Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 4789–4798
Research Article

ISSN : 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Best proximity point theorems for generalized α-proximal contractions in Banach spaces

Somayya Komal[a], Poom Kumam[a,b,c,*], Phatiphat Thounthong[d,e], Kanokwan Sitthithakerngkiet[f]

[a]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.

[b]KMUTT-Fixed Point Theory and ApplicationsResearch Group, Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand.

[c]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan.

[d]Renewable Energy Research Centre, King Mongkuts University of Technology North Bangkok (KMUTNB), Wongsawang, Bangsue, Bangkok 10800, Thailand.

[e]Department of Teacher Training in Electrical Engineering, Faculty of Technical Education, King Mongkuts University of Technology North Bangkok (KMUTNB), Wongsawang, Bangsue, Bangkok 10800, Thailand.

[f]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (KMUTNB), 1518 Pracharat 1 Rd., Wongsawang, Bangsue, Bangkok 10800, Thailand.

Communicated by B. Samet

Abstract

In this paper, we obtain the best proximity point theorem for α-proximal contraction of the first and second kinds in Banach spaces by using fixed point theorems. Also, we mention an example for justification of our results.

Keywords: Best proximity point, generalized α-proximal contraction of the first and second kinds, approximatively compactness.
2010 MSC: 47H10, 54H25.

1. Introduction

If we talk about the fixed point of any mapping T in a metric space then one can easily answer about this query that fixed point exists for a self-mapping in a metric space if $x = Tx$. Similar concept can be applied in Banach spaces for fixed point theory. No doubt, in fixed point theory, Banach contraction principle [8] is one of the very earlier results in the literature of the mathematics and non-linear analysis. Furthermore, moving ahead if we can not find a fixed point, we have a non-self-mapping $T : A \to B$ where A, B are subsets of given space then one can find best proximity point which gives us optimal

*Corresponding author
Email addresses: somayya.komal@mail.kmutt.ac.th (Somayya Komal), poom.kum@kmutt.ac.th (Poom Kumam), kanokwan.s@sci.kmutnb.ac.th (Kanokwan Sitthithakerngkiet)

doi:10.22436/jnsa.010.09.21

Received 2016-06-14

approximate solution in the sense that the error $d(x, Tx) = d(A, B)$ is minimum. Simply, it asserts that to find the solution for non-self-mapping that is to find best proximity point for any mapping is more general case than to find fixed point for any mapping. It shows that fixed point theory is the special case of best proximity point theory by taking self-mapping instead of non-self-mapping. Authors in [6] proposed different fixed point results relating to metric spaces. For further detail related to best proximity points in different spaces one can see the references [1–5, 7, 9–11].

In this paper, we obtain the best proximity point theorems via fixed point theorems for α-proximal contractions in the setting of complete norm spaces. We present an example to prove the validity of our results. Our results extend and unify many existing results in the literature. Also, it is easily verified that a normed linear space is a metric space with respect to metric d defined by $d(x, y) = \|x - y\|$, (see [12]). Banach space is a complete normed linear space.

2. Preliminaries

Definition 2.1 ([1]). Let X be a metric space, and A and B two nonempty subsets of X. Define

$$\begin{aligned} d(A, B) &= \inf\{d(a, b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \text{there exists some } b \in B \text{ such that } d(a, b) = d(A, B)\}, \\ B_0 &= \{b \in B : \text{there exists some } a \in A \text{ such that } d(a, b) = d(A, B)\}. \end{aligned}$$

Definition 2.2 ([10]). Given a non-self mapping $f : A \to B$, then an element x^* is called best proximity point of the mapping if the following condition is satisfied

$$d(x^*, fx^*) = d(A, B).$$

Definition 2.3 ([10]). Let (X, d) be a metric space and $A, B \subseteq X$. We say $T : A \to B$ is

1. generalized proximal contraction of the first kind if there exist non-negative numbers α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ such that the condition

$$d(u, Tx) = d(A, B) = d(v, Ty)$$

implies

$$d(u, v) \leqslant \alpha d(x, y) + \beta d(x, u) + \beta d(y, v) + \gamma d(x, v) + \gamma d(y, u);$$

2. generalized proximal contraction of the first kind if there exist non-negative numbers α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ such that the condition

$$d(u, Tx) = d(A, B) = d(v, Ty)$$

implies

$$d(Tu, Tv) \leqslant \alpha d(Tx, Ty) + \beta d(Tx, Tu) + \beta d(Ty, Tv) + \gamma d(Tx, Tv) + \gamma d(Ty, Tu).$$

