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On subclass of meromorphic multivalent functions associated with Liu-Srivastava operator

Saqib Hussain[a], Jamila Bibi[a], Mohsan Raza[b,*], Maslina Darus[c]

[a] COMSATS Institute of Information Technology, Abbotabad, Pakistan.

[b] Department of Mathematics, Government College University Faisalabad, Pakistan.

[c] School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia 43600, Bangi, Selangor, Malaysia.

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Abstract

In the present paper, we introduce a new subclass related to meromorphically p-valent reciprocal starlike functions associated with the Liu-Srivastava operator. Some sufficient conditions for functions belonging to this class are derived. The results presented here improve and generalize some known results.

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1. Introduction

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p}, \quad (p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured open unit disc $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$, where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we set $\Sigma_1 = \Sigma$. Let f and g be two analytic functions in the open unit disk $\mathbb{U}$, we say that the function f is subordinate to g (written as $f \prec g$) if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalent relation:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

*Corresponding author
Email addresses: saqib_math@yahoo.com (Saqib Hussain), syyedajameela@gmail.com (Jamila Bibi), mohsan976@yahoo.com (Mohsan Raza), maslina@ukm.edu.my (Maslina Darus)

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For some details see [2, 12]; see also [15].

A function $f \in \Sigma_p$ is said to be in class $\mathcal{S}_p^*(\oslash)$ of meromorphically p-valent starlike of order α if and only if

$$\Re e\left(\frac{zf'(z)}{pf(z)}\right) < -\alpha, \quad (0 \leqslant \alpha < 1).$$

It is clear that $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, the class of p-valent starlike functions. A function $f \in \mathcal{S}_p^*$ is said to be in the class $\mathcal{M}_p(\oslash)$ of meromorphically p-valent starlike of reciprocal order α if and only if

$$\Re e\left(\frac{pf(z)}{zf'(z)}\right) < -\alpha, \quad (0 \leqslant \alpha < 1\ ).$$

In particular $\mathcal{M}_1(\oslash) = \mathcal{M}(\oslash)$.

Remark 1.1. In view of the fact that

$$\Re e\,(p(z)) < 0 \Longrightarrow \Re e\left(\frac{1}{p(z)}\right) = \Re e\left(\frac{p(z)}{|p(z)|^2}\right) < 0,$$

it follows that meromorphically p-valent starlike function of reciprocal order 0 is same as a meromorphically p-valent starlike function. When $0 < \alpha < 1$, the function $f \in \Sigma_p$ is meromorphically p-valent starlike of reciprocal order α if and only if

$$\left|\frac{zf'(z)}{pf(z)} + \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}.$$

For $p = 1$, this class was studied by Sun et al. [17]. For arbitrary fixed real numbers A and B ($-1 \leqslant B < A \leqslant 1$), we denote by $P(A, B)$ the class of the functions of the form

$$q(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

which is analytic in the unit disk $\mathbb{U}$ and satisfies the condition

$$q(z) \prec \frac{1 + Az}{1 + Bz}, \quad (z \in \mathbb{U}). \tag{1.2}$$

The class $P(A, B)$ was introduced and studied by Janowski [5]. We also observe from (1.2) (see also [14]) that a function $q(z) \in P(A, B)$ if and only if

$$\left|q(z) - \frac{1 - AB}{1 - B^2}\right| < \frac{A - B}{1 - B^2}, \quad (B \neq -1), \tag{1.3}$$

and

$$\Re e\{q(z)\} > \frac{1 - A}{2}, \quad (B = -1). \tag{1.4}$$

For function $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p},$$

the Hadamard product (convolution) of f and g is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

For complex parameters α_i and β_j, where $i = 1, 2, \cdots, l$, $j = 1, 2, \cdots, m$ and $\beta_j \notin \mathbb{Z}_o^- = \{0, -1, -2, \cdots\}$, the generalized hypergeometric function ${}_l\mathcal{F}_m$ is defined by

$${}_l\mathcal{F}_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{k!\,(\beta_1)_k \cdots (\beta_m)_k} z^k,$$

where $l \leqslant m + 1$, $l, m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $(\lambda)_n$ is pochhammer symbol (or shifted factorial) defined in terms of Gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n=0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in \mathbb{N}. \end{cases}$$

Now consider the function

$$h_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) = z^{-p}\,{}_l\mathcal{F}_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)(z),$$

then the Liu-Srivastava linear operator [8, 9] $\mathcal{H}_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m) : \Sigma_p \longrightarrow \Sigma_p$ is defined by using the Hadamard product (or convolution) as

$$\begin{aligned} \mathcal{H}_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) &= h_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) * f(z) \\ &= z^{-p} + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_l)_k}{k!\,(\beta_1)_k\cdots(\beta_m)_k} a_k z^{k-p}. \end{aligned} \tag{1.5}$$

For convenience, we denote $\mathcal{H}_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m) \approxeq \mathcal{H}_{p,l,m}[\alpha_1]$.

