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Dynamics of a stochastic delay competition model with imprecise parameters

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Abstract

This paper is concerned with a two-species delay stochastic competition model with imprecise parameters. We first obtain the thresholds between persistence and extinction for each species. Then we establish sharp sufficient criteria for the existence of a unique ergodic stationary distribution of the model. The effects of imprecise parameters on the persistence, extinction and existence of the stationary distribution are revealed. Finally, we work out some numerical simulations to illustrate the theoretical results. ©2017 All rights reserved.

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1. Introduction

Competition is a common phenomenon in the natural world. On the other hand, time delay and stochastic perturbations should not be neglected (see, e.g., [3, 6, 19]). Therefore, it is interesting and important to study stochastic competition models with delay. A classical delay stochastic competition model can be expressed as follows:

$$\begin{cases} dy_1(t) = y_1(t)\Big[r_1 - c_{11}y_1(t) - c_{12}y_2(t-\tau_1)\Big]dt + \sigma_1 y_1(t)dB_1(t), \\ dy_2(t) = y_2(t)\Big[r_2 - c_{21}y_1(t-\tau_2) - c_{22}y_2(t)\Big]dt + \sigma_2 y_2(t)dB_2(t), \end{cases} \tag{1.1}$$

with initial value:

$$y(\theta) = (y_1(\theta), y_2(\theta))^T = (\eta_1(\theta), \eta_2(\theta))^T \in \Gamma, \tag{1.2}$$

where for $i, j = 1, 2,\ j \neq i$, parameters are shown in Table 1.

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Table 1: Parameters

$y_i(t)$	Population size of the ith species
$r_i > 0$	Intrinsic growth rate of the ith species
$c_{ii} > 0$	Intra-specific competition rate of the ith species
$c_{ij} > 0$	Inter-specific competition rate between the species i and j
σ_i^2	Intensity of the white noise
$\tau_i \geqslant 0$	Time delay
$(B_1(t), B_2(t))^T$	A standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$.
Γ	The family of all bounded and continuous functions from $[-\tau, 0]$ to $\mathbb{R}_+^2 = \{y = (y_1, y_2)^T \in \mathbb{R}^2 \| y_i > 0, i = 1, 2\}$, $\tau = \max\{\tau_1, \tau_2\}$.

In recent years, model (1.1) and its various generalized forms have been studied extensively, for example, persistence and extinction of model (1.1) were considered in [16]; stability in distribution of model (1.1) was exploited in [13, 15]; optimal control of model (1.1) with harvesting was investigated in [10, 12]; model (1.1) with Lévy jumps was analyzed in [14].

Model (1.1) supposes that all the parameters in the model are precisely known. However, in reality the values of all parameters can not always be known precisely due to the lack of data and mistakes in the measurement process ([21, 25]). Some authors ([20–26]) have claimed that models with imprecise parameters are more realistic. Therefore it is important to consider stochastic delay competition models with imprecise parameters and to reveal the impact of imprecise parameters on the dynamics of the models. However, to the best of our knowledge, no results of this aspect have been reported.

Motivated by these, in this paper we consider the following delay stochastic competition model with interval coefficients:

$$\begin{cases} dy_1(t) = y_1(t)\left[\hat{r}_1 - \hat{c}_{11}y_1(t) - \hat{c}_{12}y_2(t-\hat{\tau}_1)\right]dt + \sum_{i=1}^{n} \hat{\sigma}_{1i}y_1(t)dB_i(t), \\ dy_2(t) = y_2(t)\left[\hat{r}_2 - \hat{c}_{21}y_1(t-\hat{\tau}_2) - \hat{c}_{22}y_2(t)\right]dt + \sum_{i=1}^{n} \hat{\sigma}_{2i}y_2(t)dB_i(t), \end{cases}$$

with initial condition (1.2). Here, $(B_1(t), \dots, B_n(t))^T$ is an n-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$; $\hat{a}$ means the interval counterpart of a, i.e., $\hat{a} = [a_l, a_u] = \{x \in \mathbb{R} | a_l \leqslant x \leqslant a_u\}$. For biological reasons, in this paper we suppose that $r_{il} > 0$, $c_{ijl} > 0$, $\tau_{il} \geqslant 0$, $i, j = 1, 2$. For any $x \in [a_l, a_u]$, there is a $q \in [0, 1]$ such that $x = a_l^{1-q}a_u^q$. Hence we shall consider the following model:

$$\begin{cases} dy_1(t;q) = y_1(t;q)\left[r_{1l}^{1-q}r_{1u}^q - c_{11l}^{1-q}c_{11u}^q y_1(t;q) - c_{12l}^{1-q}c_{12u}^q y_2\left(t - \tau_{1l}^{1-q}\tau_{1u}^q; q\right)\right]dt \\ \qquad\qquad + \sum_{i=1}^{n} \sigma_{1il}^{1-q}\sigma_{1iu}^q y_1(t;q)dB_i(t), \\ dy_2(t;q) = y_2(t;q)\left[r_{1l}^{1-q}r_{1u}^q - c_{21l}^{1-q}c_{21u}^q y_1\left(t - \tau_{2l}^{1-q}\tau_{2u}^q; q\right) - c_{22l}^{1-q}c_{22u}^q y_2(t;q)\right]dt \\ \qquad\qquad + \sum_{i=1}^{n} \sigma_{2il}^{1-q}\sigma_{2iu}^q y_2(t;q)dB_i(t), \end{cases} \tag{1.3}$$

with initial condition (1.2). Clearly, model (1.3) contains the cases that the random noises are correlated or independent.

Remark 1.1. In this paper, we consider the Itô integral instead of Stratonovich integral because model (1.3) is an approximation to age-structured populations ([3]).

The rest of this paper is organized as follows. In Section 2 we establish the critical value between persistence and extinction for each species. In Section 3 we obtain sharp sufficient criteria for the existence of a unique ergodic stationary distribution of the model. In Section 4 we discuss the effects of imprecise parameters on the persistence, extinction and existence of the stationary distribution of the model with the help of several numerical simulations, and give some concluding remarks.