Theorem 2.4 ([1]). *Let X be a complete metric space and let $Y \subseteq X$ and $Y \neq \emptyset$. Define a mapping $F : Y \to (-\infty, \infty]$ which is proper function and bounded from below. Let $S : Y \to 2^Y \setminus \{\emptyset\}$ be a multivalued mapping such that for each $x \in Y$ there exists $y \in Sx$ with the following inequality:*

$$F(y) + d(x, y) \leqslant F(x).$$

Assume that for $z \in X$

$$\inf\{d(x, z) + d(x, Sx) : x \in Y\} = 0 \Rightarrow \ z \in Sz \cap Y.$$

Then there exists $w \in Y$ such that $w \in Sw$.

Motivated by Ardsalee and Saejung in [1], in this paper, we will introduce new notions about generalized contractions in Banach spaces and find out best proximity points by using fixed point theorem with certain conditions.

3. Fixed point theorem for generalized multivalued mappings in Banach spaces

In this part of the research article we will present fixed point theorem for multivalued mappings in Banach spaces. Furthermore, we will apply this theorem to some best proximity point problems with certain generalized contractions under Banach spaces.

Theorem 3.1. *Let X be a Banach space and let $Y \subseteq X$ and $Y \neq \emptyset$. Define a mapping $A : Y \to (-\infty, \infty]$ which is proper function and bounded below. Let $S : Y \to 2^Y \setminus \{\emptyset\}$ be a multivalued mapping such that for each $x \in Y$ there exists $y \in Sx$ with the following inequality:*

$$|A(y)| + \|x - y\| \leqslant |A(x)|. \tag{3.1}$$

Assume that for $z \in X$

$$\inf\{\|x - z\| + \|x - Sx\| : x \in Y\} = 0 \Rightarrow \; z \in Sz \cap Y. \tag{3.2}$$

Then there exists $w \in Y$ such that $w \in Sw$.

Proof. Let $x_0 \in Y$ such that $|A(x_0)| < \infty$. By inequality (3.1), there exists $x_1 \in Sx_0$ such that $|A(x_1)| + \|x_0 - x_1\| \leqslant |A(x_0)|$. By induction, we obtain a sequence $\{x_n\} \in Y$ such that

$$x_{n+1} \in Sx_n \;\text{ and }\; |A(x_{n+1})| + \|x_n - x_{n+1}\| \leqslant |A(x_n)| \;\text{ for all }\; n \geqslant 0.$$

Thus, $\{A(x_n)\}$ is a decreasing sequence. Since A is bounded below, $\lim_{n\to\infty} A(x_n) = k$ for any $k \in \mathbb{R}$. Let us assume $m \geqslant 0$. We get

$$\Sigma_{n=0}^{n=m}\|x_n - x_{n+1}\| \leqslant \Sigma_{n=0}^{n=m}|A(x_n) - A(x_{n+1})| = |A(x_0)| - |A(x_{n+1})| \leqslant |A(x_0)| - k.$$

Then $\Sigma_{n=0}^{\infty}\|x_n - x_{n+1}\| = \Sigma_{n=0}^{n=m}\|x_n - x_{n+1}\| < \infty$ and so $\{x_n\}$ is a Cauchy sequence. So $\lim_{n\to\infty} x_n = w$ for any $w \in X$. Noting

$$\lim_{n\to\infty}\|x_n - w\| = 0 \;\text{ and }\; \lim_{n\to\infty}\|x_n - Sx_n\| \leqslant \lim_{n\to\infty}\|x_n - x_{n+1}\| = 0,$$

by (3.2), we conclude that $w \in Sw \cap Y$. □

4. New results related to generalized α-proximal contraction of the first kind in Banach spaces

In this section, we will present some new notions and contraction (α-proximal contraction of the first kind) to find the best proximity point with the help of results of Ardsalee and Saejung [1], to find optimal approximate solution for multivalued mappings in Banach spaces and will express best proximity point theorems by using fixed point theorem.