The Liu-Srivastava operator is studied in [1, 13, 16], is the meromorphic analogue of the Dziok-Srivastava [3] linear operator. Special cases of the Liu-Srivastava linear operator include the meromorphic analogue of the Carlson-Shaffer linear operator $L_p(a,c) = \mathcal{H}_{p,2,1}(1,a,c)$ studied among others by Liu and Srivastava [7], Liu [6] and Yang [20]. The analogous to the Ruscheweyh derivative operator $D^{n+1} = L_p(n+p,1)$ was investigated by Yang [19]. The operator

$$J_{c,p} = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)\,dt = L_p(c,c+1), \quad (c>0),$$

was studied by Uralegaddi and Somanatha [18].

By using operator $\mathcal{H}_{p,l,m}[\alpha_1]$, we introduce the following new class.

Definition 1.2. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{M}_{[\alpha_1]}(p;\beta;\lambda;A_1,B)$, if it satisfies the subordination

$$\frac{p}{1-p\beta}\left\{\frac{(1-\lambda)\mathcal{H}_{p,l,m}[\alpha_1]f(z) + \lambda z\left(\mathcal{H}_{p,l,m}[\alpha_1]f(z)\right)'}{z\left(\mathcal{H}_{p,l,m}[\alpha_1]f(z)\right)' + \lambda z^2\left(\mathcal{H}_{p,l,m}[\alpha_1]f(z)\right)''} + \beta\right\} \prec -\frac{1+A_1 z}{1+Bz},$$

where $A_1 = (1-\alpha)A + \alpha B$, $0 \leqslant \alpha < 1$, $0 \leqslant \lambda \leqslant 1$, $-1 \leqslant B < A \leqslant 1$, $0 \leqslant p\beta < 1$ and $\mathcal{H}_{p,l,m}[\alpha_1]$ is defined in (1.5).

Remark 1.3. Using (1.3), (1.4) and for $B \neq -1$, the Definition 1.2 is equivalent to

$$\left|\frac{p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} + \beta\right\} + \frac{1-A_1B}{1-B^2}\right| < \frac{A_1-B}{1-B^2}, \tag{1.6}$$

and for $B=-1$,

$$\Re e\left\{\frac{p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} + \beta\right\}\right\} < -\frac{1-A_1}{2}, \tag{1.7}$$

also, for $B=-1$, $A_1 \neq 1$, (1.7) reduces to

$$\left|\frac{1-p\beta}{p}\left(\frac{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z) + \beta z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}\right) + \frac{1}{1-A_1}\right| < \frac{1}{1-A_1}, \tag{1.8}$$

and for $B=-1$, $A_1=1$, we obtain

$$\left|\frac{p}{1-p\beta}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} + \beta\right) + 1\right| < 1, \tag{1.9}$$

where

$$F_\lambda(z) = (1-\lambda)f(z) + \lambda z f'(z).$$

By assigning particular values to parameters the class $\mathcal{M}_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$ generalizes many previously known classes of meromorphic functions.

(i) For $\lambda = 0$, $\alpha = 0$, $l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1, \beta_1 = c$, the class $\mathcal{M}_{[\alpha_1]}(p;\beta;\lambda;A_1,B)$ coincides with the class studied in [10].

(ii) For $p = 1$, $A = 1-2\gamma$, $0 < \gamma < 1$, $\beta = 0$, $B = -1$, $a = c = 1$, the class $\mathcal{M}_{[\alpha_1]}(p;\beta;\lambda;A_1,B)$ coincides with the class studied in [17].

2. Preliminaries

We need the following lemmas for our future investigation.

Lemma 2.1 (Jack's lemma [4]). *Let the (non constant) function $\omega(z)$ be analytic in $\mathbb{U}$, with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then $z_0\omega'(z_0) = \gamma\omega(z_0)$, where γ is real number and $\gamma \geqslant 1$.*

Lemma 2.2 ([11]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that ϕ is a complex mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real x, y such that $y \leqslant -\frac{1+x^2}{2}$. If the function $p(z) = 1 + c_1z + c_2z^2 + \cdots$ is analytic in $\mathbb{U}$ and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\operatorname{Re}(p(z)) > 0$.*

Lemma 2.3 ([20]). *Let $p(z) = 1 + b_1z + b_2z^2 + \cdots$, be analytic in $\mathbb{U}$ and η be analytic and starlike (with respect to the origin) univalent in $\mathbb{U}$ with $\eta(0) = 0$. If $zp'(z) \prec \eta(z)$ then*

$$p(z) \prec 1 + \int_0^z \frac{\eta(t)}{t} dt.$$

Unless otherwise mentioned, we shall assume that $0 \leqslant \alpha < 1$, $0 \leqslant \lambda \leqslant 1$, $-1 \leqslant B < A \leqslant 1$, $0 \leqslant p\beta < 1$ and $p \in \mathbb{N}$.

