Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 4834–4845
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Global and local behavior of a class of $\xi^{(s)}$-QSO

Abdelwahab Alsarayreh[a,*], Izzat Qaralleh[b], Muhammad Zaini Ahmad[a], Basma Al-Shutnawi[b], Saba Al-Kaseasbeh[b]

[a] *Institute of Engineering Mathematics, University Malaysia Perlis Pauh Putra Main Campus, 02600 Arau, Perlis, Malaysia.*

[b] *Department of mathematics, Faculty of Science, Tafila Technical University, P. O. Box, 179, 66110, Tafila, Jordan.*

Communicated by B. Samet

Abstract

A quadratic stochastic operator (QSO) describes the time evolution of different species in biology. The main problem with regard to a nonlinear operator is to study its behavior. This has not been studied in depth; even QSOs, which are the simplest nonlinear operators, have not been studied thoroughly. This paper investigates the global behavior of an operator taken from $\xi^{(s)}$-QSO when the parameter $a = \frac{1}{2}$. Moreover, we study the local behavior of this operator at each value of a, where $0 < a < 1$.

Keywords: Quadratic stochastic operator, local behavior, global behavior.
2010 MSC: 37E99, 37N25, 39B82, 47H60, 92D25.

1. Introduction

The history of quadratic stochastic operators (QSOs) can be traced back to Bernstein's work [1]. The QSO was considered an important source of analysis for the study of dynamical properties and modelings in various fields such as biology [1, 14, 15, 18–20, 34], physics [25, 32], economics, and mathematics [15, 17, 20, 33].

One such system related to population genetics is given by a QSO [1], which is commonly used to present the time evolution of species in biology, which arises as follows: consider a population that consists of m species (or traits) $1, 2, \cdots, m$. We denote a set of all species (traits) by $I = \{1, 2, \cdots, m\}$. Let $x^{(0)} = \left(x_1^{(0)}, \cdots, x_m^{(0)}\right)$ be a probability distribution of species at an initial state and $P_{ij,k}$ be a probability that individuals in the i^{th} and j^{th} species (traits) interbreed to produce an individual from k^{th} species (trait). Then a probability distribution $x^{(1)} = \left(x_1^{(1)}, \cdots, x_m^{(1)}\right)$ of the spices (traits) in the first generation can be found as a total probability, i.e.,

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = \overline{1, m}.$$

*Corresponding author
Email addresses: abdalwahb.saraereh@gmail.com (Abdelwahab Alsarayreh), izzat_math@yahoo.com (Izzat Qaralleh), mzaini@unimap.edu.my (Muhammad Zaini Ahmad), salmashut@yahoo.com (Basma Al-Shutnawi), skasabeh@ttu.edu.jo (Saba Al-Kaseasbeh)

doi:10.22436/jnsa.010.09.25

Received 2016-09-17

This result means that the association $x^{(0)} \to x^{(1)}$ defines a mapping V called *the evolution operator*. The population evolves by starting from an arbitrary state $x^{(0)}$, then passing to the state $x^{(1)} = V(x^{(0)})$ (the first generation), then to the state $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})) = V^{(2)}(x^{(0)})$ (the second generation), and so on. Therefore, the evolution states of the population system are described by the following discrete dynamical system:

$$x^{(0)}, \quad x^{(1)} = V\left(x^{(0)}\right), \quad x^{(2)} = V^{(2)}\left(x^{(0)}\right), \quad x^{(3)} = V^{(3)}\left(x^{(0)}\right), \cdots.$$

In other words, a QSO describes a distribution of the next generation if the distribution of the current generation was given. The fascinating applications of QSO to population genetics were given in [20].

A self-contained exposition of the recent achievements and open problems in the theory of QSO were given in [11]. The main problem in nonlinear operator theory is to study the behavior of nonlinear operators. This problem was not fully addressed even in the class of QSOs, which is the simplest nonlinear operator. The difficulty of the problem depends on the given cubic matrix $(P_{ijk})_{i,j,k=1}^{m}$. An asymptotic behavior of the QSO even on the small dimensional simplex is complicated [4, 30, 31, 33, 35]. To solve this problem, many researchers always introduced a certain class of QSOs and studied their behavior. For examples, Volterra-QSO [5, 6, 10, 16, 33], permutated Volterra-QSO [8, 9], Quasi-Volterra-QSO [12], ℓ-Volterra-QSO [27, 28], non-Volterra-QSO [4, 31], strictly non-Volterra-QSO [36], F-QSO [29], and non-Volterra operators generated by product measure [7, 13, 26]. However, all these classes together would not cover a set of all QSOs. Therefore, many classes of QSOs have not been studied yet. Recently, a new class of QSOs called $\xi^{(s)}$-QSO was introduced in [21, 22, 24]. In this paper, we will continue the study of $\xi^{(s)}$-QSO. This class of operators depends on a partition of the coupled index set (the coupled trait set) $\mathbf{P}_m = \{(i,j) : i < j\} \subset I \times I$. In the case of two-dimensional simplex ($m = 3$), the coupled index set (the coupled trait set) $\mathbf{P}_3$ has five possible partitions. The dynamics of $\xi^{(s)}$-QSO that correspond to the point partition (the maximal partition) of $\mathbf{P}_3$ have been investigated in [21–24].

The paper is organized as follows. In Section 2, we present some preliminary definitions. In Section 3, we find the fixed points of V_a. In Section 4, we study the global behavior of $V_{\frac{1}{2}}$. In Section 5, we investigate the local behavior of V_a.

2. Preliminaries

Definition 2.1. QSO is a mapping of the simplex

$$S^{m-1} = \left\{ x = (x_1, \cdots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1, \; x_i \geqslant 0, \; i = \overline{1,m} \right\}$$

into itself of the form

$$x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = \overline{1,m}, \tag{2.1}$$

where $V(x) = x' = (x'_1, \cdots, x'_m)$ and $P_{ij,k}$ is a coefficient of heredity, which satisfies the following conditions

$$P_{ij,k} \geqslant 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1. \tag{2.2}$$

From the above definition we can conclude that each QSO $V : S^{m-1} \to S^{m-1}$ can be uniquely defined by a cubic matrix $\mathcal{P} = \left(P_{ijk}\right)_{i,j,k=1}^{m}$ with conditions (2.2).