Definition 4.1. Let X be a Banach space and let A, B be non-empty subsets of X with $A_0 \neq \phi$. The non-self-mapping $T : A \to B$ is said to be generalized α-proximal contraction of the first kind if there exist non-negative numbers β, γ with $2\beta + \gamma < 1$ such that the condition

$$\|u - Tx\| = d(A, B) = \|v - Ty\|$$

implies that

$$|\alpha(x, y)|\|u - v\| \leqslant \beta\|u - x\| + \gamma\|x - y\| + \beta\|y - v\|,$$

where $\alpha : X \times X \to [0, \infty)$ such that $\alpha(x, y) \geqslant 1$,

$$d(A, B) := \inf\{\|a - b\| : a \in A, \; b \in B\},$$

and

$$A_0 = \{a \in A : \|a - b\| = d(A, B)\}.$$

Likewise, define B_0.

Theorem 4.2. *Let X be a Banach space and A, B be subsets of X and $A_0 \neq \emptyset$ and closed. Define a mapping $T : A \to B$ such that $T(A_0) \subseteq B_0$ and T is generalized α-proximal contraction of the first kind. Consider that for each $x, y \in A_0$ satisfying $\|x - T(y)\| = d(A, B)$, we have $|\alpha(x, y)| \geqslant 1$. Then the following are satisfied:*

1. *there exists a unique element $x \in A$ such that $\|x - Tx\| = \|A - B\|$;*
2. *if $\{x_n\}$ is a sequence in A_0 with $\|x_{n+1} - Tx_n\| = \|A - B\|$ for each $n \geqslant 0$, then*

$$\lim_{n\to\infty} x_n = x.$$

Proof. Since $A_0 \neq \emptyset$, we take $x_0 \in A_0$, since $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$\|x_1 - T(x_0)\| = d(A, B), \text{ with } |\alpha(x_0, x_1)| \geqslant 1.$$

Again, in same manner we get $x_2 \in A_0$ such that

$$\|x_2 - T(x_1)\| = d(A, B), \text{ with } |\alpha(x_1, x_2)| \geqslant 1.$$

Repeating this process, we get $|\alpha(x_n, x_{n+1})| \geqslant 1$ for any $n \in \mathbb{N}$. For every element $x \in A_0$, we assume that

$$Sx = \{y : y \in A_0 \text{ and } \|y - Tx\| = d(A, B)\},$$

where $d(A, B) = \inf\{\|x - y\|, \text{ for } x \in A, \ y \in B\}$. It shows that $S : A_0 \to 2^{A_0} \setminus \{\emptyset\}$. Since T is generalized α-proximal contraction of the first kind, there are $\beta, \gamma \geqslant 0$ with $2\beta + \gamma < 1$ such that $\|u - Tx\| = d(A, B) = \|v - Ty\|$ implies that

$$\|u - v\| \leqslant \alpha(x, y)\|u - v\| \leqslant \beta\|u - x\| + \gamma\|x - y\| + \beta\|y - v\|$$

for all $u, v, x, y \in A$. Substitute $k = \frac{\beta+\gamma}{1-\beta}$ and $b = \frac{k+1}{2}$. Then $0 \leqslant k < b < 1$.

Claim that, for every $x, y, z \in A_0$, if $y \in Sx$ and $z \in Sy$, then

$$\|z - y\| \leqslant k\|y - x\|.$$

To verify this, let us check, for $x, y, z \in A_0$ such that $y \in Sx$ and $z \in Sy$. Then

$$\|y - Tx\| = d(A, B) = \|z - Ty\|.$$

Since T is generalized α-proximal contraction of the first kind,

$$\|z - y\| \leqslant |\alpha(x, y)|\|z - y\| \leqslant \beta\|z - y\| + \gamma\|y - x\| + \beta\|x - y\|.$$

Thus,

$$\|z - y\| \leqslant k\|y - x\|.$$

So, we proved the claim.

Further, we will express that whether the inequality (3.1) holds here. For this, let us take $x \in A_0$. Since $0 < b < 1$, we can take $y \in Sx$ so that

$$b\|x - y\| \leqslant \|x - Sx\|. \tag{4.1}$$

If $z \in Sy$, then we get by the proved claim that

$$\|y - Sy\| \leqslant \|z - y\| \leqslant k\|y - x\|. \tag{4.2}$$

By using (4.1) and (4.2), we obtain

$$\|y - Sy\| + b\|x - y\| \leqslant k\|x - y\| + \|x - Sx\|.$$

Then

$$\frac{1}{b-k}\|y-Sy\|+\|y-x\|\leqslant\frac{1}{b-k}\|x-Sx\|.$$

Suppose $F : A_0 \to [0,\infty)$ is a mapping defined as $F(x) = \frac{1}{b-k}\|x-Sx\|$ for every $x \in A_0$. So F satisfies inequality (3.1) of Theorem 3.1. Next step is to confirm that whether the inequality (3.2) of Theorem 3.1 is also satisfied or not?

