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A pinching theorem for statistical manifolds with Casorati curvatures

Chul Woo Lee[a], Dae Won Yoon[b], Jae Won Lee[b,*]

[a]Department of Mathematics, Kyungpook National University, Daegu 41566, South Korea.

[b]Department of Mathematics Education, Gyeongsang National University and RINS, Jinju 52828, South Korea.

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Abstract

With a pair of conjugate connections $\overline{\nabla}$ and $\overline{\nabla}^*$, we derive optimal Casorati inequalities with the normalized scalar curvature on submanifolds of a statistical manifold of constant curvature.

Keywords: Statistical manifolds, dual connection, Casorati curvature, δ-Casorati curvature, normalized scalar curvature.
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1. Introduction

In 1985, Amari [1] introduced the notion of statistical manifolds in his treatment of statistical inference problems in information geometry. The geometry of such manifolds is closely related to affine geometry and Hessian geometry. In such manifolds, we have the fundamental equations such as Gauss formula, Weingarten formula and the equations of Gauss, Codazzi, and Ricci in submanifolds of a statistical manifold [18]. Furuhata [5] derived a condition for the curvature of a statistical manifold to admit a kind of standard hypersurfaces.

On the other hand, it is known that the Casorati curvature of a submanifold in a Riemannian manifold is an extrinsic invariant defined as the normalized square of the length of the second fundamental form and it was preferred by Casorati over the traditional Gauss curvature ([2, 8]). Geometric meaning and the importance of such curvature were found in visual perception of shape and appearance ([7, 14, 17]). Some optimal inequalities involving Casorati curvatures were proved in [3, 4, 6, 9–13, 15, 19, 20] for several submanifolds in real, complex, and quaternionic space forms with various connections. These optimizations investigated the scalar curvature was bounded above only by Casorati curvatures.

In our paper, we establish the normalized scalar curvature is bounded by Casorati curvatures of submanifolds in a statistical manifold of constant curvature as follows.

Theorem 1.1. *Let M^n be a statistical submanifold of a statistical space form $\overline{M}^m(c)$. Then, the normalized δ-*

*Corresponding author
Email addresses: mathisu@knu.ac.kr (Chul Woo Lee), dwyoon@gnu.ac.kr (Dae Won Yoon), leejaew@gnu.ac.kr (Jae Won Lee)

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Casorati curvatures $\delta_C(n-1)$ *and* $\delta^*_C(n-1)$ *satisfy*

$$\rho \leqslant 2\delta^0_C(n-1) + \frac{1}{n-1}\mathcal{C}^0 - \frac{2n}{n-1}\|H^0\|^2 + \frac{n}{(n-1)}\overline{g}(H, H^*) + c,$$

where $2\delta^0_C(n-1) = \delta_C(n-1) + \delta^*_C(n-1)$ *and* $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$.

Theorem 1.2. *Let* M^n *be a statistical submanifold of a statistical space form* $\overline{M}^m(c)$. *Then, the normalized* δ-*Casorati curvatures* $\delta_C(n-1)$ *and* $\delta^*_C(n-1)$ *satisfy*

$$\rho \geqslant -\frac{1}{2}\delta^0_C(n-1) + \frac{2n}{n-1}\|H^0\|^2 - \frac{2}{n-1}\mathcal{C}^0 + c,$$

where $2\delta^0_C(n-1) = \delta_C(n-1) + \delta^*_C(n-1)$ *and* $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$.

2. Preliminaries

Let $(\overline{M}^m, \overline{g})$ be an m-dimensional Riemannian manifold with an affine connection $\overline{\nabla}$. Let $\overline{T}$ be the torsion tensor field of type $(1,2)$ of $\overline{\nabla}$.

Definition 2.1. A pair $(\overline{\nabla}, g)$ is called *a statistical structure* on $\overline{M}$ if

(1) $(\overline{\nabla}_X \overline{g})(Y, Z) - (\overline{\nabla}_Y \overline{g})(X, Z) = \overline{g}(\overline{T}(X, Y), Z)$ for vector fields X, Y, and Z on $\overline{M}$, and

(2) $\overline{T} = 0$.

Definition 2.2. *A statistical manifold* $(\overline{M}^m, \overline{g}, \overline{\nabla})$ is a Riemannian manifold, endowed with a pair of torsion-free affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ satisfying

$$Z\overline{g}(X, Y) = \overline{g}(\overline{\nabla}_Z X, Y) + \overline{g}\left(X, \overline{\nabla}^*_Z Y\right)$$

for any vector fields X, Y, and Z. The connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called *dual connections*.

