Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 4866–4875
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Fixed point results for multivalued mappings in G_b-cone metric spaces

Abdullah Eqal Al-Mazrooei, Jamshaid Ahmad*

Department of Mathematics, University of Jeddah, P. O. Box 80327, Jeddah 21589, Saudi Arabia.

Communicated by C. Zaharia

Abstract

The aim of this paper is to introduce the notion of generalized Hausdorff distance function on G_b-cone metric spaces and exploit it to study some fixed point results in the setting of G_b-cone metric spaces without the assumption of normality. These results improve and generalize some important known results. Some illustrative examples are also furnished to highlight the realized improvements.

Keywords: Non normal cone, fixed points, G_b-cone metric space.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

The theory of multi-valued mappings is a branch of mathematics which has received a great attention in the last decades and has various applications in convex optimization, optimal control theory, and differential inclusions. Therefore, it lies at the junction of topology, theory of functions, and nonlinear functional analysis. Nadler [32] was the author who first time introduced the notion of multivalued contraction and established some fixed point theorems. One of real generalization of Nadler's theorem was given by Mizoguchi and Takahashi in this way.

Theorem 1.1 ([29]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be multivalued mapping. If there exists a function $\varphi : (0, \infty) \to [0, 1)$ such that*

$$\limsup_{r \to t^+} \varphi(r) < 1$$

for all $t \in [0, \infty)$ and if

$$H(Tx, Ty) \leqslant \varphi(d(x, y))(d(x, y))$$

for all $x, y \in X$, then T has a fixed point in X.

*Corresponding author
Email addresses: aealmazrooei@uj.edu.sa (Abdullah Eqal Al-Mazrooei), jkhan@uj.edu.sa, jamshaid_jasim@yahoo.com (Jamshaid Ahmad)

doi:10.22436/jnsa.010.09.27

Received 2016-10-21

On the other hand, Huang and Zhang [16] introduced the concept of cone metric space with normal cone with constant K, which is generalization of metric space. After that Rezapour and Hamlbarani [34] generalized cone metric space with non-normal cone. Afterwords many researchers studied fixed point results in cone metric spaces.

Mustafa et al. in [31] generalized the metric space and introduced the notion of G-metric space which recovered the flaws of Dhage's generalization [12, 13] of metric space. In 2010, Beg et al. [9] combined G-metric space and cone metric space and gave the notion of G-cone metric space to obtain some fixed point theorems on this space.

Azam et al. [8] utilized the concept of G-cone metric space and established a fixed point theorem for multivalued mapping using the concept of generalized Hausdorff distance function which was first given by Cho and Bae [11] in 2011. Very recently, Ughade et al. [39] introduced the concept of G_b-cone metric space as a generalization of G-cone metric space and obtained some fixed point results. For more details, we refer the reader to [1–41]. In this paper, we introduce the concept of generalized Hausdorff distance function in G_b-cone metric spaces and present some fixed point theorems for multivalued mappings.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, non empty and $P \neq \{\theta\}$;
(b) $a, b \in R, a, b \geqslant 0, x, y \in P$ implies $ax + by \in P$, more generally if $a, b, c \in R$, $a, b, c \geqslant 0, x, y, z \in P$ then $ax + by + cz \in P$;
(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$.

A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preccurlyeq x \preccurlyeq y$ implies $\|x\| \leqslant K \|y\|$.

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in \text{int}(P)$ (interior of P). While $x \prec y$ means $x \preccurlyeq y$ and $x \neq y$.

Rezapour et al. [34] proved that there are no normal cones with normal constants $K < 1$ and for each $k > 1$ there are cones with normal constants $K > 1$.

Remark 1.2 ([22]). The results concerning fixed points and other results, in case of cone spaces with non-normal solid cones, cannot be provided by reducing to metric spaces, because in this case neither of the conditions of the lemmas 1-4 in [16] hold. Further, the vector cone metric is not continuous in the general case, i.e., from $x_n \to x$, $y_n \to y$ it need not follow that $d(x_n, y_n) \to d(x, y)$.

For the case of non-normal cones we have the following properties.

(PT1) If $u \preccurlyeq v$ and $v \ll w$, then $u \ll w$.
(PT2) If $u \ll v$ and $v \preccurlyeq w$, then $u \ll w$.
(PT3) If $u \ll v$ and $v \ll w$, then $u \ll w$.
(PT4) If $\theta \preccurlyeq u \ll c$ for each $c \in \text{int}(P)$, then $u = \theta$.
(PT5) If $a \preccurlyeq b + c$ for each $c \in \text{int}(P)$, then $a \preccurlyeq b$.
(PT6) If E is a real Banach space with a cone P, and if $a \preccurlyeq \lambda a$, where $a \in P$ and $0 \leqslant \lambda < 1$, then $a = \theta$.
(PT7) If $c \in \text{int}(P)$, $a_n \in \mathbb{E}$ and $a_n \to \theta$, then there exists an n_0 such that, for all $n > n_0$, we have $a_n \ll c$.