We will denote by $\mathrm{Fix}(V)$ the set of fixed points of $V : S^{m-1} \to S^{m-1}$. Given Brouwer's fixed point theorem, one has always $\mathrm{Fix}(V) \neq \emptyset$ for any QSO V. For a given point $x^{(0)} \in S^{m-1}$, a trajectory $\{x^{(n)}\}_{n=0}^{\infty}$

of $V: S^{m-1} \to S^{m-1}$ starting from $x^{(0)}$ is defined by $x^{(n+1)} = V(x^{(n)})$. By $\omega_V(x^{(0)})$, we denote a set of omega limiting points of the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$. $\{x^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} are compact. Thus, one has $\omega_V(x^{(0)}) \neq \emptyset$. Obviously, if $\omega_V(x^{(0)})$ consists of a single point, then the trajectory converges, and a limiting point is a fixed point of $V: S^{m-1} \to S^{m-1}$.

Recall that a Volterra-QSO is defined by (2.1), (2.2), and the additional assumption

$$P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i,j\}. \tag{2.3}$$

The biological treatment of condition (2.3) is clear: the offspring repeats the genotype (trait) of one of its parents.

One can see that a Volterra-QSO has the following form:

$$x'_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki}x_i\right), \quad k \in I,$$

where

$$a_{ki} = 2P_{ik,k} - 1 \ \text{ for } i \neq k \text{ and } a_{ii} = 0, \ i \in I.$$

Moreover,

$$a_{ki} = -a_{ik} \ \text{ and } \ |a_{ki}| \leqslant 1.$$

This type of operator was intensively studied in [2, 5, 6, 10, 16, 33]. Note that this operator is a discretization of the Lotka-Volterra model [19, 34] which models an interacting competing species in the population system. Such a model has received considerable attention in the fields of biology, economy, and mathematics (see for example [14, 15, 25, 34]).

The concept of ℓ-Volterra-QSO, which generalizes a notion of Volterra-QSO, was introduced in [27]. This concept is recalled as follows.

In order to introduce a new class of QSOs, we need some auxiliary notations. We fix $\ell \in I$ and assume that elements $P_{ij,k}$ of the matrix $(P_{ij,k})_{i,j,k=1}^{m}$ satisfy

$$P_{ij,k} = 0 \ \text{ if } \ k \notin \{i,j\} \ \text{ for any } \ k \in \{1,\dots,\ell\}, \ \ i,j \in I, \tag{2.4}$$

$$P_{i_0j_0,k} > 0 \ \text{ for some } (i_0,j_0), \ i_0 \neq k, \ j_0 \neq k, \ \ k \in \{\ell+1,\dots,m\}. \tag{2.5}$$

For any fixed $\ell \in I$, the QSO defined by (2.1), (2.2), (2.4), and (2.5) is called *ℓ-Volterra-QSO*.

Remark 2.2. Here, we emphasize the following points.

1. An ℓ-Volterra-QSO is a Volterra-QSO if and only if $\ell = m$.
2. No periodic trajectory exists for Volterra-QSO [5]. However, such trajectories exist for ℓ-Volterra-QSO [27].

By following [28], take $k \in \{1,...,\ell\}$, then $P_{kk,i} = 0$ for $i \neq k$ and

$$1 = \sum_{i=1}^{m} P_{kk,i} = P_{kk,k} + \sum_{i=\ell+1}^{m} P_{kk,i}.$$

With the use of $P_{ij,k} = P_{ji,k}$ and denoting $a_{ki} = 2P_{ik,k} - 1, k \neq i$, $a_{kk} = P_{kk,k} - 1$ one then obtains

$$V := \begin{cases} x'_k = x_k\left(1 + \sum\limits_{i=1}^{m} a_{ki}x_i\right), & \text{if } \ k = \overline{1,\ell}, \\ x'_k = x_k\left(1 + \sum\limits_{i=1}^{m} a_{ki}x_i\right) + \sum\limits_{\substack{i,j=1 \\ i\neq k, j\neq k}}^{m} P_{ij,k}x_ix_j, & \text{if } \ k = \overline{\ell+1,m}. \end{cases}$$

This is a canonical form of ℓ-Volterra-QSO.

Note that

$$a_{kk} \in [-1,0], \ |a_{ki}| \leqslant 1, \ a_{ki} + a_{ik} = 2(P_{ik,i} + P_{ik,k}) - 2 \leqslant 0, \ \ i,k \in I.$$

We recall that an operator V is *permuted ℓ-Volterra-QSO*, if there is a permutation τ of the set I and an ℓ-Volterra-QSO V_0 such that $(V(x))_{\tau(k)} = (V_0(x))_k$ for any $k \in I$. In other words, V can be represented as follows:

$$V_\tau := \begin{cases} x'_{\tau(k)} = x_k\left(1+\sum_{i=1}^{m} a_{ki}x_i\right), & \text{if } k=\overline{1,\ell}, \\ x'_{\tau(k)} = x_k\left(1+\sum_{i=1}^{m} a_{ki}x_i\right) + \sum_{\substack{i,j=1\\ i\neq k, j\neq k}}^{m} P_{ij,k}x_ix_j, & \text{if } k=\overline{\ell+1,m}. \end{cases}$$

By following [24], each element $x \in S^{m-1}$ is a probability distribution of the set $I = \{1, ..., m\}$. Let $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$ be vectors taken from S^{m-1}. We say that x is *equivalent to* y if $x_k = 0 \Leftrightarrow y_k = 0$. We denote this relation by $x \sim y$.

Let $\mathrm{supp}(x) = \{i : x_i \neq 0\}$ be a support of $x \in S^{m-1}$. We say that x is *singular to* y and denote by $x \perp y$, if $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$. Note that if $x, y \in S^{m-1}$ then $x \perp y$ if and only if $(x, y) = 0$, where $(\cdot, \cdot)$ stands for a standard inner product in $\mathbb{R}^m$.

We denote sets of coupled indices by

$$\mathbf{P}_m = \{(i, j) : \ i < j\} \subset I \times I, \quad \Delta_m = \{(i, i) : i \in I\} \subset I \times I.$$

For a given pair $(i, j) \in \mathbf{P}_m \cup \Delta_m$, we set a vector $\mathbb{P}_{ij} = \left(P_{ij,1}, \cdots, P_{ij,m}\right)$. Clearly, because of condition (2.2), $\mathbb{P}_{ij} \in S^{m-1}$.