2. Persistence and extinction

For the sake of convenience, we define some notations:

$$\langle f(t)\rangle = t^{-1}\int_0^t f(s)ds,\ b_i(q) = r_{il}^{1-q} r_{iu}^{q} - \sum_{k=1}^{n} \sigma_{ikl}^{2(1-q)} \sigma_{iku}^{2q}/2,\ i = 1,2,$$
$$\Delta(q) = c_{11l}^{1-q} c_{11u}^{q} c_{22l}^{1-q} c_{22u}^{q} - c_{12l}^{1-q} c_{12u}^{q} c_{21l}^{1-q} c_{21u}^{q},$$
$$\Delta_1(q) = b_1(q) c_{22l}^{1-q} c_{22u}^{q} - b_2(q) c_{12l}^{1-q} c_{12u}^{q},\ \Delta_2(q) = b_2(q) c_{11l}^{1-q} c_{11u}^{q} - b_1(q) c_{21l}^{1-q} c_{21u}^{q}.$$

To begin with, let us prepare several lemmas.

Lemma 2.1 ([17]). *Let* $\rho(t) \in C(\Omega \times [0,+\infty), \mathbb{R}_+)$.

(i) *If there are two positive constants* T *and* β_0 *such that for all* $t \geqslant T$,

$$\ln \rho(t) \leqslant \beta t - \beta_0 \int_0^t \rho(s)ds + \sum_{i=1}^{n} \alpha_i B_i(t),$$

where $\alpha_i,\ i = 1,\dots,n$ *are constants, then*

$$\begin{cases} \limsup_{t\to+\infty} \langle \rho(t)\rangle \leqslant \beta/\beta_0, & \text{a.s. (almost surely)}, \quad \text{if} \quad \beta \geqslant 0; \\ \lim_{t\to+\infty} \rho(t) = 0, & \text{a.s.}, \quad \text{if} \quad \beta < 0, \end{cases}$$

(ii) *If there are three positive constants* T, β, *and* β_0 *such that for all* $t \geqslant T$,

$$\ln \rho(t) \geqslant \beta t - \beta_0 \int_0^t \rho(s)ds + \sum_{i=1}^{n} \alpha_i B_i(t),$$

then

$$\liminf_{t\to+\infty} \langle \rho(t)\rangle \geqslant \beta/\beta_0, \quad \text{a.s.}.$$

Lemma 2.2. *For any given initial value* $\eta(\theta) \in \Gamma$, *model* (1.3) *has a unique global positive solution* $y(t;q) = (y_1(t;q), y_2(t;q))$ *on* $t \geqslant 0$ *a.s., and*

$$\limsup_{t\to+\infty} \frac{\ln y_i(t;q)}{\ln t} \leqslant 1, \quad \text{a.s.}, \ i = 1,2. \tag{2.1}$$

In addition, for all $p > 0$, *there is a positive constant* $K(p)$ *such that*

$$\limsup_{t\to\infty} \mathbb{E}\left[y_i^p(t;q)\right] \leqslant K(p), \ i = 1,2. \tag{2.2}$$

Proof. Clearly, the coefficients of model (1.3) are locally Lipschitz continuous; hence for any given initial condition $\eta(\theta) \in \Gamma$, model (1.3) has a unique local solution $y(t;q)$ for $t \in [0,\tau_e)$, where τ_e represents the explosion time. For any $t \in [0,\tau_e)$,

$$\begin{aligned} y_1(t;q) &= y_1(0)\exp\Bigg\{\int_0^t\left(b_1(q) - c_{11l}^{1-q}c_{11u}^q y_1(s;q) - c_{12l}^{1-q}c_{12u}^q y_2\left(s-\tau_{1l}^{1-q}\tau_{1u}^q;q\right)\right)ds \\ &\quad + \sum_{i=1}^n \int_0^t \sigma_{1il}^{1-q}\sigma_{1iu}^q y_1(s;q)dB_i(s)\Bigg\}, \\ y_2(t;q) &= y_2(0)\exp\Bigg\{\int_0^t\left(b_2(q) - c_{21l}^{1-q}c_{21u}^q y_1\left(s-\tau_{2l}^{1-q}\tau_{2u}^q;q\right) - c_{22l}^{1-q}c_{22u}^q y_2(s;q)\right)ds \\ &\quad + \sum_{i=1}^n \int_0^t \sigma_{2il}^{1-q}\sigma_{2iu}^q y_2(s;q)dB_i(s)\Bigg\}. \end{aligned}$$

Therefore, $y_i(t;q) > 0$ for $t \in [0,\tau_e)$, $i = 1,2$. Now let us prove $\tau_e = +\infty$. To this end, we consider the following auxiliary equations:

$$dx_i(t;q) = x_i(t;q)\left[r_{il}^{1-q}r_{iu}^q - c_{iil}^{1-q}c_{iiu}^q x_i(t;q)\right]dt + \sum_{k=1}^n \sigma_{ikl}^{1-q}\sigma_{iku}^q x_i(t;q)dB_k(t),\ i=1,2, \tag{2.3}$$

$$\begin{aligned} dz_j(t;q) &= z_j(t;q)\left[r_{jl}^{1-q}r_{ju}^q - c_{jjl}^{1-q}c_{jju}^q z_j(t;q) - c_{jil}^{1-q}c_{jiu}^q x_i\left(t-\tau_{jl}^{1-q}\tau_{ju}^q;q\right)\right]dt \\ &\quad + \sum_{k=1}^n \sigma_{jkl}^{1-q}\sigma_{jku}^q z_j(t;q)dB_k(t),\ j=1,2,\ j\neq i, \end{aligned} \tag{2.4}$$

with initial data $x(\theta) = z(\theta) = y(\theta)$. According to the comparison theorem [7], we have

$$z_i(t;q) \leqslant y_i(t;q) \leqslant x_i(t;q),\ t\in[0,\tau_e),\ i=1,2. \tag{2.5}$$

On the other hand, according to [8], the explicit solutions of equations (2.3) and (2.4) can be expressed as follows:

$$x_i(t;q) = \frac{\exp\{b_i(p)t + \sum_{k=1}^n \sigma_{ikl}^{1-q}\sigma_{iku}^q B_k(t)\}}{x_i^{-1}(0) + c_{iil}^{1-q}c_{iiu}^q\int_0^t \exp\{b_i(p)s + \sum_{k=1}^n \sigma_{ikl}^{1-q}\sigma_{iku}^q B_k(s)\}ds},\ i=1,2,$$

$$z_i(t;q) = \frac{\exp\{b_i(p)t - c_{ijl}^{1-q}c_{iju}^q\int_0^t x_j(s-\tau_{il}^{1-q}\tau_{iu}^q;q)ds + \sum_{k=1}^n \sigma_{ikl}^{1-q}\sigma_{iku}^q B_k(t)\}}{z_i^{-1}(0) + c_{iil}^{1-q}c_{iiu}^q\int_0^t \exp\{b_i(p)s - c_{ijl}^{1-q}c_{iju}^q\int_0^s x_j(\mu-\tau_{il}^{1-q}\tau_{iu}^q;q)d\mu + \sum_{k=1}^n \sigma_{ikl}^{1-q}\sigma_{iku}^q B_k(s)\}ds},\ j\neq i.$$

Note that $x_i(t;q) > 0$ and $z_i(t;q) > 0$ exist on $[0,+\infty)$, hence $\tau_e = +\infty$.