In the following we shall always assume that the cone P is solid and non-normal.

Definition 1.3 ([39]). Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preccurlyeq$ with respect to the cone P. A vector-valued function $G : X \times X \times X \to E$ is said to be a generalized cone b-metric function on X with the constant $r \geqslant 1$ if the following conditions are satisfied:

(G1) $G(x, y, z) = \theta$ if $x = y = z$;
(G2) $\theta \prec G(x, x, y)$, whenever $x \neq y$, for all $x, y \in X$;
(G3) $G(x, x, y) \preccurlyeq G(x, y, z)$, whenever $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \cdots$ (symmetric in all three variables);

(G5) $G(x, y, z) \preccurlyeq r(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$.

Then (X, G) is called a generalized cone b-metric space or more specifically a G_b-cone metric space.

The concept of a G_b-cone metric space is more general than that of a G_b-metric spaces and cone metric spaces.

Definition 1.4 ([39]). A G_b-cone metric space X is symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.5 ([39]). *Let X be a G_b-cone metric space, define $d_{G_b} : X \times X \to E$ by*

$$d_{G_b}(x, y) = G(x, y, y) + G(y, x, x).$$

Then (X, d_{G_b}) is a cone b-metric space.

It can be noted that $G(x, y, y) \preccurlyeq \frac{2r}{2r+1} d_{G_b}(x, y)$. If X is a symmetric G_b-cone metric space, then $d_{G_b}(x, y) = 2G(x, y, y)$ for all $x, y \in X$.

Definition 1.6 ([39]). Let X be a G_b-cone metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

(1) Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is N such that for all $n, m, l > N$, $G(x_n, x_m, x_l) \ll c$;

(2) convergent sequence if for every c in E with $\theta \ll c$, there is N such that for all $m, n > N$, $G(x_m, x_n, x) \ll c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.7 ([39]). A G_b-cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Proposition 1.8. *Let X be a G_b-cone metric space, then the following are equivalent:*

(i) *$\{x_n\}$ converges to x.*
(ii) *$G(x_n, x_n, x) \to \theta$, as $n \to \infty$.*
(iii) *$G(x_n, x, x) \to \theta$, as $n \to \infty$.*
(iv) *$G(x_m, x_n, x) \to \theta$, as $m, n \to \infty$.*

Lemma 1.9 ([39]). *Let $\{x_n\}$ be a sequence in a G_b-cone metric space X and if $\{x_n\}$ converges to $x \in X$, then $G(x_m, x_n, x) \to \theta$ as $m, n \to \infty$.*

Lemma 1.10 ([39]). *Let $\{x_n\}$ be a sequence in a G_b-cone metric space X and $x \in X$. If $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.11 ([39]). *Let $\{x_n\}$ be a sequence in a G_b-cone metric space X and if $\{x_n\}$ is a Cauchy sequence in X, then $G(x_m, x_n, x_l) \to \theta$, as $m, n, l \to \infty$.*

2. Main results

Denote $N(X)$, $C(X)$, $B(X)$, and $CB(X)$ the set of nonempty, closed, bounded, sequentially closed bounded subset of a G_b-cone metric space, respectively.

Let (X, G) be a G_b-cone metric space, we denote (see [8, 11])

$$s(p) = \{q \in E : p \preccurlyeq q\} \text{ for } q \in E,$$

and

$$s(a, B) = \bigcup_{b\in B} s(d_{G_b}(a, b)) = \bigcup_{b\in B} \{x \in E : d_{G_b}(a, b) \preccurlyeq x\}$$

for $a \in X$ and $B \in N(X)$.

For $A, B \in B(X)$ we denote

$$s(A,B) = \bigcup_{a\in A, b\in B} s(d_{G_b}(a,b)),$$
$$s(a,B,C) = s(a,B) + \hat{s}(B,C) + s(a,C) = \{u+v+w : u \in s(a,B), v \in \hat{s}(B,C), w \in s(a,C)\},$$

and

$$s(A,B,C) = \left(\bigcap_{a\in A} s(a,B,C)\right) \cap \left(\bigcap_{b\in B} s(b,A,C)\right) \cap \left(\bigcap_{c\in C} s(c,A,B)\right).$$

Lemma 2.1. *Let (X,G) be a G_b-cone metric space, and P be a cone in Banach space E.*

