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On quasi-linear equation problems involving critical and singular nonlinearities

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Abstract

We consider the singular boundary value problem

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = h(x)\frac{u^{-\gamma}}{|x|^{b(1-\gamma)}} + \mu\frac{u^{p^*-1}}{|x|^{bp^*}} & \text{in } \Omega\backslash\{0\}, \\ u > 0 & \text{in } \Omega\backslash\{0\}, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a bounded domain such that $0 \in \Omega$, $0 < \gamma < 1$, $0 \leqslant a < \frac{N-p}{p}$, $a \leqslant b < a+1$, $p^* := p^*(a,b) = \frac{Np}{N-(1+a-b)p}$, and $h(x)$ is a given function. Based on different assumptions, using variational methods and Ekeland's principle, we admit that this problem possesses two positive solutions. ©2017 All rights reserved.

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1. Introduction

In this paper, we are concerned with the following problem:

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = h(x)\frac{u^{-\gamma}}{|x|^{b(1-\gamma)}} + \mu\frac{u^{p^*-1}}{|x|^{bp^*}} & \text{in } \Omega\backslash\{0\}, \\ u > 0 & \text{in } \Omega\backslash\{0\}, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geqslant 3)$ with smooth boundary $\partial\Omega$ such that $0 \in \Omega$. $\mu > 0$ is a parameter, $0 < \gamma < 1$, $0 \leqslant a < \frac{N-p}{p}$, $1 < p < N$, $a \leqslant b < a+1$, and $p^* := p^*(a,b) = \frac{Np}{N-(1+a-b)p}$ is the critical Hardy-Sobolev exponent. Throughout our paper, we assume that $h \in C(\overline{\Omega})$ and $h(x) > 0$.

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Let $W_a^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_a$, where

$$\|u\|_a = \int_\Omega \left(|x|^{-ap}|\nabla u|^p dx\right)^{\frac{1}{p}}.$$

In recent years, considerable attention has been attracted to quasilinear elliptic problems [1, 2, 8–10, 12–17, 25]. In [13], Ghoussoub and Yuan studied the existence of positive solutions for the following quasi-linear equation:

$$\begin{cases} -\Delta_p u = \lambda |u|^{r-2}u + \mu \frac{|u|^{q-2}}{|x|^s}u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $1 < p < N$, $0 \leqslant s \leqslant p$, $p \leqslant q \leqslant \frac{N-s}{N-p}p$ and $p \leqslant r \leqslant \frac{Np}{N-p}$.

We would like to mention the results of [11, 18, 26], which motivated us to discuss (1.1). In [11], Deng and Huang considered the following quasilinear elliptic equation

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \frac{\mu + h(x)}{|x|^{(a+1)p}}|u|^{p-2}u + k(x)\frac{|u|^{p^*-2}u}{|x|^{bp^*}}, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $1 < p < N$, $0 \leqslant a < \frac{N-p}{p}$, $a \leqslant b < a+1$, $0 \leqslant \mu < \left(\frac{N-p}{p} - a\right)^p$, $p^* := p^*(a,b) = \frac{Np}{N-(1+a-b)p}$, and k and h are continuously bounded functions. Under some assumptions on h and k, several multiplicity theorems of (1.2) were established. Furthermore, Kang [18] studied the following quasilinear problem

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\frac{|u|^{p-2}u}{|x|^{p(a+1)}} = \frac{|u|^{p^*(a,b)-2}u}{|x|^{bp^*(a,b)}} + \lambda\frac{|u|^{q-2}u}{|x|^{dp^*(a,d)}} & \text{in } \Omega\backslash\{0\}, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $0 \in \Omega$, $N \geqslant 3$, $\lambda > 0$, $1 < p < N$, $0 \leqslant \mu < \left(\frac{N-p}{p} - a\right)^p$, $0 \leqslant a < \frac{N-p}{p}$, $a \leqslant b, d < a+1$, $p \leqslant q < p^*(a,d) = \frac{Np}{N-p(a+1-d)}$. He investigated the extremal functions and gave some estimates. We should point out that Xuan [26] has provided some properties of eigenvalues of the following quasilinear problem

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda|x|^{-(a+1)p+c}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary, $0 \in \Omega$, $1 < p < N$, $0 \leqslant a < \frac{N-p}{p}$, and $c > 0$.

On the other hand, Giacomoni et al. [14] established the multiplicity result of (1.1) when $h(x) \equiv \lambda$, $a = b = 0$, and $\mu \equiv 1$ by critical point theory and a lower-upper solution method. Moreover, Loc and Schmitt [19] studied the following singular problem:

$$\begin{cases} -\Delta_p u = a(x)g(u) + \lambda h(u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \\ u(x) > 0, & x \in \Omega, \end{cases} \tag{1.3}$$

where $p > 1$, $g(u)$ is a singular term, $a \in L^\infty(\Omega)$, λ is a parameter and $h(u)$ is a continuous function. They constructed lower-upper solutions to show the problem (1.3) has one weak solution in $W_0^{1,p}(\Omega)$. We note that when $p = 2$, the multiplicity of positive solutions for problem (1.1) has been considered by Sun and Wu [22], Sun and Li [21], and Chen and Chen [6, 7].