Now let us show (2.1) and (2.2). In fact, according to the results in [9],

$$\limsup_{t\to+\infty}\frac{\ln x_i(t;q)}{\ln t} \leqslant 1,\quad \text{a.s.},\ i=1,2,$$

and for all $p > 0$, there is a positive constant $K(p)$ such that

$$\limsup_{t\to\infty}\mathbb{E}\left[x_i^p(t;q)\right] \leqslant K(p),\ i=1,2.$$

Hence the desired assertions (2.1) and (2.2) follow from (2.5). □

The aim of this section is to study the persistence and extinction of model (1.3). If $b_i(q) < 0$, then the species i in model (1.3) goes to extinction, i.e., $\lim_{t\to+\infty} y_i(t;q) = 0$, a.s., $i = 1,2$ (the proof is standard and hence is omitted (see, e.g. [17]). Hence from now on, we always suppose that $b_1(q) > 0$ and $b_2(q) > 0$. Besides, we also suppose the following.

Assumption 2.3. $\Delta(q) > 0$, that is to say, the intra-specific competition is stronger than the inter-specific competition.

Remark 2.4. If $\Delta(q) > 0$, it is easy to show that $\Delta_1(q) < 0$ and $\Delta_2(q) < 0$ will not hold simultaneously.

Now we are in the position to state and prove our first main result.

Theorem 2.5. *Suppose that* $b_1(q) > 0$, $b_2(q) > 0$ *and Assumption* 2.3 *holds.*

(i) *If* $\Delta_1(q) > 0$ *and* $\Delta_2(q) < 0$, *then* y_2 *goes to extinction a.s. and* y_1 *is persistent in the mean a.s.:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s;q)ds = \frac{b_1(q)}{c_{11l}^{1-q}c_{11u}^{q}}, \quad a.s.; \tag{2.6}$$

(ii) *If* $\Delta_1(q) < 0$ *and* $\Delta_2(q) > 0$, *then* y_1 *goes to extinction a.s. and* y_2 *is persistent in the mean a.s.:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s;q)ds = \frac{b_2(q)}{c_{22l}^{1-q}c_{22u}^{q}}, \quad a.s.;$$

(iii) *If* $\Delta_1(q) > 0$ *and* $\Delta_2(q) > 0$, *then both* y_1 *and* y_2 *are persistent in the mean a.s.:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s;q)ds = \frac{\Delta_1(q)}{\Delta(q)}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t y_2(s;q)ds = \frac{\Delta_2(q)}{\Delta(q)}, \quad a.s.. \tag{2.7}$$

Proof. Applying Itô's formula to (2.3) yields

$$\ln x_i(t;q) - \ln x_i(0) = b_i(q)t - c_{iil}^{1-q}c_{iiu}^{q}\int_0^t x_i(s;q)ds + \sum_{k=1}^{n}\sigma_{ikl}^{1-q}\sigma_{iku}^{q}B_k(t), \ i=1,2.$$

Note that $b_i(q) > 0$, it then follows from Lemma 2.1 that

$$\frac{b_i(q)}{c_{iil}^{1-q}c_{iiu}^{q}} \leqslant \liminf_{t\to+\infty}\langle x_i(t;q)\rangle \leqslant \limsup_{t\to+\infty}\langle x_i(t;q)\rangle \leqslant \frac{b_i(q)}{c_{iil}^{1-q}c_{iiu}^{q}}, \ \text{a.s.}, \ i=1,2.$$

Therefore

$$\lim_{t\to+\infty}\langle x_i(t;q)\rangle = \lim_{t\to+\infty} t^{-1}\int_0^t x_i(s;q)ds = \frac{b_i(q)}{c_{iil}^{1-q}c_{iiu}^{q}}, \ \text{a.s.}, \ i=1,2. \tag{2.8}$$

Consequently,

$$\begin{aligned}\lim_{t\to+\infty} t^{-1}\int_{t-\tau_{jl}^{1-q}\tau_{ju}^{q}}^{t} x_i(s;q)ds &= \lim_{t\to+\infty} t^{-1}\left[\int_0^t x_i(s;q)ds - \int_0^{t-\tau_{jl}^{1-q}\tau_{ju}^{q}} x_i(s;q)ds\right] \\ &= \frac{b_i(q)}{c_{iil}^{1-q}c_{iiu}^{q}} - \frac{b_i(q)}{c_{iil}^{1-q}c_{iiu}^{q}} = 0, \ \text{a.s.}, \ i,j=1,2, \ j\neq i.\end{aligned}$$

In light of (2.5), we have

$$\lim_{t\to+\infty} t^{-1}\int_{t-\tau_{jl}^{1-q}\tau_{ju}^{q}}^{t} y_i(s;q)ds = 0, \ \text{a.s.}, \ i,j=1,2, \ j\neq i. \tag{2.9}$$

Applying Itô's formula to (1.3) yields

$$\begin{aligned}&\ln y_1(t;q) - \ln y_1(0) \\ &= b_1(q)t - c_{11l}^{1-q}c_{11u}^{q}\int_0^t y_1(s;q)ds - c_{12l}^{1-q}c_{12u}^{q}\int_0^t y_2\left(s-\tau_{1l}^{1-q}\tau_{1u}^{q};q\right)ds + \sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t) \\ &= b_1(q)t - c_{12l}^{1-q}c_{12u}^{q}\int_0^t y_2(s;q)ds + c_{12l}^{1-q}c_{12u}^{q}\left[\int_{t-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{t} y_2(s;q)ds - \int_{-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{0} y_2(s;q)ds\right] \\ &\quad - c_{11l}^{1-q}c_{11u}^{q}\int_0^t y_1(s;q)ds + \sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t).\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\ln y_2(t;q) - \ln y_2(0)\\
&= b_2(q)t - c_{21l}^{1-q}c_{21u}^{q}\int_0^t y_1\left(s-\tau_{2l}^{1-q}\tau_{2u}^{q};q\right)ds - c_{22l}^{1-q}c_{22u}^{q}\int_0^t y_2(s;q)ds + \sum_{i=1}^{n}\sigma_{2il}^{1-q}\sigma_{2iu}^{q}B_i(t)\\
&= b_2(q)t - c_{21l}^{1-q}c_{21u}^{q}\int_0^t y_1(s;q)ds + c_{21l}^{1-q}c_{21u}^{q}\left[\int_{t-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{t} y_1(s;q)ds - \int_{-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{0} y_1(s;q)ds\right]\\
&\quad - c_{22l}^{1-q}c_{22u}^{q}\int_0^t y_2(s;q)ds + \sum_{i=1}^{n}\sigma_{2il}^{1-q}\sigma_{2iu}^{q}B_i(t).
\end{aligned}\tag{2.11}$$