(i) *Let $p, q \in E$. If $p \preccurlyeq q$, then $s(q) \subset s(p)$.*
(ii) *Let $x \in X$ and $A \in N(X)$. If $0 \in s(x,A)$, then $x \in A$.*
(iii) *Let $q \in P$ and let $A, B, C \in B(X)$ and $a \in A$. If $q \in s(A,B,C)$, then $q \in s(a,B,C)$.*

Remark 2.2. Recently, Kaewcharoen and Kaewkhao [25] (see also [38]) introduced the following concepts. Let X be a G-metric space and $CB(X)$ the family of all nonempty closed bounded subsets of X. Let $H(.,.,.)$ be the Hausdorff G-distance on $CB(X)$, i.e.,

$$H_G(A,B,C) = \max\{\sup_{a\in A} G(a,B,C), \sup_{b\in B} G(b,A,C), \sup_{c\in C} G(c,A,B)\},$$
$$H_{d_G}(A,B) = \max\{\sup_{a\in A} d_G(a,B), \sup_{b\in B} d_G(b,A)\},$$

where

$$G(x,B,C) = d_G(x,B) + d_G(B,C) + d_G(x,C),$$
$$d_G(x,B) = \inf\{d_G(x,y), y \in B\},$$
$$d_G(A,B) = \inf\{d_G(a,b), a \in A, b \in B\},$$
$$G(a,b,C) = \inf\{G(a,b,c),\ c \in C\}.$$

The above expressions show a relation between H_G and H_{d_G}. Moreover, note that if (X,G) is a G_b-cone metric space, $E = R$, and $P = [0,\infty)$, then (X,G) is a G_b-metric space. Also for $A, B, C \in CB(X)$, $H_G(A,B,C) = \inf s(A,B,C)$.

Remark 2.3. Let (X,G) be a G_b-cone metric space. Then

(i) $s(\{a\},\{b\}) = s(d_{G_b}(a,b))$ for all $a, b \in X$;
(ii) if $x \in s(a,B,B)$, then $x \in 2s(d_{G_b}(a,b))$.

Proof.

(i). By definition

$$\hat{s}(\{a\},\{b\}) = \bigcup_{a\in\{a\}, b\in\{b\}} s(d_{G_b}(a,b)) = s(d_{G_b}(a,b)).$$

(ii). Let

$$\begin{aligned} & x \in s(a,B,B) \\ & \Rightarrow x \in s(a,B,B) = s(a,B) + \hat{s}(B,B) + s(a,B) \\ & \Rightarrow x \in 2s(a,B) + \hat{s}(B,B) \\ & \Rightarrow x \in 2s(d_{G_b}(a,b)) + s(\theta). \end{aligned}$$

Let $x = y + z$ for $y \in 2s(d_{G_b}(a,b))$ and $z \in s(\theta)$. Then by definition $\theta \preccurlyeq z$ and $2d_{G_b}(a,b) \preccurlyeq y$, which implies $\theta + 2d_{G_b}(a,b) \preccurlyeq y + z = x$. Hence $2d_{G_b}(a,b) \preccurlyeq x$, so $x \in 2s(d_{G_b}(a,b))$. ∎

In our main theorem we will use the concept of generalized Hausdorff distance on G_b-cone metric spaces to find fixed points of a multivalued mapping.

Theorem 2.4. *Let* (X, G) *be a complete* G_b*-cone metric space with the coefficient* $r \geqslant 1$ *and let* $T : X \longrightarrow C(X)$ *be multivalued mapping. If there exists a function* $\varphi : P \to [0, \frac{1}{r})$ *such that*

$$\lim_{n\to\infty} \sup \varphi(u_n) < \frac{1}{r}$$

for any decreasing sequence $\{u_n\}$ *in* P*, and if*

$$\varphi(G(x, y, z))G(x, y, z) \in s(Tx, Ty, Tz) \tag{2.1}$$

for all $x, y, z \in X$*, then* T *has a fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point, then $Tx_0 \neq \emptyset$. Let $x_1 \in Tx_0$. From (2.1), we have

$$\varphi(G(x_0, x_1, x_1))G(x_0, x_1, x_1) \in s(Tx_0, Tx_1, Tx_1).$$

Thus by Lemma 2.1 (iii), we get

$$\varphi(G(x_0, x_1, x_1))G(x_0, x_1, x_1) \in s(x_1, Tx_1, Tx_1).$$

By Remark 2.3, we can take $x_2 \in Tx_1$ such that

$$\varphi(G(x_0, x_1, x_1))G(x_0, x_1, x_1) \in 2s(d_{G_b}(x_1, x_2)) = s(2d_{G_b}(x_1, x_2)).$$