3. Main results

Theorem 3.1. *Let $f \in \Sigma_p$. Then $f \in \mathcal{M}_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$ if and only if*

$$\frac{p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} + \beta\right\} \prec -\frac{1+A_1z}{1+Bz}. \tag{3.1}$$

Proof. If $f \in \mathcal{M}_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$, then

$$\frac{p}{1-p\beta}\left\{\frac{(1-\lambda)\mathcal{H}_{p,l,m}[\alpha_1]f(z) + \lambda z\left(\mathcal{H}_{p,l,m}[\alpha_1]f(z)\right)'}{z\left(\mathcal{H}_{p,l,m}[\alpha_1]f(z)\right)' + \lambda z^2\left(\mathcal{H}_{p,l,m}[\alpha_1]f(z)\right)''} + \beta\right\} \prec -\frac{1+A_1z}{1+Bz}. \tag{3.2}$$

Let

$$F_\lambda(z) = (1-\lambda)f(z) + \lambda zf'(z),$$

so

$$\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z) = (1-\lambda)\mathcal{H}_{p,l,m}[\alpha_1]f(z) + \lambda z\mathcal{H}_{p,l,m}[\alpha_1]f'(z). \tag{3.3}$$

Using (3.2), (3.3) and after some simplifications we have

$$\frac{p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} + \beta\right\} \prec -\frac{1+A_1z}{1+Bz},$$

the converse is straight forward. □

Theorem 3.2. *If* $f \in \Sigma_p$ *satisfies anyone of the following conditions*

(i) *for* $B \neq -1$

$$\sum_{k=1}^{\infty} \left(\frac{|(k-p)\lambda_1| + |p(1-B^2)[(1+(k-p)\beta)\lambda_1] + (1-A_1B)(1-p\beta)(k-p)\lambda_1|}{(1-p\beta)(|A_1|-B)} \right) |\Gamma_k(\alpha_1)| |a_k| < p|1-\lambda-\lambda p|(1-|B|); \tag{3.4}$$

(ii) *for* $B = -1,\ A_1 \neq 1$

$$\sum_{k=1}^{\infty} \left(|(1+(k-p)\beta)\lambda_1| + \left|(1+(k-p)\beta)\lambda_1 + \frac{(1-A_1)(1-p\beta)(k-p)\lambda_1)}{p}\right| \right) |\Gamma_k(\alpha_1)| |a_k| < (1-p\beta)(1-|A_1|)|1-\lambda-\lambda p|; \tag{3.5}$$

(iii) *for* $B = -1,\ A_1 = 1$

$$\sum_{k=1}^{\infty} \left(|(k-p)\lambda_1| + \frac{k|\lambda_1|}{1-p\beta} \right) |\Gamma_k(\alpha_1)| |a_k| < p|\lambda_1|, \tag{3.6}$$

then $f \in \mathcal{M}_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$, *where* $\lambda_1 = 1+\lambda(k-p-1)$ *with* $\Gamma_k(\alpha_1) = \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k}$.

Proof. (i): If $B \neq -1$, by the condition (1.6) we only need to show that

$$\left| \frac{p(1-B^2)}{(1-p\beta)(A_1-B)} \left\{ \frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} + \beta \right\} + \frac{1-A_1B}{A_1-B} \right| < 1.$$

We first observe the

$$\begin{aligned} &\left| \frac{p(1-B^2)}{(1-p\beta)(A_1-B)} \left\{ \frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} + \beta \right\} + \frac{1-A_1B}{A_1-B} \right| \\ &= \left| \frac{pB(1-\lambda-\lambda p) + \sum_{k=1}^{\infty} \frac{p(1-B^2)[(1+(k-p)\beta)\lambda_1] + (1-A_1B)(1-p\beta)(k-p)\lambda_1}{(1-p\beta)(|A_1|-B)} \Gamma_k(\alpha_1) a_k z^k}{-p(1-\lambda-\lambda p) + \sum_{k=1}^{\infty} (k-p)\lambda_1 \Gamma_k(\alpha_1) a_k z^k} \right| \\ &\leqslant \frac{p|B(1-\lambda-\lambda p)| + \sum_{k=1}^{\infty} \frac{|p(1-B^2)[(1+(k-p)\beta)\lambda_1] + (1-A_1B)(1-p\beta)(k-p)\lambda_1|}{(1-p\beta)(|A_1|-B)} |\Gamma_k(\alpha_1)| |a_k| |z|^k}{p|1-\lambda-\lambda p| - \sum_{k=1}^{\infty} |(k-p)\lambda_1| |\Gamma_k(\alpha_1)| |a_k| |z|^k} \\ &< \frac{p|B(1-\lambda-\lambda p)| + \sum_{k=1}^{\infty} \frac{|p(1-B^2)[(1+(k-p)\beta)\lambda_1] + (1-A_1B)(1-p\beta)(k-p)\lambda_1|}{(1-p\beta)(|A_1|-B)} |\Gamma_k(\alpha_1)| |a_k|}{p|1-\lambda-\lambda p| - \sum_{k=1}^{\infty} |(k-p)\lambda_1| |\Gamma_k(\alpha_1)| |a_k|}. \end{aligned} \tag{3.7}$$