Let $\xi_1 = \{A_i\}_{i=1}^N$ and $\xi_2 = \{B_i\}_{i=1}^M$ be some fixed partitions of $\mathbf{P}_m$ and Δ_m, respectively, i.e. $A_i \bigcap A_j = \emptyset$, $B_i \bigcap B_j = \emptyset$, and $\bigcup_{i=1}^N A_i = \mathbf{P}_m$, $\bigcup_{i=1}^M B_i = \Delta_m$, where $N, M \leqslant m$.

Definition 2.3 ([24])**.** QSO $V : S^{m-1} \to S^{m-1}$ given by (2.1), (2.2), is called a $\xi^{(as)}$-QSO w.r.t. the partitions ξ_1, ξ_2 (where the "as" stands for absolutely continuous-singular) if the following conditions are satisfied:

(i) for each $k \in \{1, \ldots, N\}$ and any (i, j), $(u, v) \in A_k$, one has $\mathbb{P}_{ij} \sim \mathbb{P}_{uv}$;

(ii) for any $k \neq \ell$, $k, \ell \in \{1, \ldots, N\}$ and any $(i, j) \in A_k$ and $(u, v) \in A_\ell$ one has $\mathbb{P}_{ij} \perp \mathbb{P}_{uv}$;

(iii) for each $d \in \{1, \ldots, M\}$ and any (i, i), $(j, j) \in B_d$, one has $\mathbb{P}_{ii} \sim \mathbb{P}_{jj}$;

(iv) for any $s \neq h$, $s, h \in \{1, \ldots, M\}$ and any $(u, u) \in B_s$ and $(v, v) \in B_h$, one has that $\mathbb{P}_{uu} \perp \mathbb{P}_{vv}$.

Definition 2.4 ([3])**.** A fixed point $\bar{x}$ of the nonlinear system $x_{t+1} = \phi(x_t)$ is:

(i) globally (asymptotically) stable, if
$$\lim_{t\to\infty} x_t = \bar{x} \ \ \forall x_0 \in \mathbb{R}^n;$$

(ii) locally (asymptotically) stable, if there exist $\epsilon > 0$, such that
$$\lim_{t\to\infty} x_t = \bar{x}, \ \ \forall x_0 \in B_\epsilon(\bar{x}),$$
where
$$B_\epsilon(\bar{x}) = \{x \in \mathbb{R}^n : |x_i - \bar{x_i}| < \epsilon, \ \ \forall \ i = 1, 2, 3, \cdots, n\}.$$

In this paper, we are going to investigate the global behavior of $V_{\frac{1}{2}}$ and the local behavior of V_a which belongs to ℓ-Volterra-QSO.

3. Fixed points of V_a

In [24], it has been classified the $\xi^{(s)}$-QSO into 20 non-conjugate classes. In this paper we are going to continue study the behavior of these classes, one of them is $K_7 = \{V_7\}$, the operator V_7 which can be written by Tables 1 and 2.

Table 1: Coefficient of $\mathbf{P}_3$.

case	P_{12}	P_{13}	P_{23}
I_2	$(0, a, 1-a)$	$(0, a, 1-a)$	$(1, 0, 0)$

Table 2: Coefficient of Δ_3.

case	P_{11}	P_{22}	P_{33}
II_1	$(1, 0, 0)$	$(0, 1, 0)$	$(0, 0, 1)$

By combination of Tables 1 and 2, we get V_7. In this paper we will denote V_7 by V_a. Moreover, in this section we will find the fixed point of V_a,

$$V_a := \begin{cases} x' = x^2 + 2yz, \\ y' = y^2 + 2ax(1-x), \\ z' = z^2 + 2(1-a)x(1-x), \end{cases} \tag{3.1}$$

where $0 \leqslant a \leqslant 1$. This operator is an ℓ-Volterra-QSO. Let e_1, e_2, e_3 be the vertices of the simplex S^2.

Theorem 3.1. *Let $V_a : S^2 \to S^2$ be a $\xi^{(s)}$-QSO given by (3.1). Then*

$$\mathrm{Fix}(V_a) = \begin{cases} \{e_2, e_3\}, & \text{if } 0 \leqslant a \leqslant 1, \\ \{e_1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}, & \text{if } a = \frac{1}{2}, \\ \{(x^*, y^*, z^*)\}, & \text{if } a \in (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}), \end{cases}$$

where $x^* = \frac{16a^2-16a+3}{16a^2-16a+1}$, $y^* = \frac{1-4a}{16a^2-16a+1}$, *and* $z^* = 1 - x^* - y^*$.

Proof. To find the fixed points of (3.1), we should solve the following system of equations:

$$\begin{cases} x = x^2 + 2yz, \\ y = y^2 + 2ax(1-x), \\ z = z^2 + 2(1-a)x(1-x). \end{cases} \tag{3.2}$$

Now, we shall separately consider two cases. Namely, $a = \frac{1}{2}$ and $a \neq \frac{1}{2}$.

Let $a = \frac{1}{2}$. Then, by subtracting third equation from second equation in (3.2) we obtain $y - z = y^2 - z^2$. If $y \neq z$ then $y + z = 1$, therefore, $x = 0$. It follows that $y^2 = y$ and $z^2 = z$. By solving these quadratic equations we have a fixed points $\{e_2, e_3\}$. Consider now $y = z$. From the first equation of (3.2) and $x + y + z = 1$ we obtain $y = \frac{1}{2}(1-x)$. Hence, $x^2 + 2(\frac{1-x}{2})^2 = x$. Solving the last quadratic equation obtains $x \in \{1, \frac{1}{3}\}$. Thus we obtain fixed points $\{e_1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$.