For this, let us consider a sequence $\{x_n\} \in A_0$ and $z \in X$ with

$$\lim_{n\to\infty}\|x_n-z\|=0 \text{ and } \lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

Since A_0 is closed so we get $z \in A_0$ and $Tz \in T(A_0) \subset B_0$. Then we obtain an element $u \in A_0$ such that

$$\|u-Tz\|=d(A,B).$$

We choose a sequence $\{u_n\} \in A_0$ as $u_n \in Sx_n$ also

$$\|x_n-u_n\|<\|x_n-Sx_n\|+\frac{1}{n}$$

for each $n \geqslant 1$. So, $\lim_{n\to\infty}\|x_n-u_n\|=0$. We know that $u_n \in Sx_n$ for each $n \geqslant 0$,

$$\|u_n-Tx_n\|=d(A,B).$$

Also, we have $\lim_{n\to\infty}\|x_n-z\|=0$ and $\lim_{n\to\infty}\|x_n-u_n\|=0$, by using these, we obtain $\lim_{n\to\infty}u_n=z$. Using (4.1) and (4.2), and with the fact that T is generalized α-proximal contraction of the first kind, we get for every $n \geqslant 0$,

$$|\alpha(z,x_n)|\|u-u_n\|\leqslant\beta\|u-x_n\|+\gamma\|x_n-z\|+\beta\|z-u_n\|.$$

Now, by taking limit as $n\to\infty$, we have $\|u-z\|\leqslant\lim_{n\to\infty}|\alpha(z,x_n)|\|u-z\|\leqslant\beta\|u-z\|<\|u-z\|$. This is a contradiction. Thus, $u=z$ and so $\|z-Tz\|=d(A,B)$, this shows that $z\in Sz$. Hence inequality (3.2) of Theorem 3.1 is also satisfied here. To conclude this, we say by using Theorem 3.1 there is an element $w\in A_0$ such that $w\in Sw$, it means

$$\|w-Tw\|=d(A,B).$$

Now, for uniqueness, let $\|\tilde{w}-T\tilde{w}\|=d(A,B)$ for some $\tilde{w}\in A$. We know that T is generalized α-proximal contraction of the first kind, we get as

$$\|w-\tilde{w}\|\leqslant\gamma\|w-\tilde{w}\|.$$

From here we obtain $w=\tilde{w}$. Hence we get the first part. Next to prove second part, let us take a sequence $\{x_n\}$ in A_0 such that

$$\|x_{n+1}-Tx_n\|=d(A,B) \quad \text{for each} \quad n\geqslant 0.$$

Hence $x_{n+1}\in Sx_n$. From the claim, we obtain as for every $n\geqslant 0$,

$$\|x_{n+2}-x_{n+1}\|\leqslant k\|x_{n+1}-x_n\|.$$

Thus $\{x_n\}$ is a Cauchy sequence and hence $\lim_{n\to\infty}x_n = x$ for some $x\in A_0$. Since T is generalized α-proximal contraction of the first kind, we get

$$\|x_{n+1}-w\|\leqslant|\alpha(x_n,w)|\|x_{n+1}-w\|\leqslant\gamma\|x_n-w\|+\beta\|x_n-x_{n+1}\|+\beta\|w-w\|$$

for every $n\geqslant 0$. Taking limit as $n\to\infty$, we obtain $\|x-w\|\leqslant\gamma\|x-w\|$. From here we can say that $x=w$. Therefore, $\lim_{n\to\infty}x_n = w$ and now we have also proved the second part. Hence, we have proved the theorem. □

5. Some new results related to generalized α-proximal contraction of the second kind in Banach spaces

In this part, we will define generalized α-proximal contraction of the second kind and will express some results related to this kind of contraction.