Computing (2.11)$\times c_{11l}^{1-q}c_{11u}^{q}-(2.10)\times c_{21l}^{1-q}c_{21u}^{q}$, we can see that

$$\begin{aligned}
c_{11l}^{1-q}c_{11u}^{q}\ln\frac{y_2(t;q)}{y_2(0)} &= c_{11l}^{1-q}c_{11u}^{q}c_{21l}^{1-q}c_{21u}^{q}\left[\int_{t-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{t} y_1(s;q)ds - \int_{-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{0} y_1(s;q)ds\right]\\
&\quad - c_{21l}^{1-q}c_{21u}^{q}c_{12l}^{1-q}c_{12u}^{q}\left[\int_{t-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{t} y_2(s;q)ds - \int_{-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{0} y_2(s;q)ds\right]\\
&\quad + c_{21l}^{1-q}c_{21u}^{q}\ln\frac{x_1(t;q)}{x_1(0)} + \Delta_2(q)t - \Delta(q)\int_0^t y_2(s;q)ds\\
&\quad - c_{21l}^{1-q}c_{21u}^{q}\sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t) + c_{11l}^{1-q}c_{11u}^{q}\sum_{i=1}^{n}\sigma_{2il}^{1-q}\sigma_{2iu}^{q}B_i(t).
\end{aligned}\tag{2.12}$$

According to (2.1) and (2.9), for arbitrary $\varepsilon > 0$, there exists a $T > 0$ such that for $t \geqslant T$,

$$\begin{aligned}
&t^{-1}c_{21l}^{1-q}c_{21u}^{q}\ln\frac{y_1(t;q)}{y_1(0)} \leqslant \varepsilon/4,\\
&t^{-1}c_{11l}^{1-q}c_{11u}^{q}\ln y_2(0) \leqslant \varepsilon/4,\\
&t^{-1}c_{11l}^{1-q}c_{11u}^{q}c_{21l}^{1-q}c_{21u}^{q}\left[\int_{t-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{t} y_1(s;q)ds - \int_{-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{0} y_1(s;q)ds\right] \leqslant \varepsilon/4,\\
&-t^{-1}c_{21l}^{1-q}c_{21u}^{q}c_{12l}^{1-q}c_{12u}^{q}\left[\int_{t-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{t} y_2(s;q)ds - \int_{-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{0} y_2(s;q)ds\right] \leqslant \varepsilon/4.
\end{aligned}$$

When the above inequalities are used in (2.12), we can see that for $t > T$,

$$\begin{aligned}
c_{11l}^{1-q}c_{11u}^{q}\ln y_2(t;q) &\leqslant (\Delta_2(q)+\varepsilon)t - \Delta(q)\int_0^t y_2(s;q)ds\\
&\quad - c_{21l}^{1-q}c_{21u}^{q}\sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t) + c_{11l}^{1-q}c_{11u}^{q}\sum_{i=1}^{n}\sigma_{2il}^{1-q}\sigma_{2iu}^{q}B_i(t).
\end{aligned}\tag{2.13}$$

Similarly, computing (2.10)$\times c_{22l}^{1-q}c_{22u}^{q}-(2.11)\times c_{12l}^{1-q}c_{12u}^{q}$ results in

$$\begin{aligned}
c_{22l}^{1-q}c_{22u}^{q}\ln\frac{y_1(t;q)}{y_1(0)} &= c_{22l}^{1-q}c_{22u}^{q}c_{12l}^{1-q}c_{12u}^{q}\left[\int_{t-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{t} y_2(s;q)ds - \int_{-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{0} y_2(s;q)ds\right]\\
&\quad - c_{12l}^{1-q}c_{12u}^{q}c_{21l}^{1-q}c_{21u}^{q}\left[\int_{t-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{t} y_1(s;q)ds - \int_{-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{0} y_1(s;q)ds\right]\\
&\quad + c_{12l}^{1-q}c_{12u}^{q}\ln\frac{y_2(t;q)}{y_2(0)} + \Delta_1(q)t - \Delta(q)\int_0^t y_1(s;q)ds\\
&\quad + c_{22l}^{1-q}c_{22u}^{q}\sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t) - c_{12l}^{1-q}c_{12u}^{q}\sum_{i=1}^{n}\sigma_{2il}^{1-q}\sigma_{2iu}^{q}B_i(t).
\end{aligned}\tag{2.14}$$

Similar to (2.13), by virtue of (2.14) we can observe that for $t > T$,

$$c_{22l}^{1-q}c_{22u}^{q}\ln y_1(t;q) \leqslant (\Delta_1(q)+\varepsilon)t - \Delta(q)\int_0^t y_1(s;q)ds + c_{22l}^{1-q}c_{22u}^{q}\sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t) - c_{12l}^{1-q}c_{12u}^{q}\sum_{i=1}^{n}\sigma_{2il}^{1-q}\sigma_{2iu}^{q}B_i(t). \tag{2.15}$$

(i). Since $\Delta_2(q) < 0$, then we can choose ε sufficiently small such that $\Delta_2(q)+\varepsilon < 0$. Applying (i) in Lemma 2.1 to (2.13) gives $\lim_{t\to+\infty} y_2(t;q) = 0$, a.s.. The proof of (2.6) is similar to that of (2.8) and hence is omitted.

The proof of (ii) is similar to that of (i) and we also omit it.