Now, by using the inequality (3.4), we have

$$\frac{p|B(1-\lambda-\lambda p)| + \sum_{k=1}^{\infty} \frac{|p(1-B^2)[(1+(k-p)\beta)\lambda_1] + (1-A_1B)(1-p\beta)(k-p)\lambda_1|}{(1-p\beta)(|A_1|-B)} |\Gamma_k(\alpha_1)| |a_k|}{p|1-\lambda-\lambda p| - \sum_{k=1}^{\infty} |(k-p)\lambda_1| |\Gamma_k(\alpha_1)| |a_k|} < 1,$$

which, in conjunction with (3.7), completes the proof of (i) for Theorem 3.2.

(ii): If $B=-1,\ A_1\neq 1$, by the virtue of the condition (1.8), we only need to show that

$$\left|\frac{(1-A_1)(1-p\beta)}{p}\left(\frac{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)+\beta z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}\right)+1\right|<1.$$

We first observe that

$$\begin{aligned}
&\left|\frac{(1-A_1)(1-p\beta)}{p}\left(\frac{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)+\beta z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}\right)+1\right|\\
&=\left|\frac{A_1(1-p\beta)(1-\lambda-\lambda p)+\sum\limits_{k=1}^{\infty}\left((1+(k-p)\beta)\lambda_1+\frac{(1-A_1)(1-p\beta)(k-p)\lambda_1}{p}\right)\Gamma_k(\alpha_1)a_kz^k}{(1-p\beta)(1-\lambda-\lambda p)+\sum\limits_{k=1}^{\infty}(1+(k-p)\beta)\lambda_1\Gamma_k(\alpha_1)a_kz^k}\right|\\
&\leqslant\frac{|A_1||1-\lambda-\lambda p|(1-p\beta)+\sum\limits_{k=1}^{\infty}\left(\left|(1+(k-p)\beta)\lambda_1+\frac{(1-A_1)(1-p\beta)(k-p)\lambda_1}{p}\right|\right)|\Gamma_k(\alpha_1)||a_k|\left|z^k\right|}{(1-p\beta)|(1-\lambda-\lambda p)|-\sum\limits_{k=1}^{\infty}|(1+(k-p)\beta)\lambda_1||\Gamma_k(\alpha_1)||a_k||z^k|}\\
&<\frac{|A_1||1-\lambda-\lambda p|(1-p\beta)+\sum\limits_{k=1}^{\infty}\left(\left|(1+(k-p)\beta)\lambda_1+\frac{(1-A_1)(1-p\beta)(k-p)\lambda_1}{p}\right|\right)|\Gamma_k(\alpha_1)||a_k|}{(1-p\beta)|1-\lambda-\lambda p|-\sum\limits_{k=1}^{\infty}|(1+(k-p)\beta)\lambda_1||\Gamma_k(\alpha_1)||a_k|}.
\end{aligned}\tag{3.8}$$

By using the inequality (3.5), we have

$$\frac{|A_1||1-\lambda-\lambda p|(1-p\beta)+\sum\limits_{k=1}^{\infty}\left(\left|(1+(k-p)\beta)\lambda_1+\frac{(1-A_1)(1-p\beta)(k-p)\lambda_1}{p}\right|\right)|\Gamma_k(\alpha_1)||a_k|}{(1-p\beta)|1-\lambda-\lambda p|-\sum\limits_{k=1}^{\infty}|(1+(k-p)\beta)\lambda_1||\Gamma_k(\alpha_1)||a_k|}<1,$$

which, in conjunction with (3.8), completes the proof of (ii) for Theorem 3.2.