Definition 5.1. Let X be a Banach space and let A, B be non-empty subsets of X with $A_0 \neq \emptyset$. The non-self-mapping $T : A \to B$ is said to be generalized α-proximal contraction of the second kind if there exist non-negative numbers β, γ with $2\beta + \gamma < 1$ such that the condition

$$\|u - Tx\| = d(A, B) = \|v - Ty\|$$

implies that

$$|\alpha(Tx, Ty)|\|Tu - Tv\| \leqslant \beta\|Tu - Tx\| + \gamma\|Tx - Ty\| + \beta\|Ty - Tv\|,$$

where $\alpha : X \times X \to [0, \infty)$ such that $\alpha(x, y) \geqslant 1$,

$$d(A, B) := \inf\{\|a - b\| : a \in A,\ b \in B\},$$

and

$$A_0 = \{a \in A : \|a - b\| = d(A, B)\}.$$

Likewise, define B_0.

Theorem 5.2. *Let us take a Banach space $(X, \|.\|)$ and A, B be two non-empty subsets of X such that $A_0 \neq \phi$. Let us define a non-self-mapping $T : A \to B$ such that $T(A_0) \subset B_0$. If we consider that T is generalized α-proximal contraction of the second kind and $T(A_0)$ is closed subset of X. Consider that for each $x, y \in A_0$ satisfying $\|x - T(y)\| = d(A, B)$, we have $|\alpha(x, y)| \geqslant 1$. Then the following hold:*

(I) *there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$;*
(II) *if there is $\tilde{x} \in A$ such that $\|\tilde{x} - T\tilde{x}\| = d(A, B)$, then $Tx = T\tilde{x}$;*
(III) *if $\{x_n\}$ is a sequence in A_0 with $\|x_{n+1} - Tx_n\| = d(A, B)$ for each $n \geqslant 0$, then*

$$\lim_{n\to\infty} Tx_n = Tx.$$

Proof. Since $A_0 \neq \emptyset$, we take $x_0 \in A_0$, since $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$\|x_1 - T(x_0)\| = d(A, B),\ \text{with } |\alpha(x_0, x_1)| \geqslant 1.$$

Again, in same manner we get $x_2 \in A_0$ such that

$$\|x_2 - T(x_1)\| = d(A, B),\ \text{with } |\alpha(x_1, x_2)| \geqslant 1.$$

Repeating this process, we get $|\alpha(x_n, x_{n+1})| \geqslant 1$ for any $n \in \mathbb{N}$. We suppose that for each $x \in T(A_0)$,

$$Sx = \{y : y = Tu \text{ where } u \in A_0 \text{ and } \|u - x\| = d(A, B)\}.$$

It shows that $S : T(A_0) \to 2^{T(A_0)} \setminus \{\emptyset\}$. Since T is generalized α-proximal contraction of the second kind, so there are $\beta, \gamma \geqslant 0$ with $2\beta + \gamma < 1$ such that $\|u - Tx\| = d(A, B) = \|v - Ty\|$ shows that

$$\|Tu - Tv\| \leqslant |\alpha(x, y)|\|Tu - Tv\| \leqslant \beta\|Tu - Tx\| + \gamma\|Tx - Ty\| + \beta\|Ty - Tv\|,$$

for every $u, v, x, y \in A$. Put $k = \frac{\beta+\gamma}{1-\beta}$ and $b = \frac{k+1}{2}$. Then $0 \leqslant k < b < 1$. Now, we have to prove the claims of Theorem 3.1 as the first claim is about: for each $u, v, x, y \in T(A_0)$, if $u \in Sx$ and $v \in Sy$, then

$$\|u - v\| \leqslant \gamma\|x - y\| + \beta\|x - u\| + \beta\|y - v\|.$$

Now, let us prove this claim as by taking elements $u, v, x, y \in T(A_0)$ such that $u \in Sx$ and $v \in Sy$. So $u = T\tilde{u}, v = T\tilde{v}, x = T\tilde{x}, y = T\tilde{y}$ for some $\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y} \in A_0$ with

$$\|\tilde{u} - T\tilde{x}\| = d(A, B) = \|\tilde{v} - T\tilde{y}\|.$$

We know that T is generalized α-proximal contraction of the second kind, so

$$\|T\tilde{u} - T\tilde{v}\| \leqslant |\alpha(x, y)|\|T\tilde{u} - T\tilde{v}\| \leqslant \beta\|T\tilde{x} - T\tilde{u}\| + \gamma\|T\tilde{x} - T\tilde{y}\| + \beta\|T\tilde{y} - T\tilde{v}\|,$$

