is a uniform Cauchy sequence in $C[0,1]$ and since $C[0,1]$ is complete there exists $v^* \in C[0,1]$ such that $\lim_{n\to\infty} v_n = v^*$. Moreover, v^* is a fixed point of A that follows from the continuity of A.

Next we show that A has at most one fixed point. Suppose that there are two elements $x, y \in C[0,1]$ with $x = Ax$ and $y = Ay$. By (2.8), there exists $N = N(|x-y|)$ such that

$$(T(|x-y|))(t) \leqslant Nu_0(t), \quad t \in [0,1].$$

Then for $n \in \mathbb{N}$, we have

$$|x(t) - y(t)| = |(A^n x)(t) - (A^n y)(t)| \leqslant (T^n(|x-y|))(t) \leqslant N(T^{n-1}u_0)(t) \leqslant NM^{n-1}u_0(t), \quad t \in [0,1].$$

Consequently, we assert that $x = y$. This means that A has at most one fixed point. This completes the proof. □

Remark 2.2. If v^* is the unique fixed point of operator A, T_1v^* is the unique solution of boundary value problem (1.1). For any given $v_0 \in C[0,1]$, the iterative sequence $\{T_1v_n\}$ converges to T_1v^* in $C[0,1]$, where $v_n(t)$ is given by (2.10). Furthermore, it follows from (2.11) that the estimation on the convergence rate can be described by

$$\|T_1v_n - T_1v^*\| \leqslant \max_{t\in[0,1]} \int_0^1 G_1(t,s)ds\|v_n - v^*\| \leqslant \frac{NM^{n-1}}{3(1-M)}.$$

The following uniqueness result for the boundary value problem (1.1) is obtained by the Banach's contraction mapping principle. A simple comparison shows that the former theorem provides the same results with weaker conditions.

Theorem 2.3. *Suppose that there exist four nonnegative constants M_i $(i = 1,2,3,4)$ with $\frac{M_1}{8} + \frac{M_2}{3} + \frac{M_3}{2} + M_4 < 1$ such that*

$$|f(t,x_1,x_2,x_3,x_4) - f(t,y_1,y_2,y_3,y_4)| \leqslant \sum_{i=1}^{4} M_i|x_i - y_i|, \quad x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem (1.1) has a unique solution.

Proof. Note that

$$\begin{aligned}\int_t^1\int_0^1 G_1(\tau,s)dsd\tau &\leqslant \int_0^1\int_0^1 G_1(\tau,s)dsd\tau = \int_0^1\left(\frac{\tau^2}{2} - \frac{\tau^3}{6}\right)d\tau = \frac{1}{8}, \\ \int_t^1\int_0^1 G_2(\tau,s)dsd\tau &\leqslant \int_0^1\int_0^1 G_2(\tau,s)dsd\tau = \int_0^1\left(\tau - \frac{\tau^2}{2}\right)d\tau = \frac{1}{3},\end{aligned}$$

and

$$\int_t^1\int_\tau^1 1dsd\tau \leqslant \int_0^1\int_\tau^1 1dsd\tau = \int_0^1(1-\tau)d\tau = \frac{1}{2}.$$

Then for all $u, v \in C[0,1]$ and $t \in [0,1]$, we have

$$\begin{aligned}
|(Au)(t)-(Av)(t)| &\leqslant M_1\int_t^1 |(T_1u)(s)-(T_1v)(s)|ds + M_2\int_t^1 |(T_2u)(s)-(T_2v)(s)|ds \\
&\quad + M_3\int_t^1 |(T_3u)(s)-(T_3v)(s)|ds + M_4\int_t^1 |u(s)-v(s)|ds \\
&\leqslant M_1\int_t^1 T_1(|u(s)-v(s)|)ds + M_2\int_t^1 T_2(|u(s)-v(s)|)ds \\
&\quad + M_3\int_t^1 T_3(|u(s)-v(s)|)ds + M_4\int_t^1 |u(s)-v(s)|ds \\
&= (T|u-v|)(t) \\
&\leqslant \left(M_1\int_t^1\int_0^1 G_1(\tau,s)dsd\tau + M_2\int_t^1\int_0^1 G_2(\tau,s)dsd\tau + M_3\int_t^1\int_\tau^1 1dsd\tau + M_4\right)\|u-v\| \\
&\leqslant \left(\frac{M_1}{8}+\frac{M_2}{3}+\frac{M_3}{2}+M_4\right)\|u-v\|,
\end{aligned}$$

which implies $\|Au-Av\| \leqslant (\frac{M_1}{8}+\frac{M_2}{3}+\frac{M_3}{2}+M_4)\|u-v\|$. Since $\frac{M_1}{8}+\frac{M_2}{3}+\frac{M_3}{2}+M_4<1$, A is a contract operator. Therefore, by the well-known Banach's contraction mapping principle, we conclude that A has a unique fixed point which means that the boundary value problem (1.1) has a unique solution. The proof is completed. □

From the proof of Theorem 2.3, we know that

$$\|T\| \leqslant \frac{M_1}{8}+\frac{M_2}{3}+\frac{M_3}{2}+M_4<1, \tag{2.12}$$

where T is defined by (2.7). But, (2.12) may be false under the assumptions of Theorem 2.1. For example, if we take $M_2=4$, $M_1=M_3=M_4=0$, and $u(t)\equiv 1$, we have

$$|(Tu)(t)| = \max_{t\in[0,1]} M_2\int_t^1\int_0^1 G_2(\tau,s)dsd\tau = \frac{4}{3},$$

which implies that $\|T\| \geqslant \frac{4}{3}$. It seems that there are few uniqueness results if the norm of related linear operator is greater than 1. In fact, Theorems 2.1 and 2.3 conclude that $r(T)$ is less than 1, where $r(T)$ is the spectral radius of linear operator T.