Thus,

$$2d_{G_b}(x_1, x_2) \preccurlyeq \varphi(G(x_0, x_1, x_1))G(x_0, x_1, x_1).$$

Again, by (2.1), we have

$$\varphi(G(x_1, x_2, x_2))G(x_1, x_2, x_2) \in s(Tx_1, Tx_2, Tx_2),$$

and by Lemma 2.1 (iii)

$$\varphi(G(x_1, x_2, x_2))G(x_1, x_2, x_2) \in s(x_2, Tx_2, Tx_2).$$

By Remark 2.3, we can take $x_3 \in Tx_2$ such that

$$\varphi(G(x_1, x_2, x_2))G(x_1, x_2, x_2) \in 2s(d_{G_b}(x_2, x_3)).$$

Thus,

$$2d_{G_b}(x_2, x_3) \preccurlyeq \varphi(G(x_1, x_2, x_2))G(x_1, x_2, x_2).$$

It implies that

$$\begin{aligned} 2d_{G_b}(x_2, x_3) &\preccurlyeq \varphi(G(x_1, x_2, x_2))G(x_1, x_2, x_2) \\ &\preccurlyeq \varphi(G(x_1, x_2, x_2))G(x_1, x_2, x_2) + \varphi(G(x_1, x_2, x_2))G(x_2, x_1, x_1) \\ &\preccurlyeq \varphi(G(x_1, x_2, x_2))[G(x_1, x_2, x_2) + G(x_2, x_1, x_1)] \\ &= \varphi(G(x_1, x_2, x_2))d_{G_b}(x_1, x_2) \\ &\Rightarrow d_{G_b}(x_2, x_3) \preccurlyeq \frac{1}{2}\varphi(G(x_1, x_2, x_2))d_{G_b}(x_1, x_2). \end{aligned}$$

By induction we can construct a sequence $\{x_n\}$ in X such that

$$d_{G_b}(x_n, x_{n+1}) \preccurlyeq \frac{1}{2}\varphi(G(x_{n-1}, x_n, x_n))d_{G_b}(x_{n-1}, x_n), \tag{2.2}$$

$x_{n+1} \in Tx_n$ for all $n = 1,2,3\ldots$. Assume that $x_{n+1} \neq x_n$ for all $n \in N$. From (2.2) the sequence $\{d_{G_b}(x_n, x_{n+1})\}_{n\in N}$ is decreasing sequence in P. So there exists $l \in (0,1)$ such that

$$\limsup_{n\to\infty} \varphi(G(x_{n-1}, x_n, x_n)) = l.$$

Thus there exists $n_0 \in N$ such that for all $n \geqslant n_0$, $\varphi(G(x_{n-1}, x_n, x_n)) \prec l_0$ for some $l_0 \in (l,1)$. We have

$$d_{G_b}(x_n, x_{n+1}) \preccurlyeq \frac{1}{2}\varphi(d_{G_b}(x_{n-1}, x_n))d_G(x_{n-1}, x_n) \prec l_0 d_{G_b}(x_{n-1}, x_n) \prec (l_0)^n d_{G_b}(x_0, x_1)$$

for all $n \geqslant n_0$. Moreover for $m > n \geqslant n_0$, we have

$$\begin{aligned} d_{G_b}(x_n, x_m) &\preccurlyeq r[d_{G_b}(x_n, x_{n+1}) + d_{G_b}(x_{n+1}, x_m)] \\ &\preccurlyeq r d_{G_b}(x_n, x_{n+1}) + r^2[d_{G_b}(x_{n+1}, x_{n+2}) + d_{G_b}(x_{n+2}, x_m)] \\ &\vdots \\ &\preccurlyeq r d_{G_b}(x_n, x_{n+1}) + r^2 d_{G_b}(x_{n+1}, x_{n+2}) + \cdots + r^{m-n} d_{G_b}(x_{m-1}, x_m). \end{aligned}$$

Thus we have

$$\begin{aligned} d_{G_b}(x_n, x_m) &\preccurlyeq r l_0^n d_{G_b}(x_0, x_1) + r^2 l_0^{n+1} d_{G_b}(x_0, x_1) + \cdots + r^{m-n} l_0^{m-1} d_{G_b}(x_0, x_1) \\ &= r l_0^n (1 + r l_0 + \cdots + r^{m-n-1} l_0^{m-n-1}) d_{G_b}(x_0, x_1) \\ &\preccurlyeq \frac{r l_0^n}{1 - r l_0} d_{G_b}(x_0, x_1). \end{aligned}$$

According to (PT1) and (PT7) it follows that $\{x_n\}$ is a Cauchy sequence in X, by the completeness of X there exists $x^* \in X$ such that $x_n \to x^*$. Assume $k_1 \in N$ such that $d_{G_b}(x_n, x^*) \ll \frac{c}{2r}$ for all $n \geqslant k_1$.