that is,

$$\|u - v\| \leqslant \gamma\|x - y\| + \beta\|x - u\| + \beta\|y - v\|.$$

Hence, we have proved the first claim. Now, we will prove the second claim of Theorem 3.1, that is, for each $x, y, z \in T(A_0)$ if $y \in Sx$ and $z \in Sy$, then $\|z - y\| \leqslant k\|x - y\|$. For this, let us assume $x, y, z \in T(A_0)$ such that $y \in Sx$ and $z \in Sy$. Here we will use the first claim, we get

$$\|z - y\| \leqslant \gamma\|y - x\| + \beta\|y - z\| + \beta\|x - y\|.$$

So $\|z - y\| \leqslant k\|x - y\|$. Hence, the second claim has been proved.

Next, we will prove that inequality (3.1) in Theorem 3.1 holds here. Let $x \in T(A_0)$. Since $0 < b < 1$, there exists $y \in Sx$ such that

$$b\|x - y\| \leqslant \|x - Sx\|. \tag{5.1}$$

Let $z \in Sy$ then we get by using the claim II,

$$\|y - Sy\| \leqslant \|z - y\| \leqslant k\|x - y\|. \tag{5.2}$$

Now, using (5.1) and (5.2), we get

$$\|y - Sy\| + b\|x - y\| \leqslant k\|x - y\| + \|x - Sx\|.$$

Then $\frac{1}{b-k}\|y - Sy\| + \|x - y\| \leqslant \frac{1}{b-k}\|x - Sx\|$. Suppose $F : T(A_0) \to [0, \infty)$ is a mapping defined as $F(x) = \frac{1}{b-k}\|x - Sx\|$ for every $x \in T(A_0)$. So F satisfies the inequality (3.1) of Theorem 3.1 here.

Further we will prove that the inequality (3.2) of Theorem 3.1 also holds. Let $z \in X$ and $\{x_n\}$ is a sequence in $T(A_0)$ such that

$$\lim_{n\to\infty} \|x_n - z\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Since $T(A_0)$ is closed, $z \in T(A_0)$ and so we can let $\tilde{z} \in Sz$.

Now we have to show that $z = \tilde{z}$. Since $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$, we can choose a sequence $\{y_n\}$ in $T(A_0)$ so that

$$y_n \in Sx_n$$

for every $n \geqslant 0$ and

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since $\lim_{n\to\infty} \|x_n - z\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we get $\lim_{n\to\infty} y_n = z$. Using (5.1), (5.2) and claim I,

$$\|\tilde{z} - y_n\| \leqslant |\alpha(x_n, y_n)|\|\tilde{z} - y_n\| \leqslant \gamma\|z - x_n\| + \beta\|z - \tilde{z}\| + \beta\|x_n - y_n\|.$$

As $n \to \infty$, we obtain $\|\tilde{z} - z\| \leqslant \beta\|z - \tilde{z}\| < \|z - \tilde{z}\|$, which gives us a contradiction, so $z = \tilde{z}$. Thus, inequality (3.2) in Theorem 3.1 also holds here. Using Theorem 3.1, there exists $w \in T(A_0)$ such that $w \in Sw$, that is, there exists $w^* \in A_0$ such that $w = Tw^*$ and

$$\|w^* - Tw^*\| = d(A, B).$$

So we have best proximity point for T. Thus, we have proved the first part (I). Next, to prove uniqueness of best proximity point of T, that is second part (II) of Theorem 5.2, for this $v \in A$ be another best proximity point of T such that $\|v - Tv\| = d(A, B)$. Since T is generalized α-proximal contraction of the second kind,

$$\|Tv - Tw^*\| \leqslant |\alpha(Tv, Tw^*)|\|Tv - Tw^*\| \leqslant \gamma\|Tv - Tw^*\| + \beta\|Tv - Tv\| + \beta\|Tw^* - Tw^*\|.$$