Theorem 2.4. *Suppose that there exist four nonnegative constants* M_i $(i=1,2,3,4)$ *satisfying*

$$|f(t,x_1,x_2,x_3,x_4)-f(t,y_1,y_2,y_3,y_4)| \leqslant \sum_{i=1}^4 M_i|x_i-y_i|, \quad x_i,y_i\in\mathbb{R},$$

and $r(T)<1$. *Then the boundary value problem* (1.1) *has a unique solution.*

Proof. Set $\varepsilon=\dfrac{1}{2}(1-r(T))$. Since $r(T)<1$, then by Gelfand's formula, we have that there exists a natural number N such that for $n\geqslant N$,

$$\| T^n \| \leqslant [r(T)+\varepsilon]^n = [\frac{r(T)+1}{2}]^n.$$

For every $x\in C[0,1]$, define

$$\|x\|^* = \sum_{i=1}^N [\frac{r(T)+1}{2}]^{N-i}\|T^{i-1}x\|, \tag{2.13}$$

where $T^0 = I$ is the identity operator. By (2.13), it is easy to see

$$[\frac{r(T)+1}{2}]^N \|x\| \leqslant \|x\|^* \leqslant \sum_{i=1}^{N} [\frac{r(T)+1}{2}]^{N-i} \|T^{i-1}\| \cdot \|x\|,$$

which implies $\|\cdot\|^*$ is a norm in $C[0,1]$ and equivalent with the norm $\|\cdot\|$.

Then for all $u, v \in C[0,1]$, by (2.13), we have

$$\begin{aligned}
\|Au - Av\|^* &= \sum_{i=1}^{N} [\frac{r(T)+1}{2}]^{N-i} \|T^{i-1}(Au - Av)\| \\
&= \sum_{i=1}^{N} [\frac{r(T)+1}{2}]^{N-i} \max_{t\in[0,1]} |(T^{i-1}(Au - Av))(t)| \\
&\leqslant \sum_{i=1}^{N} [\frac{r(T)+1}{2}]^{N-i} \max_{t\in[0,1]} |(T^{i-1}(T|u - v|))(t)| \\
&= \frac{r(T)+1}{2} \sum_{i=1}^{N-1} [\frac{r(T)+1}{2}]^{N-i-1} \|(T^{i}|u - v|)\| + \|(T^{N}|u - v|)\| \\
&\leqslant \frac{r(T)+1}{2} \sum_{i=1}^{N-1} [\frac{r(T)+1}{2}]^{N-i-1} \|(T^{i}|u - v|)\| + [\frac{r(T)+1}{2}]^{N} \|u - v\| \\
&= \frac{r(T)+1}{2} \sum_{i=1}^{N} [\frac{r(T)+1}{2}]^{N-i} \|T^{i-1}(|u - v|)\| = \frac{r(T)+1}{2} \|u - v\|^*.
\end{aligned}$$

Thus the Banach contraction mapping principle implies that A has a unique fixed point x^* in $C[0,1]$ which means that the boundary value problem (1.1) has a unique solution. The proof is completed. □

Remark 2.5. From the above three theorems, we know that the conditions of Theorem 2.4 imposed on M_i are optimal. Correspondingly, it is difficult to calculate the value of $r(T)$ and the convergence rate of iterative sequences.

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References

[1] A. R. Aftabizadeh, *Existence and uniqueness theorems for fourth-order boundary value problems*, J. Math. Anal. Appl., **116** (1986), 415–426. 1

[2] R. P. Agarwal, D. O'Regan, *Twin solutions to singular boundary value problems*, Proc. Amer. Math. Soc., **128** (2000), 2085–2094.

[3] R. P. Agarwal, D. O'Regan, V. Lakshmikantham, *Singular* $(p, n-p)$ *focal and* (n, p) *higher order boundary value problems*, Nonlinear Anal., **42** (2000), 215–228. 1

[4] Y.-J. Cui, *Uniqueness of solution for boundary value problems for fractional differential equations*, Appl. Math. Lett., **51** (2016), 48–54. 1

[5] Y.-J. Cui, Y.-M. Zou, *An existence and uniqueness theorem for a second order nonlinear system with coupled integral boundary value conditions*, Appl. Math. Comput., **256** (2015), 438–444. 1

[6] C. P. Gupta, *Existence and uniqueness theorems for a bending of an elastic beam equation*, Appl. Anal., **26** (1988), 289–304. 1

[7] L. Guo, L. Liu, Y.-H. Wu, *Uniqueness of iterative positive solutions for the singular fractional differential equations with integral boundary conditions*, Boundary Value Problems, **2016** (2016), 20 pages. 1

[8] G. Infante, P. Pietramala, *A cantilever equation with nonlinear boundary conditions*, Electron. J. Qual. Theory Differ. Equ., **2009** (2009), 14 pages. 1
[9] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, Netherlands, (1964).
[10] Y.-X. Li, *Existence of positive solutions for the cantilever beam equations with fully nonlinear terms*, Nonlinear Anal. Real World Appl., **27** (2016), 221–237.
[11] F. Minhós, T. Gyulov, A. I. Santos, *Lower and upper solutions for a fully nonlinear beam equation*, Nonlinear Anal., **71** (2009), 281–292.
[12] Q.-L. Yao, *Monotonically iterative method of nonlinear cantilever beam equations*, Appl. Math. Comput., **205** (2008), 432–437.
[13] Q.-L. Yao, *Local existence of multiple positive solutions to a singular cantilever beam equation*, J. Math. Anal. Appl., **363** (2010), 138–154. 1
[14] Y. Zou, G.-P. He, *On the uniqueness of solutions for a class of fractional differential equations*, Appl. Math. Lett., **74** (2017), 68–73. 1