(iii): If $B=1,\ A_1=1$, by virtue of the condition (1.9), we only need to show that

$$\left|\frac{p}{1-p\beta}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+1\right|<1,\qquad(z\in\mathbb{U}).$$

We first observe that

$$\begin{aligned}
\left|\frac{p}{1-p\beta}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+1\right|&=\left|\frac{\sum\limits_{k=1}^{\infty}\frac{k\lambda_1}{1-p\beta}\Gamma_k(\alpha_1)a_kz^k}{-p(1-\lambda-\lambda p)+\sum\limits_{k=1}^{\infty}(k-p)\lambda_1\Gamma_k(\alpha_1)a_kz^k}\right|\\
&\leqslant\frac{\sum\limits_{k=1}^{\infty}\frac{k|\lambda_1|}{1-p\beta}|\Gamma_k(\alpha_1)||a_k||z|^k}{p|1-\lambda-\lambda p|-\sum\limits_{k=1}^{\infty}|(k-p)\lambda_1||\Gamma_k(\alpha_1)||a_k||z|^k}\\
&<\frac{\sum\limits_{k=1}^{\infty}\frac{k|\lambda_1|}{1-p\beta}|\Gamma_k(\alpha_1)||a_k|}{p|1-\lambda-\lambda p|-\sum\limits_{k=1}^{\infty}|(k-p)\lambda_1||\Gamma_k(\alpha_1)||a_k|}.
\end{aligned}\tag{3.9}$$

Now, by using the inequality (3.6), we have

$$\frac{\sum\limits_{k=1}^{\infty}\frac{k|\lambda_1|}{1-p\beta}|\Gamma_k(\alpha_1)||a_k|}{p|1-\lambda-\lambda p|-\sum\limits_{k=1}^{\infty}|(k-p)\lambda_1||\Gamma_k(\alpha_1)||a_k|}<1,$$

which, in conjunction with (3.9), completes the proof of (iii), for Theorem 3.2. □

Theorem 3.3. *If* $f \in \Sigma_p$ *satisfies anyone of the following conditions:*

(i) *for* $B \neq -1$,

$$\left|L_{p,l,m}^{\alpha_1}(F(z))\right| < \frac{(1-p\beta)(A_1-B)}{(1-p\beta)(A_1-B)+(1+|B|)}; \tag{3.10}$$

(ii) *for* $B = -1, \ -1 < A_1 \leqslant 0$

$$\left|L_{p,l,m}^{\alpha_1}(F(z))\right| < \frac{(1-p\beta)(1-A_1)(1+A_1)}{2p\beta(1+A_1)+2(1-A_1)};$$

(iii) *for* $B = -1, \ A_1 = 1$

$$\left|L_{p,l,m}^{\alpha_1}(F(z))\right| < \frac{1-p\beta}{2-p\beta},$$

then $f \in \mathcal{M}_{[\alpha_1]}p; \alpha; \beta; \lambda; A, B)$, *where*

$$L_{p,l,m}^{\alpha_1}(F(z)) = 1 + \frac{z\left(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)\right)''}{\left(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)\right)'} - \frac{z\left(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)\right)'}{\left(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)\right)}.$$

Proof. (i) If $B \neq -1$, let

$$\omega(z) = \frac{1 + \frac{1+|B|}{1+|B|+A_1-B}\cdot\frac{p}{1-p\beta}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)}{1-\frac{1+|B|}{1+|B|+A_1-B}} - 1, \quad (z \in \mathbb{U}), \tag{3.11}$$

then the function ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. Using (3.11) and after some simplifications, we obtain

$$\frac{p\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} = \frac{(1-p\beta)(A_1-B)\omega(z)-(1+|B|)}{1+|B|}. \tag{3.12}$$

Differentiating both sides of (3.12), logarithmically we get

$$L_{p,l,m}^{\alpha_1}(F(z)) = -\frac{(1-p\beta)(A_1-B)z\omega'(z)}{(1-p\beta)(A_1-B)\omega(z)-(1+|B|)}. \tag{3.13}$$

By virtue of (3.10) and (3.13), we find that

$$\left|L_{p,l,m}^{\alpha_1}(F(z))\right| = (1-p\beta)(A_1-B)\left|\frac{z\omega'(z)}{(1-p\beta)(A_1-B)\omega(z)-(1+|B|)}\right|,$$

and

$$\left|L_{p,l,m}^{\alpha_1}(F(z))\right| < \frac{(1-p\beta)(A_1-B)}{(1-p\beta)(A_1-B)+(1+|B|)}.$$

Next, we claim that $|\omega(z)| < 1$. Indeed, if not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\leqslant|z_0|}|\omega(z)| = |\omega(z_0)| = 1, \quad (z_0 \in \mathbb{U}).$$