Assume $a \neq \frac{1}{2}$. Replacing z by $1 - x - y$ in equation (1) of (3.2) we can write (1) as

$$x^2 + 2y - 2yx - 2y^2 - x = 0. \tag{3.3}$$

Rewriting the second equation of (3.2) gives

$$y^2 + 2ax - 2ax^2 - y = 0. \tag{3.4}$$

By adding (3.3) to (3.4), we obtain the following equation

$$x((1-4a)x + (4a-1) - 2y) = 0. \tag{3.5}$$

The solutions of equation (3.5) are $\{0, \frac{2y-4a+1}{1-4a}\}$. If $x = 0$, then it follows from equation (3.3) that $y^2 - y = 0$. Therefore, $y \in \{0, 1\}$. Then we have the fixed points $\{e_2, e_3\}$. If $x = \frac{2y-4a+1}{1-4a}$ then $y = \frac{1}{2}(x-1)(1-4a)$. By inserting this value in first equation of (3.2) we obtain

$$(x-1)(x+\frac{1}{2}(x-1)(1-4a)(-3+4a)) = 0.$$

So, $x \in \{1, \frac{16a^2-16a+3}{16a^2-16a+1}\}$. If $x = 1$, then we have the fixed point e_1. If $x = x^*$ then one obtains point $x^* = \frac{16a(a-1)+3}{16a(a-1)+1}, y^* = \frac{1-4a}{16a(a-1)+1}$ and $z^* = 1 - x^* - y^*$. We are now in the position to find out the conditions of the parameter a that makes the last fixed point lies in the simplex. To achieve this goal, we have to use the $0 < \frac{x+3}{x+1} < 1$ whenever $x \in [\frac{1}{3}, \frac{3}{4}]$

Therefore, if $a \in [\frac{1}{3}, \frac{3}{4}]$, then we obtain the fixed point (x^*, y^*, z^*). If $a = \frac{1}{3}$ one has the fixed point e_2, and if $a = \frac{1}{2}$, then we obtain the fixed point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The value of $a = \frac{3}{4}$ gives the fixed point e_3. This process completes the proof. □

4. Global behavior of $V_{\frac{1}{2}}$

In this section we will study the global behavior of V_a when $a = \frac{1}{2}$. Moreover, let us define the following regions:

$$\begin{aligned}
A_1 &:= \{(x,y,z) \in S^2 : 0 \leqslant x,y,z \leqslant \frac{1}{2}\},\\
A_2 &:= \{(x,y,z) \in S^2 : \frac{1}{2} \leqslant x < 1\},\\
A_3 &:= \{(x,y,z) \in S^2 : \frac{1}{2} \leqslant y < 1\},\\
B_1 &:= \{(x,y,z) \in S^2 :\ 0 \leqslant x,y,z \leqslant \frac{1}{2},\ \ x \leqslant z\},\\
B_2 &:= \{(x,y,z) \in S^2 :\ 0 \leqslant x,y,z \leqslant \frac{1}{2},\ \ z \leqslant x\},\\
B_3 &:= \{(x,y,z) \in S^2 :\ 0 \leqslant x,y,z \leqslant \frac{1}{2},\ \ y \leqslant x\},\\
B_4 &:= \{(x,y,z) \in S^2 :\ 0 \leqslant x,y,z \leqslant \frac{1}{2},\ \ \frac{1}{3} \leqslant x \leqslant z\},\\
B_5 &:= \{(x,y,z) \in S^2 :\ 0 \leqslant x,y,z \leqslant \frac{1}{2},\ \ 0 \leqslant z \leqslant x \leqslant \frac{1}{3} \leqslant y \leqslant \frac{1}{2}\}.
\end{aligned}$$

Proposition 4.1. *The following statements hold for $V_{\frac{1}{2}}$:*

(1) *the region A_1 is invariant;*
(2) *if $x^{(0)} \notin \mathrm{Fix}(V_{\frac{1}{2}})$, and $x^{(0)} \in A_2$, then, $V^{(n)}(x^{(0)})$ goes to $A_1 \cup A_3$;*
(3) *if $x^{(0)} \notin \mathrm{Fix}(V_{\frac{1}{2}})$, and $x^{(0)} \in A_3$, then, $V^{(n)}(x^{(0)})$ goes to A_1.*

Proof.
(1). Let $x^{(0)} = (x,y,z) \in A_1$, then $0 \leqslant x,y,z \leqslant \frac{1}{2}$. We want to show that $0 \leqslant x', y', z' \leqslant \frac{1}{2}$. To achieve this objective, we can easily show that $-1 \leqslant 3x-1 \leqslant \frac{1}{2}$ by squaring the last inequality, we obtain $0 \leqslant (3x-1)^2 \leqslant 1$. Given $(y-z)^2 \geqslant 0$, it follows that $(3x-1)^2 - 3(y-z)^2 \leqslant 1$. Thus, $9x^2 - 6x + 1 - 3(y-z)^2 \leqslant 1$. Adding 2 for both sides and dividing the resulting inequality by 3, one finds that

$$3x^2 - 2x + 1 - (y-z)^2 \leqslant 1.$$

Therefore,

$$2x^2 + (1-x)^2 - (y-z)^2 \leqslant 1.$$

Replacing $(1-x)$ by $(y+z)$, we have

$$2x^2 + (y+z)^2 - (y-z)^2 \leqslant 1,$$

then $2x^2 + 4yz \leqslant 1$, which implies that

$$x' = x^{(2)} + 2yz \leqslant \frac{1}{2}.$$

Now, let us prove that if $0 \leqslant y \leqslant \frac{1}{2}$ then $0 \leqslant y' \leqslant \frac{1}{2}$.
Consider $y' = y^2 + x - x^2$. Given that $0 \leqslant y \leqslant \frac{1}{2}$, $0 \leqslant y^2 \leqslant \frac{1}{4}$, the following is obtained:

$$0 \leqslant x - x^2 \leqslant \frac{1}{4}.$$

Therefore,

$$0 \leqslant y' = y^2 + x - x^2 \leqslant \frac{1}{2}.$$

Hence, $0 \leqslant y' \leqslant \frac{1}{2}$.

Performing the same process for z' we have $0 \leqslant z' \leqslant \frac{1}{2}$, thus, A_1 is an invariant region.

(2). By contrast, suppose that A_2 is invariant, which means that if $\frac{1}{2} \leqslant x \leqslant 1$, then $\frac{1}{2} \leqslant x^{(n)} \leqslant 1$. However,

$$x' = x^2 + 2yz \leqslant x^2 + 2xz = x(x+2z),$$

then $\frac{x'}{x} \leqslant 1$. Therefore, $x^{(n)}$ decreasing sequence and bounded. Then $x^{(n)}$ converges and should go to a fixed point which means $\frac{1}{2} \leqslant x^{(n)} \leqslant 1$. However, this region does not have any fixed point, which contradicts our assumption. Hence, there is $n_k \in \mathbb{N}$ such that $x^{(n_k)}$ goes to the invariant region $A_1 \cup A_3$, and never come back to A_2.