Now we are in the position to prove (iii). Note that $\Delta_2(q) > 0$, according to (2.13) and Lemma 2.1, we obtain

$$\limsup_{t\to+\infty}\langle y_2(t;q)\rangle \leqslant \frac{\Delta_2(q)+\varepsilon}{\Delta(q)}, \quad \text{a.s..}$$

As an application of the arbitrariness of ε, one can observe that

$$\limsup_{t\to+\infty}\langle y_2(t;q)\rangle \leqslant \frac{\Delta_2(q)}{\Delta(q)}, \quad \text{a.s..} \tag{2.16}$$

In the same way, by (2.15), Lemma 2.1, and the arbitrariness of ε, we have

$$\limsup_{t\to+\infty}\langle y_1(t;q)\rangle \leqslant \frac{\Delta_1(q)}{\Delta(q)}, \quad \text{a.s..} \tag{2.17}$$

Let $\varepsilon < c_{11l}^{1-q}c_{11u}^{q}\frac{\Delta_1(q)}{\Delta(q)}$. Substituting (2.9) and (2.16) into (2.10) results in that for sufficiently large t,

$$\begin{aligned}
t^{-1}\ln y_1(t;q) &= t^{-1}\ln y_1(0) + b_1(q) - c_{11l}^{1-q}c_{11u}^{q}\langle y_1(t;q)\rangle - c_{12l}^{1-q}c_{12u}^{q}\langle y_2(t;q)\rangle + \sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t)/t \\
&\quad + c_{12l}^{1-q}c_{12u}^{q}t^{-1}\left[\int_{t-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{t} y_2(s;q)ds - \int_{-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{0} y_2(s;q)ds\right] \\
&\geqslant b_1(q) - \varepsilon - c_{11l}^{1-q}c_{11u}^{q}\langle y_1(t;q)\rangle - c_{12l}^{1-q}c_{12u}^{q}\limsup_{t\to+\infty}\langle y_2(t;q)\rangle + \sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t)/t \\
&\geqslant b_1(q) - \varepsilon - c_{11l}^{1-q}c_{11u}^{q}\langle y_1(t;q)\rangle - c_{12l}^{1-q}c_{12u}^{q}\frac{\Delta_2(q)}{\Delta(q)} + \sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t)/t \\
&= c_{11l}^{1-q}c_{11u}^{q}\frac{\Delta_1(q)}{\Delta(q)} - \varepsilon - c_{11l}^{1-q}c_{11u}^{q}\langle y_1(t;q)\rangle + \sum_{i=1}^{n}\sigma_{1il}^{1-q}\sigma_{1iu}^{q}B_i(t)/t.
\end{aligned}$$

By virtue of (ii) in Lemma 2.1 and the arbitrariness of ε, one can observe that

$$\liminf_{t\to+\infty}\langle y_1(t;q)\rangle \geqslant \frac{\Delta_1(q)}{\Delta(q)}, \quad \text{a.s..}$$

In the same way, substituting (2.9) and (2.17) into (2.11), we obtain

$$\liminf_{t\to+\infty}\langle y_2(t;q)\rangle \geqslant \frac{\Delta_2(q)}{\Delta(q)}, \quad \text{a.s..}$$

Then the desired assertion (2.7) follows. □

3. Stationary distribution

In this section, we will consider the existence of a unique stationary distribution of model (1.3), i.e., to prove that there is a probability measure ν with support $\mathbb{R}^2_+$ such that for any initial data $\eta \in \Gamma$, the transition probability $p(t, \eta, \cdot; q)$ of $y(t; q)$ converges weakly to ν as $t \to +\infty$ (see e.g. [18]). To this end, we introduce a technical assumption.

Assumption 3.1. $c_{11l}^{1-q}c_{11u}^{q} > c_{21l}^{1-q}c_{21u}^{q}$, $c_{22l}^{1-q}c_{22u}^{q} > c_{12l}^{1-q}c_{12u}^{q}$.

Remark 3.2. Clearly, under Assumption 3.1, we have $\Delta(q) > 0$ and Remark 2.4.

Theorem 3.3. *Let Assumption 3.1 holds.*

(a) *If $\Delta_1(q) > 0$ and $\Delta_2(q) > 0$, then the distribution of $y(t; q)$ weakly converges to a unique distribution ν which is ergodic:*

$$\int_{\mathbb{R}^2_+} \xi_i \nu(d\xi_1, d\xi_2; q) = \lim_{t\to+\infty} \langle y_i(t; q)\rangle = \frac{\Delta_i(q)}{\Delta(q)}, \ i = 1, 2. \tag{3.1}$$

(b) *If $\Delta_1(q)\Delta_2(q) < 0$, then model (1.3) does not have a stationary distribution.*

Proof. (a). The proof is divided into three parts.

Part 1. In this part, let us prove

$$\lim_{t\to\infty} \mathbb{E}\Big|y_i(t; q; \eta) - y_i(t; q; \psi)\Big| = 0, \ i = 1, 2, \tag{3.2}$$

where $y(t; q; \eta)$ and $y(t; q; \psi)$ are any two solutions of (1.3) with initial values $\eta(\theta) \in \Gamma$ and $\psi(\theta) \in \Gamma$, respectively. Define

$$\begin{aligned} V(t; q) = &\sum_{i=1}^{2} \Big|\ln y_i(t; q; \eta) - \ln y_i(t; q; \psi)\Big| + c_{12l}^{1-q}c_{12u}^{q} \int_{t-\tau_{1l}^{1-q}\tau_{1u}^{q}}^{t} \Big|y_2(s; q; \eta) - y_2(s; q; \psi)\Big| ds \\ &+ c_{21l}^{1-q}c_{21u}^{q} \int_{t-\tau_{2l}^{1-q}\tau_{2u}^{q}}^{t} \Big|y_1(s; q; \eta) - y_1(s; q; \psi)\Big| ds. \end{aligned}$$