Remark 2.3.

(a) $\left(\overline{\nabla}^*\right)^* = \overline{\nabla}$.

(b) If $(\overline{\nabla}, \overline{g})$ is a statistical structure, then so is $\left(\overline{\nabla}^*, \overline{g}\right)$.

(c) Any torsion-free affine connection $\overline{\nabla}$ has always a dual connection satisfying

$$\overline{\nabla} + \overline{\nabla}^* = 2\overline{\nabla}^0,$$

where $\overline{\nabla}^0$ is the Levi-Civita connection for $\overline{M}$.

Let $\overline{R}$ and $\overline{R}^*$ be the curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively.

Definition 2.4. A statistical structure $(\overline{\nabla}, g)$ is said to be *of constant curvature* $c \in \mathbb{R}$ if $\overline{R}(X, Y)Z = c\{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y\}$ for any vector fields X, Y, and Z. A statistical structure $(\overline{\nabla}, g)$ with constant curvature 0 is called *a Hessian structure*.

By direct calculation, the curvature tensor fields $\overline{R}$ and $\overline{R}^*$ satisfy

$$\overline{g}\left(\overline{R}^*(X, Y)Z, W\right) = -\overline{g}\left(Z, \overline{R}(X, Y)W\right).$$

Therefore, if $(\overline{\nabla}, g)$ is a statistical structure of constant curvature c, so is $\left(\overline{\nabla}^*, \overline{g}\right)$.

For submanifolds in statistical manifolds, we have pairs of induced connections ∇, ∇^*, second fundamental forms h, h^*, shape operators A, A^*, and normal connections D, D^* satisfying equations analogous to the Gauss and the Weingarten ones for $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively. Moreover, the induced metric g is unique, and (∇, g) and (∇^*, g) are induced dual statistical structures on the submanifolds. The fundamental equations for statistical submanifolds are given by Vos ([18]).

Let M be an n-dimensional submanifold of a statistical manifold $(\overline{M}, \overline{g})$ and g the induced metric on M. Then for any vector fields X, Y, the Gauss formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y).$$

Let $\overline{R}$ and R be the curvature tensor fields of $\overline{\nabla}$ and ∇, respectively. Then, we have Gauss formula given by

$$\overline{g}\left(\overline{R}\left(X, Y\right) Z, W\right) = g\left(R\left(X, Y\right) Z, W\right) + \overline{g}\left(h\left(X, Z\right), h^*\left(Y, W\right)\right) - \overline{g}\left(h^*\left(X, W\right), h\left(Y, Z\right)\right). \tag{2.1}$$

If $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space T_pM and $\{e_{n+1}, \dots, e_m\}$ is an orthonormal basis of the normal space $T_p^{\perp}M$, then the scalar curvature τ at p is defined as

$$\tau(p) = \sum_{1 \leqslant i < j \leqslant n} g\left(R(e_i, e_j)e_j, e_i\right)$$

and the normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

We denote by H and H^* the mean curvature vectors, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \qquad H^*(p) = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i), \tag{2.2}$$

and we also set

$$h_{ij}^{\alpha} = \overline{g}(h(e_i, e_j), e_\alpha), \quad h_{ij}^{*\,\alpha} = \overline{g}(h^*(e_i, e_j), e_\alpha),$$

$i, j \in \{1, \dots, n\}$, $\alpha \in \{n+1, \dots, m\}$.

Then it is well-known that the squared mean curvatures of the submanifold M in $\overline{M}$ are defined by

$$\|H\|^2 = \frac{1}{n^2}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2, \quad \|H^*\|^2 = \frac{1}{n^2}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{*\,\alpha}\right)^2,$$

and the squared norms of h and h^* over dimension n are denoted by $\mathcal{C}$ and $\mathcal{C}^*$ are called the Casorati curvatures of the submanifold M, respectively. Therefore we have

$$\mathcal{C} = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{\alpha}\right)^2 \text{ and } \mathcal{C}^* = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{*\,\alpha}\right)^2.$$