We now show that $x^* \in Tx^*$. Using (2.1) for $x_n, x^* \in X$, we have

$$\varphi(G(x_n, x^*, x^*))G(x_n, x^*, x^*) \in s(Tx_n, Tx^*, Tx^*).$$

By Lemma 2.1 (iii) we have

$$\varphi(G(x_n, x^*, x^*))G(x_n, x^*, x^*) \in s(x_{n+1}, Tx^*, Tx^*).$$

Thus there exists $v_n \in Tx^*$, such that

$$\varphi(G(x_n, x^*, x^*))G(x_n, x^*, x^*) \in 2s(d_{G_b}(x_{n+1}, v_n)).$$

It implies that

$$\begin{aligned} 2d_{G_b}(x_{n+1}, v_n) &\preccurlyeq \varphi(G(x_n, x^*, x^*))G(x_n, x^*, x^*), \\ 2d_{G_b}(x_{n+1}, v_n) &\preccurlyeq \varphi(G(x_n, x^*, x^*))G(x_n, x^*, x^*) + \varphi(G(x_n, x^*, x^*))G(x^*, x_n, x_n) \\ &\preccurlyeq \varphi(G(x_n, x^*, x^*))[G(x_n, x^*, x^*) + G(x^*, x_n, x_n)] = \varphi(G(x_n, x^*, x^*))d_{G_b}(x_n, x^*). \end{aligned}$$

Thus we have

$$d_{G_b}(x_{n+1}, v_n) \preccurlyeq \frac{1}{2}\varphi(G(x_n, x^*, x^*))d_{G_b}(x_n, x^*).$$

Now consider

$$\begin{aligned} d_{G_b}(x^*, v_n) &\preccurlyeq r d_{G_b}(x_{n+1}, x^*) + r d_{G_b}(x_{n+1}, v_n) \\ &\preccurlyeq r d_{G_b}(x_{n+1}, x^*) + \frac{r}{2}\varphi(G(x_n, x^*, x^*))d_{G_b}(x_n, x^*) \\ &\prec r d_{G_b}(x_{n+1}, x^*) + r d_{G_b}(x_n, x^*) \\ &\prec r d_{G_b}(x_{n+1}, x^*) + r\varphi(G(x_n, x^*, x^*))d_{G_b}(x_n, x^*), \\ d_G(x^*, v_n) &\ll \frac{rc}{2r} + \frac{rc}{2r} = c, \end{aligned}$$

for all $n \geqslant k_1$. Therefore $\lim_{n\to\infty} v_n = x^*$. Since Tx^* is closed so $x^* \in Tx^*$. □

Corollary 2.5. *Let (X, G) be a complete G_b-cone metric space with the coefficient $r \geqslant 1$ and let $T : X \longrightarrow C(X)$ be multivalued mapping. If there exists a constant $k \in [0, \frac{1}{r})$ such that*

$$kG(x, y, z) \in s(Tx, Ty, Tz)$$

for all $x, y, z \in X$, then T has a fixed point.

Corollary 2.6. *Let (X, G) be a complete G_b-metric space with the coefficient $r \geqslant 1$ and let $T : X \longrightarrow CB(X)$ be multivalued mapping. If there exists a function $\varphi : P \to [0, \frac{1}{r})$ such that*

$$\limsup_{n\to\infty} \varphi(u_n) < \frac{1}{r}$$

for any decreasing sequence $\{u_n\}$ in P, and if

$$H(Tx, Ty, Tz) \leqslant \varphi(G(x, y, z))G(x, y, z)$$

for all $x, y, z \in X$, then T has a fixed point.

Corollary 2.7. *Let (X, G) be a complete Gb-metric space with the coefficient $r \geqslant 1$ and let $T : X \longrightarrow CB(X)$ be multivalued mapping. If there exists a constant $k \in [0, 1)$ such that*

$$H(Tx, Ty, Tz) \leqslant kG(x, y, z)$$

for all $x, y, z \in X$, then T has a fixed point.

Next corollary is Nadler's multivalued contraction theorem in G-cone metric space.

Corollary 2.8. *Let (X, G) be a complete G-cone metric space and let $T : X \longrightarrow CB(X)$ be multivalued mapping. If there exists a constant $k \in [0, 1)$ such that*

$$kG(x, y, z) \in s(Tx, Ty, Tz)$$

for all $x, y, z \in X$, then T has a fixed point in X.

Corollary 2.9 ([8])**.** *Let (X, G) be a complete G-cone metric space and let $T : X \longrightarrow C(X)$ be multivalued mapping. If there exists a function $\varphi : P \to [0, 1)$ such that*

$$\limsup_{n\to\infty} \varphi(u_n) < 1$$

for any decreasing sequence $\{u_n\}$ in P, and if

$$\varphi(G(x, y, z))G(x, y, z) \in s(Tx, Ty, Tz)$$

for all $x, y, z \in X$, then T has a fixed point.