According to Itô's formula, one can see that

$$\begin{aligned} dV(t; q) = &\,\mathrm{sgn}\Big(y_1(t; q; \eta) - y_1(t; q; \psi)\Big)\Big[- c_{11l}^{1-q}c_{11u}^{q}\Big(y_1(t; q; \eta) - y_1(t; q; \psi)\Big) \\ &- c_{12l}^{1-q}c_{12u}^{q}\Big(y_2\Big(t - \tau_{1l}^{1-q}\tau_{1u}^{q}; q; \eta\Big) - y_2\Big(t - \tau_{1l}^{1-q}\tau_{1u}^{q}; q; \psi\Big)\Big)\Big] dt \\ &+ \mathrm{sgn}\Big(y_2(t; q; \eta) - y_2(t; q; \psi)\Big)\Big[- c_{21l}^{1-q}c_{21u}^{q}\Big(y_1\Big(t - \tau_{2l}^{1-q}\tau_{2u}^{q}; q; \eta\Big) \\ &- y_1\Big(t - \tau_{2l}^{1-q}\tau_{2u}^{q}; q; \psi\Big)\Big) - c_{22l}^{1-q}c_{22u}^{q}\Big(y_2(t; q; \eta) - y_2(t; q; \psi)\Big)\Big] dt \\ &+ \sum_{i=1}^{2} \sum_{j=1, j\neq i}^{2} c_{ijl}^{1-q}c_{iju}^{q}\Big|y_j(t; q; \eta) - y_j(t; q; \psi)\Big| dt \\ &- \sum_{i=1}^{2} \sum_{j=1, j\neq i}^{2} c_{ijl}^{1-q}c_{iju}^{q}\Big|y_j\Big(t - \tau_{il}^{1-q}\tau_{iu}^{q}; q; \eta\Big) - y_j\Big(t - \tau_{il}^{1-q}\tau_{iu}^{q}; q; \psi\Big)\Big| dt \\ \leqslant &- \sum_{i=1}^{2} c_{iil}^{1-q}c_{iiu}^{q}\Big|y_i(t; q; \eta) - y_i(t; q; \psi)\Big| dt \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{2}\sum_{j=1,j\neq i}^{2} c_{ijl}^{1-q}c_{iju}^{q}\left|y_j\left(t-\tau_{il}^{1-q}\tau_{iu}^{q};q;\eta\right)-y_j\left(t-\tau_{il}^{1-q}\tau_{iu}^{q};q;\psi\right)\right|dt\\
&+\sum_{i=1}^{2}\sum_{j=1,j\neq i}^{2} c_{ijl}^{1-q}c_{iju}^{q}\left|y_j(t;q;\eta)-y_j(t;q;\psi)\right|dt\\
&-\sum_{i=1}^{2}\sum_{j=1,j\neq i}^{2} c_{ijl}^{1-q}c_{iju}^{q}\left|y_j\left(t-\tau_{il}^{1-q}\tau_{iu}^{q};q;\eta\right)-y_j\left(t-\tau_{il}^{1-q}\tau_{iu}^{q};q;\psi\right)\right|dt\\
=&-\left[c_{11l}^{1-q}c_{11u}^{q}-c_{21l}^{1-q}c_{21u}^{q}\right]\left|y_1(t;q;\eta)-y_1(t;q;\psi)\right|dt\\
&-\left[c_{22l}^{1-q}c_{22u}^{q}-c_{12l}^{1-q}c_{12u}^{q}\right]\left|y_2(t;q;\eta)-y_2(t;q;\psi)\right|dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}(V(t;q))\leqslant V(0)&-\left[c_{11l}^{1-q}c_{11u}^{q}-c_{21l}^{1-q}c_{21u}^{q}\right]\int_0^t\mathbb{E}\left|y_1(s;q;\eta)-y_1(s;q;\psi)\right|ds\\
&-\left[c_{22l}^{1-q}c_{22u}^{q}-c_{12l}^{1-q}c_{12u}^{q}\right]\int_0^t\mathbb{E}\left|y_2(s;q;\eta)-y_2(s;q;\psi)\right|ds.
\end{aligned}
$$

It then follows from $V(t;q)\geqslant 0$ that

$$
\begin{aligned}
&\left[c_{11l}^{1-q}c_{11u}^{q}-c_{21l}^{1-q}c_{21u}^{q}\right]\int_0^t\mathbb{E}\left|y_1(s;q;\eta)-y_1(s;q;\psi)\right|ds\\
&+\left[c_{22l}^{1-q}c_{22u}^{q}-c_{12l}^{1-q}c_{12u}^{q}\right]\int_0^t\mathbb{E}\left|y_2(s;q;\eta)-y_2(s;q;\psi)\right|ds\leqslant V(0)<\infty.
\end{aligned}
$$

Thereby,

$$
\mathbb{E}\left|y_i(t;q;\eta)-y_i(t;q;\psi)\right|\in L^1[0,\infty),\ i=1,2.
$$

According to model (1.3), one obtains

$$
\begin{aligned}
\mathbb{E}(y_i(t;q))=y_i(0)+\int_0^t&\left[r_{il}^{1-q}r_{iu}^{(q)}\mathbb{E}(y_i(s;q))-c_{iil}^{1-q}c_{iiu}^{q}\mathbb{E}(y_i^2(s;q))\right.\\
&\left.-c_{ijl}^{1-q}c_{iju}^{q}\mathbb{E}\left(y_i(s;q)y_j\left(s-\tau_{il}^{1-q}\tau_{iu}^{q};q\right)\right)\right]ds,\ i,j=1,2,\ j\neq i.
\end{aligned}
$$

That is to say, $\mathbb{E}(y_i(t;q))$ is continuously differentiable. In addition, by virtue of (2.2),

$$
\frac{d\mathbb{E}(y_i(t;q))}{dt}\leqslant r_{il}^{1-q}r_{iu}^{(q)}\mathbb{E}(y_i(t;q))\leqslant K_1,\ i=1,2,
$$

where K_1 is a positive constant. Therefore, $\mathbb{E}(y_i(t;q))$ is uniformly continuous. An application of Barbalat's conclusion [2] leads to the required assertion (3.2).

Part 2. In this part let us prove that there is a unique probability measure ν with support R_+^2 such that for any initial data $\eta\in\Gamma$, the transition probability $p(t,\eta,\cdot;q)$ of $y(t;q)$ converges weakly to ν as $t\to+\infty$.