Suppose now that L is a k-dimensional subspace of T_pM, $k \geqslant 2$ and let $\{e_1, \dots, e_k\}$ be an orthonormal basis of L. Then, the Casorati curvatures $\mathcal{C}\left(L\right)$ and $\mathcal{C}^*\left(L\right)$ of L are defined as

$$\mathcal{C}\left(L\right) = \frac{1}{k}\sum_{\alpha=n+1}^{n}\sum_{i,j=1}^{k}\left(h_{ij}^{\alpha}\right)^2, \text{ and } \mathcal{C}^*\left(L\right) = \frac{1}{k}\sum_{\alpha=n+1}^{n}\sum_{i,j=1}^{k}\left(h^{*\,\alpha}_{ij}\right)^2.$$

The normalized δ-Casorati curvatures $\delta_C(n-1)$ and $\widehat{\delta}_C(n-1)$ of the submanifold M^n are defined as

$$\left[\delta_C(n-1)\right]_p = \frac{1}{2}\mathcal{C}_p + \frac{(n+1)}{2n}\inf\{\mathcal{C}(L) | L \text{ a hyperplane of } T_pM\},$$

and

$$\left[\widehat{\delta}_C(n-1)\right]_p = 2\mathcal{C}_p - \frac{(2n-1)}{2n}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}.$$

Moreover, the dual normalized δ^*-Casorati curvatures $\delta^*_C(n-1)$ and $\widehat{\delta}^*_C(n-1)$ of the submanifold M^n are defined as

$$[\delta^*_C(n-1)]_p = \frac{1}{2}\mathcal{C}^*_p + \frac{(n+1)}{2n}\inf\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\},$$

and

$$\left[\widehat{\delta}^*_C(n-1)\right]_p = 2\mathcal{C}^*_p - \frac{(2n-1)}{2n}\sup\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\}.$$

The following lemma plays a key role in the proof of our theorem.

Lemma 2.5 ([16]). *Let*

$$\Gamma = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = k\}$$

be a hyperplane of $\mathbb{R}^n$, *and* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *a quadratic form given by*

$$f(x_1, x_2, \cdots, x_n) = a\sum_{i=1}^{n-1}(x_i)^2 + b\,(x_n)^2 - 2\sum_{1\leqslant i<j\leqslant n} x_i x_j, \qquad a > 0,\ b > 0.$$

Then, by the constrained extremum problem, f *has a global solution as follows,*

$$x_1 = x_2 = \cdots = x_{n-1} = \frac{k}{a+1}, \quad x_n = \frac{k}{b+1} = \frac{k(n-1)}{(a+1)b} = (a-n+2)\frac{k}{a+1},$$

provided that

$$b = \frac{n-1}{a-n+2}.$$

3. The proofs of main theorems

proof of Theorem 1.1. Let $p \in M$ and the set $\{e_1, e_2, \ldots, e_n\}$ and $\{e_{n+1}, e_{n+2}, \ldots, e_m\}$ be orthonormal bases of T_pM and $T^\perp_pM$, respectively. From (2.1), we have

$$n(n-1)c = 2\tau(p) - n^2\bar{g}(H, H^*) + \sum_{i,j=1}^{n}\bar{g}(h^*(e_i, e_j), h(e_i, e_j)), \tag{3.1}$$

where H and H^* are the mean curvature vector fields defined by (2.2).

Since $2H^0 = H + H^*$ and the definition of Casorati curvature, $4\|H^0\|^2 = \|H\|^2 + \|H^*\|^2 + 2\bar{g}(H, H^*)$, it follows that

$$n(n-1)c = 2\tau(p) - 2n^2\|H^0\|^2 + \frac{n^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) + 2n\mathcal{C}^0 - \frac{n}{2}\left(\mathcal{C} + \mathcal{C}^*\right).$$

We consider now the following quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = n(n-1)\mathcal{C}^0 + (n-1)(n+1)\mathcal{C}^0(L) + \frac{1}{2}n\left(\mathcal{C} + \mathcal{C}^*\right) - \frac{n^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) - 2\tau(p) + n(n-1)c,$$

where L is a hyperplane of T_pM. Without loss of generality, we can assume that L is spanned by

$e_1, \ldots, e_{n-1}$. Using (3.1), we derive

$$\begin{aligned}\frac{1}{2}\mathcal{P} &= \sum_{\alpha=n+1}^{m}\sum_{i=1}^{n-1}\left[n\left(h_{ii}^{0\alpha}\right)^2 + (n+1)\left(h_{in}^{0\alpha}\right)^2\right]\\ &\quad + \sum_{\alpha=n+1}^{m}\left[2(n+1)\sum_{1=i<j}^{n-1}\left(h_{ij}^{0\alpha}\right)^2 - 2\sum_{1\leqslant i<j}^{n} h_{ii}^{0\alpha}h_{jj}^{0\alpha} + \frac{n-1}{2}\left(h_{nn}^{0\alpha}\right)^2\right]\\ &\geqslant \sum_{\alpha=n+1}^{m}\left[\sum_{i=1}^{n-1} n\left(h_{ii}^{0\alpha}\right)^2 + \frac{n-1}{2}\left(h_{nn}^{0\alpha}\right)^2 - 2\sum_{1\leqslant i<j}^{n} h_{ii}^{0\alpha}h_{jj}^{0\alpha}\right].\end{aligned}$$