By Remark 2.2, we have the following results of [38].

Corollary 2.10 ([38])**.** *Let (X, G) be a complete G-metric space and let $T : X \longrightarrow CB(X)$ be multivalued mapping. If there exists a function $\varphi : [0, +\infty) \to [0, 1)$ such that*

$$\limsup_{r\to t^+} \varphi(r) < 1$$

for any $t \geqslant 0$, and if

$$H_G(Tx, Ty, Tz) \leqslant \varphi(G(x, y, z))G(x, y, z)$$

for all $x, y, z \in X$, then T has a fixed point in X.

Corollary 2.11 ([38]). *Let* (X, G) *be a complete G-metric space and let* $T : X \longrightarrow CB(X)$ *be multivalued mapping. If there exists a constant* $k \in [0, 1)$ *such that*

$$H_G(Tx, Ty, Tz) \leqslant kG(x, y, z)$$

for all $x, y, z \in X$ *, then* T *has a fixed point in* X.

In the following we formulate an illustrative example regarding our main theorem.

Example 2.12. Let $X = [0, 1]$, $E = C[0, 1]$ be endowed with the strongly locally convex topology $\tau(E, E^*)$, and let $P = \{x \in E : 0 \leqslant x(t),\ t \in [0, 1]\}$. Then the cone is $\tau(E, E^*)$-solid, and non-normal with respect to the topology $\tau(E, E^*)$. Define $G : X \times X \times X \to E$ by

$$G(x, y, z)(t) = \max\{|x - y|^p\ ,\ |y - z|^p\ ,\ |x - z|^p\}e^t,$$

where $p \geqslant 1$. Then G is a G_b-cone metric on X.

Consider a mapping $T : X \to C(X)$ defined by$Tx = [0, \frac{1}{3}x]$. Let $\varphi(t) = \frac{2}{3^p}$ for all $t \in P$, where $p \geqslant 1$. The contractive condition of main theorem is trivial for the case when $x = y = z = 0$. Suppose without any loss of generality that all x, y and z are nonzero and $x < y < z$. Then

$$G(x, y, z) = |x - z|^p\, e^t,$$

and

$$d_{G_b}(x, y) = 2\,|x - y|^p\, e^t.$$

Now

$$s(x, Ty) = \begin{cases} 0, & \text{if } x \leqslant \frac{y}{3}, \\ \left|x - \frac{y}{3}\right|^p e^t, & \text{if } x > \frac{y}{3}, \end{cases} \qquad s(y, Tz) = \begin{cases} 0, & \text{if } y \leqslant \frac{z}{3}, \\ \left|y - \frac{z}{3}\right|^p e^t, & \text{if } y > \frac{z}{3}. \end{cases}$$

Now for $s(x, Ty) = 0 = s(y, Tz)$, we have

$$s(x, Ty, Tz) = \bigcap_{x \in Tx} s(x, Ty, Tz) = s(x, Ty) + \hat{s}(Ty, Tz) + s(x, Tz) = s(0)$$

and

$$s(y, Tx, Tz) = s(y, Tx) + \hat{s}(Tx, Tz) + s(y, Tz) = s(2\left|y - \frac{x}{3}\right|^p e^t)$$

and

$$\bigcap_{y \in Ty} s(y, Tx, Tz) = s(2\left|\frac{y}{3} - \frac{x}{3}\right|^p e^t).$$

Similarly

$$s(z, Tx, Ty) = s(z, Tx) + \hat{s}(Tx, Ty) + s(z, Ty) = s(2\left|z - (\frac{x}{3} + \frac{y}{3})\right|^p e^t),$$

and

$$\bigcap_{z \in Tz} s(z, Tx, Ty) = s(2\left|\frac{z}{3} - (\frac{x}{3} + \frac{y}{3})\right|^p e^t) = s(2\left|\frac{z}{3} - \frac{x}{3} - \frac{y}{3}\right|^p e^t).$$

So we have

$$\begin{aligned} s(Tx, Ty, Tz) &= \left(\bigcap_{x \in Tx} s(x, Ty, Tz)\right) \cap \left(\bigcap_{y \in Ty} s(y, Tx, Tz)\right) \cap \left(\bigcap_{z \in Tz} s(z, Tx, Ty)\right) \\ &= (s(0)) \cap \left(s(2\left|\frac{y}{3} - \frac{x}{3}\right|^p e^t)\right) \cap \left(s(2\left|\frac{z}{3} - \frac{x}{3} - \frac{y}{3}\right|^p e^t)\right). \end{aligned}$$

Now we discuss the following three possible cases.