Applying Lemma 2.1 to $\omega(z)$ at the point z_0, we have

$$z_0\omega'(z_0) = \gamma\omega(z_0), \quad (\gamma \geqslant 1).$$

By writing

$$\omega(z_0) = e^{i\theta}, \qquad (0 \leqslant \theta \leqslant 2\pi),$$

and setting $z = z_0$ in (3.13), we get

$$\left|L^{\alpha_1}_{p,l,m}(F(z_0))\right| = (1-p\beta)(A_1-B)\left|\frac{\gamma}{(1-p\beta)(A_1-B)-(1+|B|)e^{-i\theta}}\right|,$$

which implies

$$\left|L^{\alpha_1}_{p,l,m}(F(z_0))\right| \geqslant (1-p\beta)(A_1-B)\left|\frac{1}{(1-p\beta)(A_1-B)-(1+|B|)e^{-i\theta}}\right|.$$

This implies that

$$\left|L^{\alpha_1}_{p,l,m}(F(z))\right|^2 \geqslant \frac{[(1-p\beta)(A_1-B)]^2}{[(1-p\beta)(A_1-B)]^2+(1+|B|)^2-2(1-p\beta)(A_1-B)(1+|B|)\cos\theta}. \tag{3.14}$$

Since the right hand side of (3.14) takes its minimum value for $\cos\theta = -1$, we have

$$\left|L^{\alpha_1}_{p,l,m}(F(z_0))\right|^2 \geqslant \frac{[(1-p\beta)(A_1-B)]^2}{[(1-p\beta)(A_1-B)+(1+|B|)]^2}.$$

This implies that

$$\left|\frac{p}{1-p\beta}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+1\right| < \frac{A_1-B}{1+|B|},$$

then, we have

$$\begin{aligned}
&\left|\frac{p}{1-p\beta}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+\frac{1-A_1B}{1-B^2}\right| \\
&\leqslant \left|\frac{p}{1-p\beta}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+1\right| + \left|\frac{1-A_1B}{1-B^2}-1\right| \\
&< \frac{A_1-B}{(1+|B|)} + \frac{|B|(A_1-B)}{1-B^2} \\
&= \frac{A_1-B}{1-B^2}, \qquad (B \neq -1).
\end{aligned}$$

Therefore, we conclude that $f(z) \in \mathcal{M}_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$ for $B \neq -1$.

Using similar arguments as in proof of (i), (ii) and (iii) can be easily verified. □

Theorem 3.4. *If* $f \in \Sigma_p$ *satisfies*

$$\Re e\left(L^{\alpha_1}_{p,l,m}(F(z))\right) < \begin{cases} \frac{\beta_2}{2(1-p\beta)(A_1-B)}, & B + \frac{1-B}{2(1-p\beta)} \leqslant A_1 \leqslant 1, \\ \frac{(1-p\beta)(A_1-B)}{2\beta_2}, & B < A_1 \leqslant B + \frac{1-B}{2(1-p\beta)}, \end{cases} \tag{3.15}$$

then $f(z) \in M_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$, *where* $\beta_2 = (1-A_1) + p\beta(A_1-B)$.

Proof. Let

$$g(z) = \frac{\frac{-p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right\} - \frac{1-A_1}{1-B}}{1-\frac{1-A_1}{1-B}}. \tag{3.16}$$

Then g is analytic in $\mathbb{U}$. Using (3.16), we have

$$\frac{-p\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'} = \frac{(1-p\beta)(A_1-B)g(z)+\beta_2}{1-B}. \tag{3.17}$$

Differentiating (3.17) logarithmically, we obtain

$$-L^{\alpha_1}_{p,l,m}\left(F\left(z\right)\right)=\frac{\left(1-p\beta\right)\left(A_1-B\right)zg'\left(z\right)}{\left(1-p\beta\right)\left(A_1-B\right)g\left(z\right)+\beta_2}=\prec\left(g\left(z\right),zg'\left(z\right);z\right),$$

where

$$\prec\left(r,s;t\right)=\frac{\left(1-p\beta\right)\left(A_1-B\right)s}{\left(1-p\beta\right)\left(A_1-B\right)r+\beta_2}.$$

For all real x and y satisfying $y\leqslant-\frac{1+x^2}{2}$, we have

$$\begin{aligned}\Re e\left(\prec\left(ix,y;z\right)\right)&=\frac{\left(1-p\beta\right)\left(A_1-B\right)\beta_2 y}{(\beta_2)^2+\left[\left(1-p\beta\right)\left(A_1-B\right)\right]^2x^2}\\&\leqslant-\frac{1+x^2}{2}.\frac{\left(1-p\beta\right)\left(A_1-B\right)\beta_2}{(\beta_2)^2+\left[\left(1-p\beta\right)\left(A_1-B\right)\right]^2x^2}\\&\leqslant\begin{cases}-\frac{\beta_2}{2(1-p\beta)(A_1-B)}, & B+\frac{1-B}{2(1-p\beta)}\leqslant A_1\leqslant 1,\\ -\frac{(1-p\beta)(A_1-B)}{2\beta_2}, & B<A_1\leqslant B+\frac{1-B}{2(1-p\beta)}.\end{cases}\end{aligned}$$