For $\alpha = n+1, \cdots, m$, let us consider the quadratic form $f_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$f_\alpha\left(h_{11}^{0\alpha}, \cdots, h_{nn}^{0\alpha}\right) = \sum_{i=1}^{n-1} n\left(h_{ii}^{0\alpha}\right)^2 + \frac{n-1}{2}\left(h_{nn}^{0\alpha}\right)^2 - 2\sum_{1\leqslant i<j}^{n} h_{ii}^{0\alpha}h_{jj}^{0\alpha}, \tag{3.2}$$

and the constrained extremum problem

$$\min f_\alpha,$$

subject to the component of traceH^0:

$$F : h_{11}^{0\alpha} + \cdots + h_{nn}^{0\alpha} = k^\alpha,$$

where k^α is constant. Comparing (3.2) with the quadratic function in Lemma 2.5, we see that

$$a = n, \qquad b = \frac{n-1}{2}.$$

Therefore, we have the critical point $\left(h_{11}^{0\alpha}, \ldots, h_{nn}^{0\alpha}\right)$, given by

$$h_{11}^{0\alpha} = h_{22}^{0\alpha} = \cdots = h_{n-1n-1}^{0\alpha} = \frac{k^\alpha}{n+1}, \qquad h_{nn}^{0\alpha} = \frac{2k^\alpha}{n+1},$$

is a global minimum point by Lemma 2.5. Moreover, $f_\alpha\left(h_{11}^{0\alpha}, \ldots, h_{nn}^{0\alpha}\right) = 0$. Therefore, we have

$$\mathcal{P} \geqslant 0$$

and this implies

$$2\tau(p) \leqslant n(n-1)\mathcal{C}^0 + (n-1)(n+1)\mathcal{C}^0(L) + \frac{1}{2}n\left(\mathcal{C} + \mathcal{C}^*\right) - \frac{n^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) + n(n-1)c.$$

Therefore, we derive

$$\frac{1}{2}\rho \leqslant \delta_{\mathcal{C}}^0(n-1) + \frac{1}{4(n-1)}\left(\mathcal{C} + \mathcal{C}^*\right) - \frac{n}{4(n-1)}\left(\|H\|^2 + \|H^*\|^2\right) + \frac{1}{2}c.$$

□

The proof of Theorem 1.2. We consider a quadratic polynomial in the components of the second fundamental form:

$$\mathcal{Q} = -\frac{n(n-1)}{4}\left(\mathcal{C} + \mathcal{C}^*\right) - \frac{(n-1)(n+1)}{4}\left(\mathcal{C}(L) + \mathcal{C}^*(L)\right) - 2\tau(p) + 2n^2\|H^0\|^2 - 2n\mathcal{C}^0 + n(n-1)c,$$

where L is a hyperplane of T_pM. Without loss of generality, we can assume that L is spanned by $e_1, \dots, e_{n-1}$. Using (3.1), we derive