Denote by $\mathcal{P}(t,\eta,A;q)$ the probability of $y(t;q;\eta)\in A$. According to (2.2) and Chebyshev's inequality, one can observe that the family of $\{p(t,\eta,\cdot;q)\}$ is tight. Let $\mathbb{P}(\Gamma;q)$ be all the probability measures on Γ. For all $P_1,P_2\in\mathbb{P}$, define

$$
D_H(P_1,P_2)=\sup_{h\in H}\left|\int_{\mathbb{R}_+^2}h(x)P_1(dx)-\int_{\mathbb{R}_+^2}h(x)P_2(dx)\right|,
$$

where

$$H=\left\{h:\Gamma\to\mathbb{R}\Big||h(x)-h(z)|\leqslant\|x-z\|,|h(\cdot)|\leqslant 1\right\}.$$

For any $h\in H$ and $t, s>0$,

$$\begin{aligned}\Big|\mathbb{E}h(y(t+s;q;\eta))-\mathbb{E}h(y(t;q;\eta))\Big| &= \Big|\mathbb{E}\Big[\mathbb{E}\Big(h(y(t+s;q;\eta))|\mathcal{F}_s\Big)\Big]-\mathbb{E}h(y(t;q;\eta))\Big|\\ &= \Big|\int_{\mathbb{R}^2_+}\mathbb{E}h(y(t;q;\xi))p(s,\eta,d\xi)-\mathbb{E}h(y(t;q;\eta))\Big|\\ &\leqslant \int_{\mathbb{R}^2_+}\Big|\mathbb{E}h(y(t;q;\xi))-\mathbb{E}h(y(t;q;\eta))\Big|p(s,\eta,d\xi;q).\end{aligned}$$

It then follows from (3.2) that there is a T such that for $t\geqslant T$,

$$\sup_{h\in H}\Big|\mathbb{E}h(y(t;q;\xi))-\mathbb{E}h(y(t;q;\eta))\Big|\leqslant\epsilon.$$

In other words,

$$\Big|\mathbb{E}h(y(t+s;q;\eta))-\mathbb{E}h(y(t;q;\eta))\Big|\leqslant\epsilon.$$

An application of the arbitrariness of h yields

$$\sup_{h\in H}\Big|\mathbb{E}h(y(t+s;q;\eta))-\mathbb{E}h(y(t;q;\eta))\Big|\leqslant\epsilon.$$

Consequently,

$$D_H\Big(p(t+s,\eta,\cdot;q),p(t,\eta,\cdot;q)\Big)\leqslant\epsilon,\quad\forall\, t\geqslant T,\ s>0.$$

Namely, $\{p(t,\lambda,\cdot;q):\ t\geqslant 0\}$ is Cauchy in $\mathbb{P}$, where $\lambda=\lambda(\theta)\equiv(0.2,0.2)^T$, $\theta\in[-\tau,0]$. That is to say there is a unique $\nu(\cdot;q)\in\mathbb{P}(\Gamma)$ such that

$$\lim_{t\to+\infty}D_H\Big(p(t,\lambda,\cdot;q),\nu(\cdot;q)\Big)=0.$$

By (3.2),

$$\lim_{t\to+\infty}D_H\Big(p(t,\eta,\cdot;q),p(t,\lambda,\cdot;q)\Big)=0.$$

Thereby

$$\lim_{t\to+\infty}D_H\Big(p(t,\eta,\cdot;q),\nu(\cdot;q)\Big)\leqslant\lim_{t\to+\infty}D_H\Big(p(t,\eta,\cdot;q),p(t,\lambda,\cdot;q)\Big)+\lim_{t\to+\infty}D_H\Big(p(t,\lambda,\cdot;q),\nu(\cdot;q)\Big)=0.$$

According to (iii) in Theorem 2.5, then we obtain the desired assertion.

Part 3. In this part, let us prove ν is ergodic and (3.1) holds. According to the uniqueness of ν and Corollary 3.4.3 in [5], we can obtain that $\nu(\cdot)$ is strong mixing. Hence Theorem 3.2.6 in [5] means $\nu(\cdot)$ is ergodic. An application of (3.3.2) in [5] gives

$$\lim_{t\to\infty}t^{-1}\int_0^t y_i(s;q)ds=\int_{\mathbb{R}^2_+}\xi_i\nu(d\xi_1,d\xi_2;q),\ i=1,2.$$

This together with (2.7) means (3.1).

(b). Note that $\Delta_1(q)\Delta_2(q)<0$, without loss of generality, we suppose that $\Delta_1(q)<0,\ \Delta_2(q)>0$. According to (ii) in Theorem 2.5, the species 1 goes to extinction. Hence model (1.3) does not have a stationary distribution. □

4. Numerical simulations, discussions, and conclusions

In this paper, we proposed and studied a two-species delay stochastic competition model with imprecise parameters. Our main results are Theorem 2.5 and Theorem 3.3. Theorem 2.5 establishes the threshold between persistence and extinction for each species. Theorem 3.3 gives the sharp sufficient criteria for the existence of a unique ergodic stationary distribution of the model. To the best of our knowledge, this paper is the first attempt to study stochastic delay population models with imprecise parameters.

Our results show that the imprecise parameters have close relationships with the persistence, extinction and the existence of stationary distribution of the model. To see these more clearly, let us consider the following example (in the following example and simulations, the values of parameters are chosen hypothetically). We choose $r_{1l} = 0.2$, $r_{1u} = 0.3$, $r_{2l} = 0.15$, $r_{2u} = 0.2$, $c_{11l} = 0.4$, $c_{11u} = 0.5$, $c_{12l} = 0.1$, $c_{12u} = 0.2$, $c_{21l} = 0.2$, $c_{21u} = 0.3$, $c_{22l} = 0.3$, $c_{22u} = 0.4$, $\sigma_{11l}^2 = 0.1$, $\sigma_{11u}^2 = 0.2$, $\sigma_{1jl} = \sigma_{1ju} = 0$, $j = 2, \ldots, n$, $\sigma_{22l}^2 = 0.2$, $\sigma_{22u}^2 = 0.1$, $\sigma_{2kl} = \sigma_{2ku} = 0$, $k = 1, 3, \ldots, n$, $\tau_{1l} = \tau_{2l} = 5$, $\tau_{1u} = \tau_{2u} = 6$. Hence Assumption 3.1 holds. The only difference between the conditions of Figs. 1 and 2 is that the value of q is different.

(A) In Fig. 1, we choose $q = 0.8$, then $\Delta_1(0.8) = 0.0503 > 0$ and $\Delta_2(0.8) = 0.0054 > 0$. According to (a) in Theorem 3.3, both y_1 and y_2 are persistent and the model has a unique ergodic stationary distribution ν and:

$$\int_{\mathbb{R}_+^2} \xi_1 \nu(d\xi_1, d\xi_2; 0.8) = \lim_{t\to+\infty} \langle y_1(t; 0.8)\rangle = \frac{\Delta_1(0.8)}{\Delta(0.8)} = 0.3799,$$
$$\int_{\mathbb{R}_+^2} \xi_2 \nu(d\xi_1, d\xi_2; 0.8) = \lim_{t\to+\infty} \langle y_2(t; 0.8)\rangle = \frac{\Delta_2(0.8)}{\Delta(0.8)} = 0.0404.$$

Fig. 1 (a) is a sample path of the model and Fig. 1 (b) is the distribution of the solution at $t = 8000$ (see e.g., [4]).