(3). This is proved by using the same process that was used to prove (2). Suppose that A_3 is invariant. Then, it follows that if $\frac{1}{2} \leqslant y \leqslant 1$. Then $\frac{1}{2} \leqslant y' \leqslant 1$. It is easy to see that $y > x$, then one can easily check that the sequence $\{y^{(n)}\}$ is decreasing and bounded. Hence, $\{y^{(n)}\}$ will go to a fixed point, but no fixed point has this property in this region. Therefore, in both cases there is $n_k \in \mathbb{N}$ such that $x^{(n_k)}$ tends to the invariant region A_1, and this finding is precisely the assertion of the proposition. □

Proposition 4.2. *The following statements hold true for* $V_{\frac{1}{2}}$*:*

(i) *if* $x^{(0)} \in B_1$, *then there exists* $n_k \in \mathbb{N}$ *such that* $V^{(n_k)}(x) \in B_2$. *Moreover,* B_2 *is invariant;*
(ii) *if* $x^{(0)} \in B_3$, *then there exists* $n_k \in \mathbb{N}$ *such that* $V^{(n_k)}(x) \in B_4$. *Moreover,* $B_4 \cup B_5$ *is invariant;*
(iii) *if* $x^{(0)} \in B_4$, *then there exists* $n_k \in \mathbb{N}$ *such that* $V^{(n_k)}(x) \in B_5$*;*
(iv) B_5 *is invariant.*

Proof.

(i). By contrast, suppose that B_1 is invariant, which means that if $x \leqslant z$ then $x^{(n)} \leqslant z^{(n)}$. However,

$$y^2 - y + x - x^2 \leqslant 0.$$

Therefore, the sequence $\{y^{(n)}\}$ is decreasing and bounded. Thus, it should go to fixed point, but $\frac{1}{3} \leqslant y^{(n)} \leqslant \frac{1}{2}$. Hence, $y^{(n)} \to \frac{1}{3}$. Moreover, from Proposition 4.1, we have $y^{(n)} - z^{(n)}$ is decreasing sequence and bounded, then it will go to a fixed point. Therefore $y^{(n)} - z^{(n)} \to \frac{1}{3}$. Hence, $z^{(n)} \to 0$, which implies that $x^{(n)} \to \frac{2}{3}$. But $(\frac{2}{3}, \frac{1}{3}, 0)$ is not a fixed point, which follows that n_k exists, such that $V^{(n_k)}(x)$ goes to $B_2 \cup B_3 \subset B_2$.

Now, let us consider $x' - z' = x^2 + 2yz - z^2 - x + x^2$ and $x' - z' = 2x^2 + 2yz - z^2 - x$. One can easily check that the minimum value of $x' - z'$ occurs when $x^{(0)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the minimum value is zero. Therefore $x' > z'$ whenever $x > z$. Thus, B_2 is invariant.

(ii). Suppose the assertion is false. Then B_3 is invariant, and it follows that

$$0 \leqslant z^{(n)} \leqslant y^{(n)} \leqslant x^{(n)} \leqslant \frac{1}{2}.$$

Clearly, the sequence $\{z^{(n)}\}$ is decreasing and bounded in this region. Hence, the sequence is convergent and goes to a fixed point. However, no fixed point has this property in this region, which contradicts our expected findings. Therefore, there exists $n_k \in \mathbb{N}$ such that $V^{(n_k)}(x) \to B_4$. Now, we are going to show that $B_4 \cup B_5$ is invariant. It follows from finding the minimum value of $f(x, y) = y' - x' = y^2 + x - x^2 - x^2 - 2y(1 - x - y)$ on $B_4 \cup B_5$ that the minimum value of f is zero. Hence $y^{(n)} > x^{(n)}$, and $B_4 \cup B_5$ is invariant.

(iii). Suppose contrary to our claim that B_4 is invariant then $x^{(n)} \geqslant \frac{1}{2}$ for all $n \in \mathbb{N}$. Thus, the sequence $\{z^{(n)}\}$ is decreasing and bounded, therefore, it is convergent and goes to fixed point 0. Clearly, the sequence $\{y^{(n)}\}$ is a bounded and increasing sequence. Hence, $y^{(n)} \to \frac{1}{3}$, which means that $x^{(n)} \to \frac{2}{3}$ but $(\frac{2}{3}, \frac{1}{3}, 0)$ is not a fixed point. Thus, $n_k \in N$ exists such that $V^{(n_k)}(x)$ goes to B_5.

(iv). Let $x \leqslant \frac{1}{3}$. Then, $x' = x^2 + 2yz$ has a maximum value when $x^{(0)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Therefore, $x' \leqslant \frac{1}{3}$, it follows that $x \leqslant \frac{1}{3}$. Thus, $x^{(n)} \leqslant \frac{1}{3}$. Using $x \geqslant z$ we have $x^{(n)} \geqslant z^{(n)} \geqslant 0$ because $y^{(n)} + x^{(n)} + z^{(n)} = 1$, then we obtain $y^{(n)} \geqslant \frac{1}{3} \geqslant x^{(n)}$ and we conclude that B_5 is invariant, this completes the proof. □

Theorem 4.3. *Let* $x^{(0)} = (x, y, z) \notin \mathrm{Fix}(V_{\frac{1}{2}})$, *then* $\omega_{V_{\frac{1}{2}}}(x^{(0)}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Proof. From Propositions 4.1 and 4.2, we only need to study the behavior of $V_{\frac{1}{2}}$ over B_5. We know that $y' = y^2 + x(1 - x)$. Then, we obtain $y' \leqslant y(y + 2x)$. Therefore, the sequence $\{y^{(n)}\}$ is bounded and decreasing; thus it should go to a fixed point. Hence, we have $y^{(n)} \to \frac{1}{3}$. However, the sequence $|y^{(n)} - z^{(n)}| \to 0$. Hence, $z^{(n)} \to \frac{1}{3}$. Using $x^{(n)} = 1 - y^{(n)} - z^{(n)}$ we obtain $x^{(n)} \to \frac{1}{3}$, which is the desired conclusion. □

5. Local behavior of V_a

In this section, we will study the local behavior of V_a around each fixed point through linearization around each fixed point of V_a where $0 < a < 1$.