Then $\|Tv - Tw^*\| \leqslant \gamma\|Tv - Tw^*\| < \|Tv - Tw^*\|$ implies that $Tv = Tw^*$. Hence we have proved the first two parts of Theorem 5.2, now it only remains that the last part left to be proved. For this, let $\{x_n\}$ be a sequence in A_0 such that $\|x_{n+1} - Tx_n\| = d(A, B)$ for all $n \geqslant 0$. So we get $Tx_{n+1} \in STx_n$. By using claim II of this theorem, we have for all $n \geqslant 0$,

$$\|Tx_{n+2} - Tx_{n+1}\| \leqslant k\|Tx_{n+1} - Tx_n\|.$$

Thus $\{Tx_n\}$ is a Cauchy sequence and so $\lim_{n\to\infty} Tx_n = s$ for any $s \in X$. Since $w \in Sw$ and $Tx_{n+1} \in STx_n$, we get

$$\|w - Tx_{n+1}\| \leqslant |\alpha(w, tx_n)|\|w - Tx_{n+1}\| \leqslant \gamma\|w - Tx_n\| + \beta\|w - w\| + \beta\|Tx_n - Tx_{n+1}\|.$$

By taking limit as $n \to \infty$, we obtain $\|w - s\| \leqslant \gamma\|w - s\|$. So $w = s$. It means that $\lim_{n\to\infty} Tx_n = w$. Thus, theorem has been proved. □

Definition 5.3. Let $(X, \|.\|)$ be a Banach space and let A, B be non-empty subsets of X. The set B is said to be approximately compact with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $\lim_{n\to\infty} \|x - y_n\| = d(x, B)$ for some $x \in A$ has a convergent subsequence.

Theorem 5.4. *Let $(X, \|.\|)$ be a Banach space and A, B be non-empty closed subsets of X such that A_0 is non-empty. Assume a continuous mapping $T : A \to B$ which is generalized α-proximal contraction of the second kind such that $T(A_0) \subset B_0$. If A is approximately compact with respect to B, then the following hold:*

1. *there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$;*
2. *if there is $\tilde{x} \in A$ such that $\|\tilde{x} - T\tilde{x}\| = d(A, B)$, then $Tx = T\tilde{x}$;*
3. *if $\{x_n\}$ is a sequence in A_0 with $\|x_{n+1} - Tx_n\| = d(A, B)$ for each $n \geqslant 0$, then*

$$\lim_{n\to\infty} Tx_n = Tx.$$

Proof. Let define mappings $S : T(A_0) \to 2^{T(A_0)} \setminus \{\phi\}$ and $F : T(A_0) \to [0, \infty)$ as in the proof of Theorem 5.2 likewise. Proceeding the same proof lines with above mentioned mappings, we proved that the inequality (3.1) of the Theorem 3.1 is satisfied.

Next, to prove that second inequality (3.2) of Theorem 3.1 also holds. For this, assume that a sequence $\{x_n\} \in T(A_0)$ and let $z \in X$. Suppose that

$$\lim_{n\to\infty} \|x_n - z\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{5.3}$$

Since $x_n \in T(A_0) \subset T(A) \subset B$ and B is closed, $z \in B$. We take a sequence $\{y_n\}$ in $T(A_0)$ so that $y_n \in Sx_n$ for every $n \geqslant 0$ and

$$\|x_n - y_n\| = 0. \tag{5.4}$$

We know that $y_n \in Sx_n$ for each $n \geqslant 0$, we may write $y_n = Tu_n$ for some $u_n \in A_0$ with

$$\|u_n - x_n\| = d(A, B).$$

We have

$$d(A, B) \leqslant \|u_n - z\| \leqslant \|u_n - x_n\| + \|x_n - z\| = d(A, B) + \|x_n - z\|.$$

Thus $\lim_{n\to\infty} \|u_n - z\| = d(A, B)$. Since A is approximately compact with respect to B, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u$ for some $u \in A$. Since T is continuous, we get $Tu_{n_k} \to Tu$. By using (5.3) and (5.4), we obtain $y_n \to z$ and so

$$Tu = \lim_{k\to\infty} Tu_{n_k} = \lim_{n\to\infty} Tu_n = \lim_{n\to\infty} y_n = z.$$

Therefore,

$$\|u - Tu\| = \|u - z\| = \lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = d(A, B),$$

that is, $Tu \in STu$ or $z \in Sz \cap T(A_0)$. Therefore, the inequality (3.2) in Theorem 3.1 holds here. Now by using Theorem 3.1, we can say, there is $w \in T(A_0)$ such that $w \in Sw$, it means $w = T\tilde{w}$ for any $\tilde{w} \in A_0$ with $\|\tilde{w} - T\tilde{w}\| = d(A, B)$. Hence, the first part of this theorem 5.4 is proved and the rest of it can be proved from the conclusions from Theorem 5.2. □