(B) In Fig. 2, we choose $q = 0.5$, then $\Delta_1(0.5) > 0$ and $\Delta_2(0.5) = -0.0037 < 0$. It then follows from (b) in Theorem 3.3 that y_2 goes to extinction and y_1 is persistent:

$$\lim_{t\to+\infty} \langle y_1(t; 0.5)\rangle = \frac{b_1(0.5)}{c_{11l}^{0.5} c_{11u}^{0.5}} = 0.3873.$$

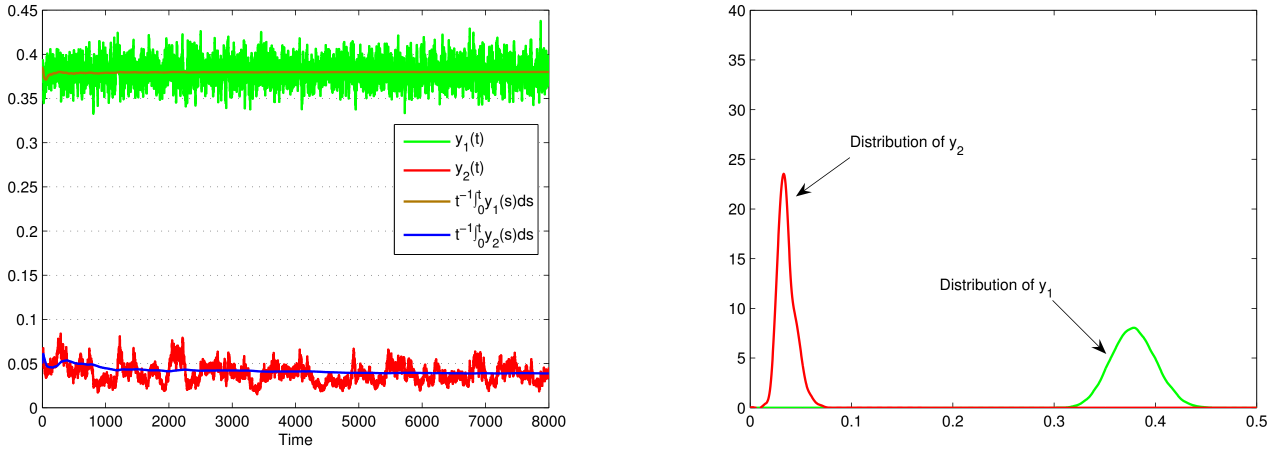
And hence the model does not have a stationary distribution.

Comparing Figs. 1 and 2 we can see that the imprecise parameters have close relationships with the persistence, extinction, and the existence of stationary distribution of the model. In fact, in the above example, if $q > \frac{\ln 1.8}{\ln 1.5}$, then the model has a unique ergodic stationary distribution; if $q < \frac{\ln 1.8}{\ln 1.5}$, then the model does not have a stationary distribution.

Some topics deserve further consideration. Firstly, it is interesting to study some more realistic but more complex systems, for example, reaction diffusion ([1]). Secondly, it is interesting to investigate food-chain models or cooperation models [11, 27]. Thirdly, in this paper we only consider the two-species model. It is of interest to investigate the n-species model. In fact, we also attempt to investigate this problem. However, we can not establish the critical values between persistence and extinction for each species at present stage. Finally, it is useful to study what happens if Assumption 2.3 is not satisfied. In this case, $\Delta_1(q) > 0$ and $\Delta_2(q) > 0$ will not hold simultaneously. Similar to the proof of Theorem 2.5, we can show that

(i') if $\Delta_1(q) > 0$ and $\Delta_2(q) < 0$, then (i) in Theorem 2.5 holds;
(ii') if $\Delta_1(q) < 0$ and $\Delta_2(q) > 0$, then (ii) in Theorem 2.5 holds.

However, what happens if $\Delta_1(q) < 0$ and $\Delta_2(q) < 0$ are still unknown. We leave all of these for future work.



(a) (b)

Figure 1: Model (1.3) with $q = 0.8$, $r_{1l} = 0.2$, $r_{1u} = 0.3$, $r_{2l} = 0.15$, $r_{2u} = 0.2$, $c_{11l} = 0.4$, $c_{11u} = 0.5$, $c_{12l} = 0.1$, $c_{12u} = 0.2$, $c_{21l} = 0.2$, $c_{21u} = 0.3$, $c_{22l} = 0.3$, $c_{22u} = 0.4$, $\sigma^2_{11l} = 0.1$, $\sigma^2_{11u} = 0.2$, $\sigma_{1jl} = \sigma_{1ju} = 0$, $j = 2, \ldots, n$, $\sigma^2_{22l} = 0.2$, $\sigma^2_{22u} = 0.1$, $\sigma_{2kl} = \sigma_{2ku} = 0$, $k = 1, 3, \ldots, n$, $\tau_{1l} = \tau_{2l} = 5$, $\tau_{1u} = \tau_{2u} = 6$, $\psi_1(\theta) = 0.38 + 0.05 \sin\theta$, $\psi_2(\theta) = 0.05 + 0.03 \sin\theta$, $\theta \in [-6, 0]$. (a) is a sample path; (b) is the distribution of the solution at $t = 8000$.

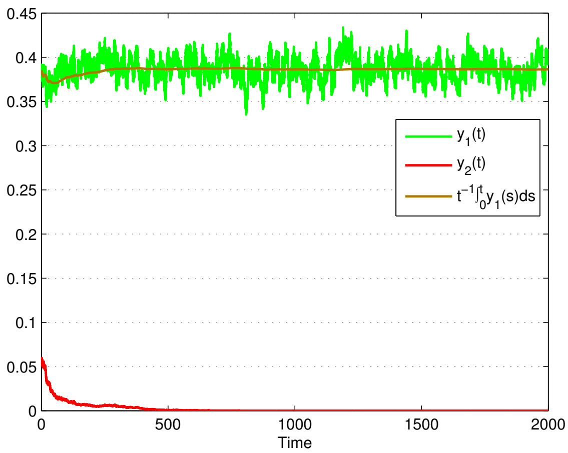


Figure 2: Model (1.3) with $q = 0.5$, other parameters are the same with those in Fig. 1.

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