Following [3], suppose that the dynamical system has a fixed point $\bar{x} = (x^*, y^*, z^*)$, namely, there exists $\bar{x} \in S^2$ such that $\bar{x} = \phi(\bar{x})$. A Taylor expansion of $x_{it+1} = \phi(x_t)$ around the fixed point $\bar{x}$, obtains

$$x_{it+1} = \phi^i(x_t) = \phi^i(\bar{x}) + \Sigma_{j=1}^{n} \phi_j^i(\bar{x})(x_j t - \bar{x_j}) + \cdots + R_n,$$

where $\phi_j^i(\bar{x})$ is the partial derivatives of the function $\phi^i(x_t)$ with respect to x_{jt} evaluated at $\bar{x}$, i.e, $\phi_j^i(\bar{x}) = \frac{\partial \phi^i(x_t)}{\partial x_{it}}$. Thus, the linearized equation around the fixed point $\bar{x}$ is given by

$$x_{it+1} = \phi_1^i(\bar{x}) x_{1t} + \phi_2^i(\bar{x}) x_{2t} + \cdots + \phi_n^i(\bar{x}) x_{nt} + \phi^i(\bar{x}) - \Sigma_{j=1}^{n} \phi_j^i(\bar{x}) \bar{x_j}.$$

The linearized system, is therefore:

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \\ \vdots \\ x_{nt+1} \end{bmatrix} = \begin{bmatrix} \phi_1^1(\bar{x}) & \phi_2^1(\bar{x}) & \cdots & \phi_n^1(\bar{x}) \\ \phi_1^2(\bar{x}) & \phi_2^2(\bar{x}) & \cdots & \phi_n^2(\bar{x}) \\ \vdots & & & \\ \phi_1^n(\bar{x}) & \phi_2^n(\bar{x}) & \cdots & \phi_n^n(\bar{x}) \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{bmatrix} + \begin{bmatrix} \phi^1(\bar{x}) - \sum_{j=1}^{n} \phi_j^1(\bar{x}) \bar{x_j} \\ \phi^2(\bar{x}) - \sum_{j=1}^{n} \phi_j^2(\bar{x}) \bar{x_j} \\ \vdots \\ \phi^n(\bar{x}) - \sum_{j=1}^{n} \phi_j^n(\bar{x}) \bar{x_j} \end{bmatrix}.$$

Thus, the nonlinear system has been approximated locally by a linear system

$$x_{t+1} = Ax_t + B,$$

where

$$A \equiv \begin{bmatrix} \phi_1^1(\bar{x}) & \cdots & \phi_n^1(\bar{x}) \\ \phi_1^2(\bar{x}) & \cdots & \phi_n^2(\bar{x}) \\ \vdots & \ddots & \\ \phi_1^n(\bar{x}) & \cdots & \phi_n^n(\bar{x}) \end{bmatrix} \equiv D\phi(\bar{x})$$

is the Jacobian matrix of $\phi(x_t)$ evaluated at $\bar{x}$, and

$$B \equiv \begin{bmatrix} \phi^1(\bar{x}) - \sum_{j=1}^n \phi_j^1(\bar{x})\bar{x_j} \\ \phi^2(\bar{x}) - \sum_{j=1}^n \phi_j^2(\bar{x})\bar{x_j} \\ \vdots \\ \phi^n(\bar{x}) - \sum_{j=1}^n \phi_j^n(\bar{x})\bar{x_j} \end{bmatrix}.$$

Theorem 5.1. *Let $V_a : S^2 \to S^2$ be a $\xi^{(s)}$-QSO given by (3.2). Then the following statements hold true.*

(i) *The local behavior of V_a is stable around the following fixed points*

$$\begin{cases} (\frac{1}{3}\frac{1}{3}, \frac{1}{3}) & : \ a = \frac{1}{2}, \\ (x^*, y^*, z^*) & : \ a \in (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}), \\ \{e_3\} & : \ a \in (0, \frac{1}{4}], \\ \{e_2\} & : \ a \in [\frac{3}{4}, 1). \end{cases}$$

(ii) *If $a = \frac{1}{2}$, then, the local behavior V_a is unstable around $\{e_1, e_2, e_3\}$.*

(iii) *If $a \in (0, \frac{1}{4})$, then, the local behavior V_a is unstable around $\{e_2\}$.*

(iv) *If $a \in (\frac{3}{4}, 1)$, then, the local behavior V_a is unstable around $\{e_3\}$.*

Proof.

(i). Let $a = \frac{1}{2}$. Then, we linearize V_a around the fixed points when $a = \frac{1}{2}$. Thus the linearization of $V_{\frac{1}{2}}$ around the fixed point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ has the following form:

$$\begin{cases} x' = \frac{2}{3}x + \frac{2}{3}y + \frac{2}{3}z - \frac{1}{3}, \\ y' = \frac{1}{3}x + \frac{2}{3}y, \\ z' = \frac{1}{3}x + \frac{2}{3}z. \end{cases} \tag{5.1}$$

We can rewrite first equation of system (5.1) as $\frac{2}{3}(x+y+z) - \frac{1}{3} = \frac{1}{3}$. It follows that $x^{(n)}$ goes to $\frac{1}{3}$. Therefore, by induction of equation (2) in (5.1) we obtain $y^{(n)} = (\frac{2}{3})^n y + \frac{1}{9}\sum_{k=1}^n (\frac{2}{3})^{k-1}$. Thus, the limit of the last expression is taking into account that the sum of convergent geometric series is $\frac{1}{3}$. Hence $\lim_{n\to\infty} y^{(n)} = \frac{1}{3}$. Consequently, $z^{(n)} = 1 - x^{(n)} + y^{(n)} = \frac{1}{3}$. Hence, $V_{\frac{1}{2}}$ is stable around $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Moreover, let $a \in (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$. Then, we can rewrite the linearization of V_a around (x^*, y^*, z^*) as follows:

$$\begin{cases} x' = (2x^* - 2y^*)x + (2z^* - 2y^*)y + 2y^* + x^* - 2(x^*)^2 - 4y^*x^*, \\ y' = 2a(1-2x^*)x + 2y^*y + y^* - 2a(1-2x^*) - 2(y^*)^2. \end{cases}$$