We know put

$$\Omega=\left\{\xi:\Re e(\xi)>\begin{cases}-\frac{\beta_2}{2(1-p\beta)(A_1-B)}, & B+\frac{1-B}{2(1-p\beta)}\leqslant A_1\leqslant 1,\\ -\frac{(1-p\beta)(A_1-B)}{2\beta_2}, & B<A_1\leqslant B+\frac{1-B}{2(1-p\beta)},\end{cases}\right\},$$

then $\prec(ix,y;z)\notin\Omega$ for all real x,y such that $y\leqslant-\frac{1+x^2}{2}$. Moreover, in view of (3.15), we know that $\prec\left(g\left(z\right),zg'\left(z\right);z\right)\in\Omega$. Thus by Lemma 2.2 we deduce that

$$\Re e\left(g\left(z\right)\right)>0,\quad\left(z\in\mathbb{U}\right),$$

which shows that the desired assertion of Theorem 3.4 holds. □

Theorem 3.5. *If* $f\in\Sigma_p$ *satisfies*

$$\Re e\left\{\frac{p\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right)}{z(\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right))'}\left(1+\eta\frac{z(\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right))''}{\left(\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right)\right)'}\right)\right\}>-\frac{1}{2}\delta_1\eta+p\eta-\left(1-\eta\right)\frac{\beta_2}{1-B},$$

then $f\in M_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$ *for* $\eta\geqslant0$, *where* $\delta_1=\left(1-p\beta\right)\left(\frac{A_1-B}{1-B}\right)$.

Proof. Let

$$h\left(z\right)=\frac{\frac{-p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right\}-\frac{1-A_1}{1-B}}{1-\frac{1-A_1}{1-B}}.\tag{3.18}$$

Then h is analytic in $\mathbb{U}$. It follows from (3.18) that

$$\frac{-p\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right)}{z(\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right))'}=\frac{\left(1-p\beta\right)\left(A_1-B\right)h\left(z\right)+\beta_2}{1-B},\tag{3.19}$$

and

$$1+\eta\frac{z(\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right))''}{\left(\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right)\right)'}=\frac{P+Qh\left(z\right)+Rzh'\left(z\right)}{\left(1-p\beta\right)\left(A_1-B\right)h\left(z\right)+\beta_2},\tag{3.20}$$

where

$$P=-p\eta\left(1-B\right)+\left(1-\eta\right)\left[\left(1-A_1\right)+p\beta\left(A_1-B\right)\right],$$

$$Q=(1-p\beta)(A_1-B)(1-\eta), \quad R=-(1-p\beta)(A_1-B)\eta,$$

combining (3.19) and (3.20), we get

$$\begin{aligned}\frac{-p\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}\left(1+\eta\frac{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))''}{(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}\right)&=-p\eta+(1-\eta)\frac{\beta_2}{(1-B)}\\&\quad+\delta_1(1-\eta)h(z)-\delta_1\eta zh'(z)\\&=\phi\left(h(z),zh'(z);z\right),\end{aligned}$$

where

$$\phi(r,s;t)=-\delta_1\eta s+\delta_1(1-\eta)r-p\eta+(1-\eta)\frac{\beta_2}{1-B}.$$

Rest of the proof follows by working in similar way as in Theorem 3.4 □

Theorem 3.6. *If* $f\in\Sigma_p$ *satisfies anyone of the following conditions:*

(i) *for* $B\neq-1$,

$$\left|\left\{\frac{p(1-B^2)}{(1-p\beta)(A_1-B)}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+\frac{1-A_1B}{A_1-B}\right\}'\right|\leqslant\eta|z|^\tau;$$

(ii) *for* $B=-1,\ A_1\neq1$,

$$\left|\left(1+\frac{(1-A_1)(1-p\beta)}{p}\left(\frac{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)+\beta z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}\right)\right)'\right|\leqslant\eta|z|^\tau;$$

(iii) *for* $B=-1,\ A_1=1$,

$$\left|\left\{\frac{p}{(1-p\beta)}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+1\right\}'\right|\leqslant\eta|z|^\tau,$$

then $f\in M_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$, *for* $0<\eta\leqslant\tau+1$ *and* $\tau\geqslant0$.