Example 5.5. Let us take $\mathbb{R}^2$ with usual norm and $A = \{(0, a) : 0 \leqslant a\}$ and $B = \{(1, b) : 0 \leqslant b\}$. We take $A_0 = A$ and $B_0 = B$. Define a mapping $T : A \to B$ as $T(0, a) = (1, g(a))$, where $g(a) = \frac{3}{4} + \frac{3}{a+4}$ and $(0, a) \in A$. Also,

$$\|g(a) - g(b)\| = \|\frac{3}{a+4} - \frac{3}{b+4}\| \leqslant \frac{3}{4}\|a - b\|.$$

Since we know that $T(A_0) = T(A) = \{(1, x) : x \in [0, \frac{3}{4})\}$ is not closed, it is obvious that A is approximately compact with respect to B and also T is continuous. Define $\alpha : X \times X \to \mathbb{R}^+$ as

$$\alpha(x, y) = 1, \text{ if } x, y \in (0, a) \text{ where } a \geqslant 0,$$

otherwise 0. We will prove that T is generalized α-proximal contraction of the second type. Let us take $u, v, x, y \in A$ such that $\|u - Tx\| = \|v - Ty\| = d(A, B)$. Then we can say that $x = (0, a_1)$ and $y = (0, a_2)$ for some $a_1, a_2 \geqslant 0$. Thus, $u = Tx$ that is, $u = (1, g(a_1))$ and $v = Ty$ implies that $v = (1, g(a_2))$. We write as

$$\|Tu - Tv\| = \|g^2(a_1) - g^2(a_2)\| \leqslant \frac{3}{4}\|g(a_1) - g(a_2)\| = \frac{3}{4}\|Tx - Ty\|.$$

Therefore, T is generalized α-proximal contraction of the second type. Thus, claims of Theorem 5.4 satisfied.

6. Conclusions

In this article the authors introduced the new notions of generalized α-proximal contractions of the first and second kinds in Banach spaces. These contractions and theorems in this paper motivated the new techniques for finding the best proximity points in Banach spaces.

Acknowledgments

This project was supported by Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Cluster (CLASSIC), Faculty of Science, KMUTT. Somayya Komal was supported by the Petchra Pra Jom Klao Doctoral Scholarship Academic for Ph.D. Program at KMUTT. Moreover, this research work was financially supported by King Mongkuts University of Technology North Bangkok. Contract No. KMUTNB-60-ART-085.

References

[1] P. Ardsalee, S. Saejung, *Best proximity point theorems via fixed point theorems for multivalued mappings*, Fixed Point Theory Appl., **2016** (2016), 11 pages. 1, 2.1, 2.4, 2, 4

[2] A. Fernández-León, *Best proximity points for proximal contractions*, J. Nonlinear Convex Anal., **15** (2014), 313–324.

[3] M. Jleli, E. Karapınar, B. Samet, *A short note on the equivalence between 'best proximity' points and 'fixed point' results*, J. Inequal. Appl., **2014** (2014), 9 pages.

[4] M. Jleli, B. Samet, *Best proximity points for $\alpha\psi$-proximal contractive type mappings and applications*, Bull. Sci. Math., **137** (2013), 977–995.

[5] M. Jleli, B. Samet, *Remarks on the paper: Best proximity point theorems: an exploration of a common solution to approximation and optimization problems*, Appl. Math. Comput., **228** (2014), 366–370. 1

[6] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japon., **44** (1996), 381–391. 1

[7] S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. 1

[8] S. Sadiq Basha, *Extensions of Banach's contraction principle*, Numer. Funct. Anal. Optim., **31** (2010), 569–576. 1

[9] S. Sadiq Basha, *Best proximity points: optimal solutions*, J. Optim. Theory Appl., **151** (2011), 210–216. 1

[10] S. Sadiq Basha, N. Shahzad, *Best proximity point theorems for generalized proximal contractions*, Fixed Point Theory Appl., **2012** (2012), 9 pages. 2.2, 2.3

[11] B. Samet, *Some results on best proximity points*, J. Optim. Theory Appl., **159** (2013), 281–291. 1

[12] W. Takahashi, *Nonlinear functional analysis*, Fixed point theory and its applications, Yokohama Publishers, Yokohama, (2000). 1