Let us define the following matrix

$$\tilde{A} = \begin{pmatrix} 2x^* - 2y^* & 2z^* - 2y^* \\ 2a(1-2x^*) & 2y^* \end{pmatrix},$$

by using $\tilde{A} = QDQ^{-1}$ where D is a diagonal matrix given as follows:

$$\tilde{A} = \begin{pmatrix} f(a) & 0 \\ 0 & g(a) \end{pmatrix},$$

where $f(a) = \frac{16a^2-16a+3+\sqrt{1+64a-320a^2+512a^3-256a^4}}{16a^2-16a+1}$ and $g(a) = \frac{-3+16a-16a^2+\sqrt{1+64a-320a^2+512a^3-256a^4}}{16a^2-16a+1}$. One can easily check $0 < |f(a)| < 1$ and $0 < |g(a)| < 1$. However,

$$x^{(n)} = QD^nQ^{-1}(x-\bar{x}) + \bar{x}.$$

Then, $\lim_{n\to\infty} x^{(n)} = \bar{x}$ where $\bar{x} = (x^*, y^*, z^*)$. Therefore, V_a is stable around (x^*, y^*, z^*).

We now look at the stability of V_a around e_2 and e_3. V_a is linearized around e_3. Then we have

$$\begin{cases} x' = 2y, \\ y' = 2ax, \\ z' = 2(1-a)x + 2z - 1. \end{cases} \tag{5.2}$$

After induction of the first equation of (5.2), we obtain the sub-sequences $x^{(2n)} = (4a)^n x$, $x^{(2n+1)} = (4a)^n 2y$, $y^{(2n)} = (4a)^n y$ and $y^{(2n-1)} = (4a)^n \frac{x}{2}$. It is easy to see that the sequences are convergent when $a \in \left(0, \frac{1}{4}\right]$ and go to zero, so, in general x^n, y^n converge to zero, therefore, z^n converges to one. Hence, the behavior of V_a around e_3 is stable when $a \in \left(0, \frac{1}{4}\right]$. Finally, the linearization of V_a around e_2 has the following form:

$$\begin{cases} x' = 2z, \\ y' = 2ax + 2y - 1, \\ z' = 2(1-a)x. \end{cases} \tag{5.3}$$

From the induction of the first and third equations of the system (5.3) we obtain the following sub-sequences: $x^{(2n)} = 2^{(2n)}(1-a)^n x$, $x^{(2n+1)} = 2^{(2n+1)}(1-a)^n z$, $z^{(2n)} = 2^{(2n)}(1-a)z$ and $z^{(2n-1)} = 2^{(2n-1)}(1-a)x$. One can easily show that if $a \in \left[\frac{3}{4}, 1\right)$, then the sequences x^n, z^n are convergent and go to zero. Therefore, the sequence z^n is convergent and goes to one.

The sequences are clearly divergent when $a \in (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$. Hence, the behavior of V_a around e_3 is unstable when $a \in (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$.

(ii). Now, let us look at the stability of $V_{\frac{1}{2}}$ around e_1. The linearization of $V_{\frac{1}{2}}$ takes the following form

$$\begin{cases} x' = 2x - 1, \\ y' = -x + 1, \\ z' = -x + 1. \end{cases} \tag{5.4}$$

On the basis of second and third equations of system (5.4), we obtain $z^{(n)} = y^{(n)} = 2^{n-1}(1-x)$, which is a divergent sequence. Therefore, $V_{\frac{1}{2}}$ is unstable around e_1. We now study the local behavior of $V_{\frac{1}{2}}$ around e_2. The linearization of $V_{\frac{1}{2}}$ around e_2 is

$$\begin{cases} x' = 2x - 1, \\ y' = -x + 1, \\ z' = -x + 1. \end{cases}$$

Here, we consider two subsequences $x^{(2n)} = 2^n x$ and $x^{(2n-1)} = 2^n x$, which are both divergent. Thus, $V_{\frac{1}{2}}$ is unstable around e_2. In the same manner, we can see that $V_{\frac{1}{2}}$ is unstable around e_3.

(iii). From proof of (i), we can easily see that the induction of linearization around e_2 gives the following subsequences: $x^{(2n)} = 2^{(2n)}(1-a)^n x$, $x^{(2n+1)} = 2^{(2n+1)}(1-a)^n z$, $z^{(2n)} = 2^{(2n)}(1-a)z$, and $z^{(2n-1)} = 2^{(2n-1)}(1-a)x$. Clearly if $a \in \left(0, \frac{1}{4}\right)$, then x^n and z^n are divergent. Hence, the local behavior of V_a is unstable around e_2.

(iv). From proof of (i), we can easily see that the induction of linearization around e_3 gives the following subsequences: $x^{(2n)} = (4a)^n x$, $x^{(2n+1)} = (4a)^n 2y$, $y^{(2n)} = (4a)^n y$ and $y^{(2n-1)} = (4a)^n \frac{x}{2}$. Clearly if $a \in \left(\frac{3}{4}, 1\right)$, then x^n and y^n are divergent. Hence, the local behavior of V_a is unstable around e_3. □

Acknowledgment

The authors are grateful to Prof. Farrukh Mukhamedov for his valuable comments and remarks, that improved the presentation of this paper.