Proof. (i): If $B\neq-1$, we define the function $\Psi(z)$ by

$$\Psi(z)=z\left\{\frac{p(1-B^2)}{(1-p\beta)(A_1-B)}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+\frac{1-A_1B}{A_1-B}\right\},$$

then $\Psi(z)$ is regular in $\mathbb{U}$ and $\Psi(0)=0$. The condition of theorem gives us that

$$\left|\left(\frac{\Psi(z)}{z}\right)'\right|\leqslant\eta|z|^\tau.$$

It follows that

$$\left|\left(\frac{\Psi(z)}{z}\right)\right|=\left|\int_0^z\left(\frac{\Psi(t)}{t}\right)'dt\right|\leqslant\int_0^{|z|}\eta|t|^\tau d|t|=\frac{\eta}{\tau+1}|z|^{\tau+1}.$$

This implies that

$$\left|\left(\frac{\Psi(z)}{z}\right)\right|\leqslant\frac{\eta}{\tau+1}|z|^{\tau+1}<1,\quad(0<\eta\leqslant\tau+1,\ \tau\geqslant0).$$

Therefore, by the definition of $\Psi(z)$, we conclude that

$$\left|\frac{p\left(1-B^2\right)}{(1-p\beta)(A_1-B)}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+\frac{1-A_1B}{A_1-B}\right|<1,$$

which is equivalent to

$$\left|\frac{p}{(1-p\beta)}\left(\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right)+\frac{1-A_1B}{A_1-B}\right|<\frac{A_1-B}{1-B^2}.$$

Therefore, we conclude that $f(z)\in\mathcal{M}_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$.

(ii): If $B=-1$, $A_1\neq 1$, we define the function

$$\Psi(z)=z\left(1+\frac{(1-A_1)(1-p\beta)}{p}\left(\frac{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)+\beta z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}\right)\right),$$

then $\Psi(z)$ is regular in $\mathbb{U}$ and $\Psi(0)=0$.

Using similar arguments as in proof of (i) and (ii), condition (iii) can be easily verified. □

Theorem 3.7. *If* $f\in\Sigma_p$ *satisfies*

$$\left|\frac{\frac{1-p\beta}{p}z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)+\beta z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}\left(1+M^{\alpha_1}_{p,l,m}(F(z))\right)\right|<\frac{A_1-B}{1-A_1},$$

then $f\in\mathcal{M}_{[\alpha_1]}(p;\alpha;\beta;\lambda;A,B)$, *for* $-1\leqslant B<A_1<\frac{1+B}{2}$, *where*

$$M^{\alpha_1}_{p,l,m}(F(z))=\frac{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))''}{(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}-\frac{z\left(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)+\beta z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'\right)'}{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)+\beta z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}.$$

Proof. Let

$$q(z)=\frac{-p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z)}{z(\mathcal{H}_{p,l,m}[\alpha_1]F_\lambda(z))'}+\beta\right\}. \tag{3.21}$$

Then $q(z)$ is analytic in $\mathbb{U}$. The condition of theorem gives us that

$$\left|z\left(\frac{1}{q(z)}\right)'\right|<\frac{A_1-B}{1-A_1},$$

that is,

$$z\left(\frac{1}{q(z)}\right)'\prec\frac{A_1-B}{1-A_1}z. \tag{3.22}$$

An application of Lemma 2.3 to (3.22) yields

$$q(z)\prec\frac{1-A_1}{1-A_1+(A_1-B)z}=F(z). \tag{3.23}$$

By noting that

$$\begin{aligned}\Re\left(1+\frac{zF''(z)}{F'(z)}\right)&=\Re\left(\frac{1-A_1-(A_1-B)z}{1-A_1+(A_1-B)}\right)\\&\geqslant\frac{1-A_1-(A_1-B)}{1-A_1+(A_1-B)}\\&>0\ \left(-1\leqslant B<A_1<\frac{1+B}{2}\right),\end{aligned}$$

which implies that the region $F(\mathbb{U})$ is symmetric with respect to the real axis and F is convex univalent in

$\mathbb{U}$. Therefore, we have

$$\mathfrak{Re}\left(F\left(z\right)\right) > F\left(1\right) = \frac{1-A_1}{1-B}. \tag{3.24}$$

Combining (3.21), (3.23) and (3.24), we deduce that for $\left(-1 \leqslant B < A_1 < \frac{1+B}{2}\right)$,

$$\mathfrak{Re}\left\{\frac{p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right)}{z(\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right))'}+\beta\right\}\right\} < -\frac{1-A_1}{1-B},$$

which is equivalent to

$$\frac{p}{1-p\beta}\left\{\frac{\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right)}{z(\mathcal{H}_{p,l,m}\left[\alpha_1\right]F_\lambda\left(z\right))'}+\beta\right\} \prec -\frac{1+A_1 z}{1+Bz}.$$

This evidently completes the proof of Theorem 3.7. □

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