$$\begin{aligned}
-2\mathcal{Q} = & \sum_{\alpha=n+1}^{m} \sum_{i=1}^{n-1} \left[n\,(h_{ii}^{\alpha})^2 + (n+1)\,(h_{in}^{\alpha})^2 \right] \\
& + \sum_{\alpha=n+1}^{m} \left[2(n+1) \sum_{1 \leqslant i<j}^{n-1} \left(h_{ij}^{\alpha}\right)^2 - 2 \sum_{1 \leqslant i<j}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} + \frac{n-1}{2}\,(h_{nn}^{\alpha})^2 \right] \\
& + \sum_{\alpha=n+1}^{m} \sum_{i=1}^{n-1} \left[n\,(h_{ii}^{*\alpha})^2 + (n+1)\,(h_{in}^{*\alpha})^2 \right] \\
& + \sum_{\alpha=n+1}^{m} \left[2(n+1) \sum_{1 \leqslant i<j}^{n-1} \left(h_{ij}^{*\alpha}\right)^2 - 2 \sum_{1 \leqslant i<j}^{n} h_{ii}^{*\alpha} h_{jj}^{*\alpha} + \frac{n-1}{2}\,(h_{nn}^{*\alpha})^2 \right] \\
\geqslant & \sum_{\alpha=n+1}^{m} \left[\sum_{i=1}^{n-1} n\,(h_{ii}^{\alpha})^2 + \frac{n-1}{2}\,(h_{nn}^{\alpha})^2 - 2 \sum_{1 \leqslant i<j}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} \right] \\
& + \sum_{\alpha=n+1}^{m} \left[\sum_{i=1}^{n-1} n\,(h_{ii}^{*\alpha})^2 + \frac{n-1}{2}\,(h_{nn}^{*\alpha})^2 - 2 \sum_{1 \leqslant i<j}^{n} h_{ii}^{*\alpha} h_{jj}^{*\alpha} \right].
\end{aligned}$$

For $\alpha = n+1, \dots, m$, let us consider the quadratic form $g_\alpha : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}
g_\alpha\left(h_{11}^{\alpha}, \cdots, h_{nn}^{\alpha}, h_{11}^{*\alpha}, \cdots, h_{nn}^{*\alpha}\right) = & \sum_{i=1}^{n-1} n\,(h_{ii}^{\alpha})^2 + \frac{n-1}{2}\,(h_{nn}^{\alpha})^2 - 2 \sum_{1 \leqslant i<j}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} \\
& + \sum_{i=1}^{n-1} n\,(h_{ii}^{*\alpha})^2 + \frac{n-1}{2}\,(h_{nn}^{*\alpha})^2 - 2 \sum_{1 \leqslant i<j}^{n} h_{ii}^{*\alpha} h_{jj}^{*\alpha},
\end{aligned}$$

and the constrained extremum problem $\min g_\alpha$, subject to $G : h_{11}^{\alpha} + \cdots + h_{nn}^{\alpha} = k^{\alpha}$ and $h_{11}^{*\alpha} + \cdots + h_{nn}^{*\alpha} = l^{\alpha}$, where k^{α} and l^{α} are constant. From Lemma 2.5, we have the critical point $\left(h_{11}^{\alpha}, \dots, h_{nn}^{\alpha}, h_{11}^{*\alpha}, \dots, h_{nn}^{*\alpha}\right)$, given by

$$\begin{aligned}
h_{11}^{\alpha} &= \cdots = h_{n-1n-1}^{\alpha} = \frac{k^{\alpha}}{n+1}, & h_{nn}^{\alpha} &= \frac{2k^{\alpha}}{n+1}, \\
h_{11}^{*\alpha} &= \cdots = h_{n-1n-1}^{*\alpha} = \frac{l^{\alpha}}{n+1}, & h_{nn}^{*\alpha} &= \frac{2l^{\alpha}}{n+1},
\end{aligned}$$

is a global minimum point. Moreover, $g_\alpha\left(h_{11}^{\alpha}, \dots, h_{nn}^{\alpha}, h_{11}^{*\alpha}, \dots, h_{nn}^{*\alpha}\right) = 0$. Therefore, we have

$$-2\mathcal{Q} \geqslant 0 \qquad \Rightarrow \qquad \mathcal{Q} \leqslant 0,$$

and this implies

$$2\tau(p) \geqslant -\frac{n(n-1)}{4}\,(\mathcal{C} + \mathcal{C}^*) - \frac{(n-1)(n+1)}{4}\,(\mathcal{C}(L) + \mathcal{C}^*(L)) + 2n^2 \|H^0\|^2 - 2n\mathcal{C}^0 + n(n-1)c.$$

Therefore we derive

$$2\rho \geqslant -\delta_C(n-1) - \delta_C^*(n-1) + \frac{4n}{n-1}\|H^0\|^2 - \frac{4}{n-1}\mathcal{C}^0 + 2c$$

for every tangent hyperplane L of M. □

Remark 3.1.

(1) Theorem 1.1 shows the normalized scalar curvature is bounded above by Casorati curvatures.
(2) Theorem 1.2 shows the normalized scalar curvature is bounded below by Casorati curvatures.

Remark 3.2. The normalized scalar curvature is bounded by the normalized Casorati curvatures $\widehat{\delta}_C(n-1)$ and $\widehat{\delta}^*{}_C(n-1)$ with similar proof of Theorems 1.1 and 1.2.

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