(i). If $s(Tx,Ty,Tz) = s(2\left|\frac{z}{3}-\frac{x}{3}-\frac{y}{3}\right|^p e^t)$, then we have

$$\begin{aligned} 2\left|\frac{z}{3}-\frac{x}{3}-\frac{y}{3}\right|^p e^t &\leqslant 2\left|\frac{z}{3}-\frac{x}{3}\right|^p e^t \quad \text{for } t\in[0,1] \\ &= \frac{2}{3^p}|z-x|^p e^t = \frac{2}{3^p}\max\{|x-y|^p, |y-z|^p, |x-z|^p\}e^t = \frac{2}{3^p}G(x,y,z). \end{aligned}$$

So by definition we have

$$\frac{2}{3^p}G(x,y,z) \in s(2\left|\frac{z}{3}-\frac{x}{3}-\frac{y}{3}\right|^p e^t) = s(Tx,Ty,Tz).$$

(ii). If $s(Tx,Ty,Tz) = s(2\left|\frac{y}{10}-\frac{x}{10}\right| e^t)$, then we have

$$\begin{aligned} 2\left|\frac{y}{3}-\frac{x}{3}\right|^p e^t &\leqslant 2\left|\frac{z}{3}-\frac{x}{3}\right|^p e^t \quad \text{for } t\in[0,1] \\ &= \frac{2}{3^p}|z-x| e^t = \frac{2}{3^p}\max\{|x-y|^p, |y-z|^p, |x-z|^p\}e^t = \frac{2}{3^p}G(x,y,z). \end{aligned}$$

So by definition we have

$$\frac{2}{3^p}G(x,y,z) \in s(Tx,Ty,Tz).$$

(iii). If $s(Tx,Ty,Tz) = s(0)$, then

$$0 \leqslant 2\left|\frac{z}{3}-\frac{x}{3}\right| e^t = \frac{2}{3^p}|z-x|^p e^t = \frac{2}{3^p}\max\{|x-y|^p, |y-z|^p, |x-z|^p\}e^t = \frac{2}{3^p}G(x,y,z).$$

So by definition we have

$$\frac{2}{3^p}G(x,y,z) \in s(Tx,Ty,Tz).$$

Hence, all the conditions of main theorem are obviously satisfied and 0 is a fixed point of mapping T.

Acknowledgment

This article was funded by the Deanship of Scientific Research (DSR), University of Jeddah, Jeddah. Therefore, authors acknowledges with thanks DSR, UOJ for financial support.

References

[1] J. Ahmad, A. S. Al-Rawashdeh, A. Azam, *Fixed point results for $\{\alpha,\xi\}$-expansive locally contractive mappings*, J. Inequal. Appl., **2014** (2014), 10 pages. 1

[2] J. Ahmad, A. Al-Rawashdeh, A. Azam, *New fixed point theorems for generalized F-contractions in complete metric spaces*, Fixed Point Theory Appl., **2015** (2015), 18 pages.

[3] J. Ahmad, N. Hussain, A. Azam, M. Arshad, *Common fixed point results in complex valued metric space with applications to system of integral equations*, J. Nonlinear Convex Anal., **29** (2015), 855–871.

[4] A. Al-Rawashdeh, J. Ahmad, *Common fixed point theorems for JS-contractions*, Bull. Math. Anal. Appl., **8** (2016), 12–22.

[5] Z. Aslam, J. Ahmad, N. Sultana, *New common fixed point theorems for cyclic compatible contractions*, J. Math. Anal., **8** (2017), 1–12.

[6] A. Azam, M. Arshad, I. Beg, *Common fixed points of two maps in cone metric spaces*, Rend. Circ. Mat. Palermo, **57** (2008), 433–441.

[7] A. Azam, M. Arshad, I. Beg, *Existence of fixed points in complete cone metric spaces*, Int. J. Mod. Math., **5** (2010), 91–99.

[8] A. Azam. N. Mehmood, *Fixed point theorems for multivalued mappings in G-cone metric spaces*, J. Inequal. Appl., **2013** (2013), 12 pages. 1, 2, 2.9

[9] I. Beg, M. Abbas, T. Nazir, *Generalized cone metric spaces*, J. Nonlinear Sci. Appl., **3** (2010), 21–31. 1