References

[1] S. Bernstein, *Solution of a mathematical problem connected with the theory of heredity*, Ann. Math. Statistics, **13** (1942), 53–61. 1
[2] A. Dohtani, *Occurrence of chaos in higher-dimensional discrete-time systems*, SIAM J. Appl. Math., **52** (1992), 1707–1721. 2
[3] O. Galor, *Discrete dynamical systems*, Springer, Berlin, (2007). 2.4, 5
[4] R. N. Ganikhodzhaev, *A family of quadratic stochastic operators that act in* S^2, (Russian); Dokl. Akad. Nauk UzSSR, **1** (1989), 3–5. 1
[5] R. N. Ganikhodzhaev, *Quadratic stochastic operators, Lyapunov functions and tournaments*, (Russian); translated from Mat. Sb., **183** (1992), 119–140, Russian Acad. Sci. Sb. Math., **76** (1992), 489–506. 1, 2, 2.2
[6] R. N. Ganikhodzhaev, *A chart of fixed points and Lyapunov functions for a class of discrete dynamical systems*, (Russian); translated from Mat. Zametki, **56** (1994), 40–49, Math. Notes, **56** (1994), 1125–1131. 1, 2
[7] N. N. Ganikhodzhaev, *An application of the theory of Gibbs distributions to mathematical genetics*, Doklady Math., **61** (2000), 321–323. 1
[8] R. N. Ganikhodzhaev, R. É. Abdirakhmanova, *Description of quadratic automorphisms of a finite-dimensional simplex*, (Russian); Uzbek. Mat. Zh., **1** (2002), 7–16. 1
[9] R. N. Ganikhodzhaev, A. M. Dzhurabaev, *The set of equilibrium states of quadratic stochastic operators of type* V_π, (Russian); Uzbek. Mat. Zh., **3** (1998), 23–27. 1
[10] R. N. Ganikhodzhaev, D. B. Éshmamatova, *Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories*, (Russian); Vladikavkaz. Mat. Zh., **8** (2006), 12–28. 1, 2
[11] R. Ganikhodzhaev, F. Mukhamedov, U. Rozikov, *Quadratic stochastic operators and processes: results and open problems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **14** (2011), 270–335. 1
[12] N. N. Ganikhodzhaev, R. T. Mukhitdinov, *On a class of measures corresponding to quadratic operators*, (Russian); Dokl. Akad. Nauk Rep. Uzb., **3** (1995), 3–6. 1
[13] N. N. Ganikhodzhaev, U. A. Rozikov, *On quadratic stochastic operators generated by Gibbs distributions*, Regul. Chaotic Dyn., **11** (2006), 467–473. 1
[14] J. Hofbauer, V. Hutson, W. Jansen, *Coexistence for systems governed by difference equations of Lotka-Volterra type*, J. Math. Biol., **25** (1987), 553–570. 1, 2
[15] J. Hofbauer, K. Sigmund, *The theory of evolution and dynamical systems*, Mathematical aspects of selection, Translated from the German, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, (1988). 1, 2
[16] R. D. Jenks, *Quadratic differential systems for interactive population models*, J. Differential Equations, **5** (1969), 497–514. 1, 2
[17] H. Kesten, *Quadratic transformations: A model for population growth*, II, Advances in Appl. Probability, **2** (1970), 179–228. 1
[18] S.-T. Li, D.-M. Li, G.-K. Qu, *On stability and chaos of discrete population model for a single-species with harvesting*, J. Harbin Univ. Sci. Tech., **6** (2006), 021. 1
[19] A. J. Lotka, *Undamped oscillations derived from the law of mass action*, J. Amer. Chem. Soc., **42** (1920), 1595–1599. 2
[20] Y. I. Lyubich, *Mathematical structures in population genetics*, Translated from the 1983 Russian original by D. Vulis and A. Karpov, Biomathematics, Springer-Verlag, Berlin, (1992). 1
[21] F. Mukhamedov, A. H. M. Jamal, *On* ξ^s*-quadratic stochastic operators in 2-dimensional simplex*, Proc. of the 6th IMT-GT Inter. Conf. on Math., Stat. and Its Appl., Universiti Tunku Abdul Rahman, Kuala Lumpur, Malaysia, (2010), 159–172. 1
[22] F. Mukhamedov, M. Saburov, A. H. M. Jamal, *On dynamics of* ξ^s*-quadratic stochastic operators*, Inter. Jour. Modern Phys., Conf. Ser., **9** (2012), 299–307. 1
[23] F. Mukhamedov, M. Saburov, I. Qaralleh, *Classification of* $\xi^(s)$*-quadratic stochastic operators on 2D simplex*, J. Phys., Conf. Ser., **435** (2013), 8 pages.
[24] F. Mukhamedov, M. Saburov, I. Qaralleh, *On* $\xi^{(s)}$*-quadratic stochastic operators on two-dimensional simplex and their behavior*, Abstr. Appl. Anal., **2013** (2013), 12 pages. 1, 2, 2.3, 3
[25] M. Plank, *Hamiltonian structures for the* n*-dimensional Lotka-Volterra equations*, J. Math. Phys., **36** (1995), 3520–3543. 1, 2
[26] U. A. Rozikov, N. B. Shamsiddinov, *On non-Volterra quadratic stochastic operators generated by a product measure*, Stoch. Anal. Appl., **27** (2009), 353–362. 1
[27] U. A. Rozikov, A. Zada, *On dynamics of* ℓ*-Volterra quadratic stochastic operators*, Int. J. Biomath., **3** (2010), 143–159. 1, 2, 2.2

[28] U. A. Rozikov, A. Zada, *ℓ-Volterra quadratic stochastic operators: Lyapunov functions, trajectories*, Appl. Math. Inf. Sci., **6** (2012), 329–335. 1, 2

[29] U. A. Rozikov, U. U. Zhamilov, F-*quadratic stochastic operators*, (Russian); translated from Mat. Zametki, **83** (2008), 606–612, Math. Notes, **83** (2008), 554–559. 1

[30] M. Saburov, *Some strange properties of quadratic stochastic Volterra operators*, World Appl. Sci. J., **21** (2013), 94–97. 1

[31] P. R. Stein, S. M. Ulam, *Non-linear transformation studies on electronic computers*, Rozprawy Mat., **39** (1964), 66 pages. 1

[32] F. E. Udwadia, N. Raju, *Some global properties of a pair of coupled maps: quasi-symmetry, periodicity, and synchronicity*, Phys. D, **111** (1998), 16–26. 1

[33] S. M. Ulam, *Problems in modern mathematics*, Science Editions John Wiley & Sons, Inc., New York, (1964). 1, 2

[34] V. Volterra, *Lois de fluctuation de la population de plusieurs espèces coexistant dans le même milieu*, Association Franc. Lyon., **1926** (1927), 96–98. 1, 2

[35] M. I. Zaharevič, *The behavior of trajectories and the ergodic hypothesis for quadratic mappings of a simplex*, (Russian); Uspekhi Mat. Nauk, **33** (1978), 207–208. 1

[36] U. U. Zhamilov, U. A. Rozikov, *On the dynamics of strictly non-Volterra quadratic stochastic operators on a two-dimensional simplex*, (Russian); translated from Mat. Sb., **200** (2009), 81–94, Sb. Math., **200** (2009), 1339–1351. 1