[10] C.-M. Chen, *On set-valued contractions of Nadler type in* tvs-G-*cone metric spaces*, Fixed Point Theory Appl., **2012** (2012), 8 pages.
[11] S.-H. Cho, J.-S. Bae, *Fixed point theorems for multivalued maps in cone metric spaces*, Fixed Point Theory Appl., **2011** (2011), 7 pages. 1, 2
[12] B. C. Dhage, *Generalised metric spaces and mappings with fixed point*, Bull. Calcutta Math. Soc., **84** (1992), 329–336. 1
[13] B. C. Dhage, *Generalized metric spaces and topological structure*, I, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), **46** (2000), 3–24. 1
[14] C. Di Bari, P. Vetro, *ϕ-pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, **57** (2008), 279–285.
[15] C. Di Bari, P. Vetro, *Weakly ϕ-pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, **58** (2009), 125–132.
[16] L.-G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468–1476. 1, 1.2
[17] N. Hussain, M. Abbas, *Common fixed point results for two new classes of hybrid pairs in symmetric spaces*, Appl. Math. Comput., **218** (2011), 542–547.
[18] N. Hussain, J. Ahmad, *New Suzuki-Berinde type fixed point results*, Carpathian J. Math., **33** (2017), 59–72.
[19] N. Hussain, J. Ahmad, A. Azam, *Generalized fixed point theorems for multi-valued α-ψ-contractive mappings*, J. Inequal. Appl., **2014** (2014), 15 pages.
[20] N. Hussain, J. Ahmad, M. A. Kutbi, *Fixed point theorems for generalized Mizoguchi-Takahashi graphic contractions*, J. Funct. Spaces, **2016** (2016), 7 pages.
[21] N. Hussain, E. Karapınar, P. Salimi, P. Vetro, *Fixed point results for G^m-Meir-Keeler contractive and G-(α,ψ)-Meir-Keeler contractive mappings*, Fixed Point Theory Appl., **2013** (2013), 14 pages.
[22] S. Janković, Z. Kadelburg, S. Radenović, *On cone metric spaces: a survey*, Nonlinear Anal., **74** (2011), 2591–2601. 1.2
[23] M. Jleli, B. Samet, *Remarks on* G-*metric spaces and fixed point theorems*, Fixed Point Theory Appl., **2012** (2012), 7 pages.
[24] Z. Kadelburg, S. Radenović, *Some results on set-valued contractions in abstract metric spaces*, Comput. Math. Appl., **62** (2011), 342–350.
[25] A. Kaewcharoen, A. Kaewkhao, *Common fixed points for single-valued and multi-valued mappings in* G-*metric spaces*, Int. J. Math. Anal. (Ruse), **5** (2011), 1775–1790. 2.2
[26] A. Latif, W. A. Albar, *Fixed point results in complete metric spaces*, Demonstratio Math., **41** (2008), 145–150.
[27] A. Latif, N. Hussain, J. Ahmad, *Coincidence points for hybrid contractions in cone metric spaces*, J. Nonlinear Convex Anal., **17** (2016), 899–906.
[28] A. Latif, F. Y. Shaddad, *Fixed point results for multivalued maps in cone metric spaces*, Fixed Point Theory Appl., **2010** (2010), 11 pages.
[29] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., **141** (1989), 177–188. 1.1
[30] Z. Mustafa, B. Sims, *Some remarks concerning* D-*metric spaces*, International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, (2004), 189–198.
[31] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297. 1
[32] S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–478. 1
[33] W. Onsod, T. Saleewong, J. Ahmad, A. E. Al-Mazrooei, P. Kumam, *Fixed points of a θ-contraction on metric spaces with a graph*, Commun. Nonlinear Anal., **2** (2016), 139–149.
[34] S. Rezapour, R. Hamlbarani, *Some notes on the paper: "Cone metric spaces and fixed point theorems of contractive mappings" [J. Math. Anal. Appl.,* **332** *(2007), 1468–1476] by L.-G. Huang and X. Zhang*, J. Math. Anal. Appl., **345** (2008), 719–724. 1, 1
[35] W. Shatanawi, *Some common coupled fixed point results in cone metric spaces*, Int. J. Math. Anal. (Ruse), **4** (2010), 2381–2388.
[36] W. Shatanawi, *On w-compatible mappings and common coupled coincidence point in cone metric spaces*, Appl. Math. Lett., **25** (2012), 925–931.
[37] W. Shatanawi, *Some coincidence point results in cone metric spaces*, Math. Comput. Modelling, **55** (2012), 2023–2028.
[38] N. Tahat, H. Aydi, E. Karapinar, W. Shatanawi, *Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in* G-*metric spaces*, Fixed Point Theory Appl., **2012** (2012), 9 pages. 2.2, 2, 2.10, 2.11
[39] M. Ughade, R. D. Daheriya, *Fixed point results for contraction mappings in G_b-cone metric spaces*, Gazi Univ. J. Sci., **28** (2015), 659–673. 1, 1.3, 1.4, 1.5, 1.6, 1.7, 1.9, 1.10, 1.11
[40] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, **56** (2007), 464–468.
[41] T. X. Wang, *Fixed-point theorems and fixed-point stability for multivalued mappings on metric spaces*, Nanjing Daxue Xuebao Shuxue Bannian Kan, **6** (1989), 16–23. 1