In this paper, we will establish some existence and multiplicity theorems for (1.1) when $\mu \in (0, \mu^*)$ for some $\mu^* > 0$ and give the lower bounds for $\mu^* = \mu^*(\Omega, \gamma, p^*, h(x)) > 0$.

2. Preliminaries

Throughout this paper, define $\|u\|_s^s = \int_\Omega (|x|^{-b}|u|)^s dx$. We denote by the first eigenfunction e_1 with $-\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda_1 \frac{e_1^{p-1}}{|x|^{bp}}$ in Ω, $e_1 |_{\partial\Omega}= 0$, $0 \leqslant e_1 \leqslant 1$.

We set

$$S_a = \inf\left\{ \frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\left(\int_\Omega |x|^{-bp^*}|u|^{p^*} dx\right)^{\frac{p}{p^*}}} \mid u \in W_a^{1,p}(\Omega),\ u \neq 0 \right\}.$$

The infimum can be achieved by the function $U^*(x)$, where U^* is the radially symmetric ground state of the following limiting problem

$$\begin{cases} -\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \frac{u^{p^*-1}}{|x|^{bp^*}} & \text{in } \mathbb{R}^N\backslash\{0\}, \\ u(x) > 0 \quad \text{in } \mathbb{R}^N\backslash\{0\}, \\ u \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty. \end{cases}$$

The functional associated to (1.1) is

$$I_\mu(u) = \frac{1}{p}\int_\Omega |x|^{-ap}|\nabla u|^p - \frac{1}{1-\gamma}\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma} - \frac{\mu}{p^*}\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}, \quad \forall\, u \in W_a^{1,p}(\Omega).$$

We introduce the constraint set

$$\mathcal{N}_\mu = \left\{ t(u)u:\ u \in W_a^{1,p}(\Omega)\backslash\{0\} \right\},$$

where $t(u)$ are the zeros of the following map

$$\begin{aligned} t \longrightarrow \varphi(t,u) &= \frac{1}{t^{p^*-1}}\frac{d}{dt}I_\mu(tu) \\ &= t^{p-p^*}\int_\Omega |x|^{-ap}|\nabla u|^p - t^{-\gamma-p^*+1}\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma} - \mu\int_\Omega \frac{|u|^{p^*}}{|x|^{bp^*}}. \end{aligned} \tag{2.1}$$

In order to obtain our results, split $\mathcal{N}_\mu$ into the following three parts

$$\mathcal{N}_\mu^+ = \left\{ v = t(u)u \in \mathcal{N}_\mu:\ (p-p^*)\|v\|_a^p + (p^*+\gamma-1)\int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma}dx > 0 \right\},$$

$$\mathcal{N}_\mu^0 = \left\{ v = t(u)u \in \mathcal{N}_\mu:\ (p-p^*)\|v\|_a^p + (p^*+\gamma-1)\int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma}dx = 0 \right\},$$

$$\mathcal{N}_\mu^- = \left\{ v = t(u)u \in \mathcal{N}_\mu:\ (p-p^*)\|v\|_a^p + (p^*+\gamma-1)\int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma}dx < 0 \right\}.$$

A function u is called a solution of (1.1) if $u \in W_a^{1,p}(\Omega)$ such that $u(x) > 0$ a.e. in Ω and

$$\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\phi - \int_\Omega \frac{h(x)(|x|^{-b})^{1-\gamma}}{u^\gamma}\phi - \mu\int_\Omega \frac{u^{p^*-1}}{|x|^{bp^*}}\phi = 0, \quad \forall\, \phi \in W_a^{1,p}(\Omega).$$

Lemma 2.1. *Assume that* $\mu \in (0,T)$, *where*

$$T = \left(\frac{\gamma-1+p}{\gamma+p^*-1}\right)\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{p^*-p}{p-1+\gamma}}\left(\frac{1}{\|h\|_\infty}\right)^{\frac{p^*-p}{p-1+\gamma}}\left(\frac{S_\lambda}{|\Omega|^{\frac{p(1+a-b)}{N}}}\right)^{\frac{\gamma+p^*-1}{p-1+\gamma}},$$

then $\mathcal{N}_\mu^0 = \{0\}$. *Furthermore, for every* $u \in W_a^{1,p}(\Omega)\backslash\{0\}$, $\varphi(t,u)$ *has exactly two zeros* $t^\mp(u)$ *such that*

$$0 < t^-(u) < t^+(u),\quad t^-(u)u \in \mathcal{N}_\mu^+,\quad t^+(u)u \in \mathcal{N}_\mu^-.$$

Proof.

(1) Define $\varphi : (0,\infty)\times\{W_a^{1,p}(\Omega)\backslash\{0\}\} \longrightarrow \mathbb{R}$ by (2.1). Let $\varphi'(t,u)=0$, then

$$t=\left[\frac{(p^*-p)\|u\|_a^p}{(\gamma+p^*-1)\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx}\right]^{-\frac{1}{p-1+\gamma}} := t_{\max,u}.$$

We can deduce that $\varphi'(t,u)>0$ when $0<t<t_{\max,u}$ and $\varphi'(t,u)<0$ when $t>t_{\max,u}$. Furthermore, we have

$$\begin{aligned}
\varphi(t_{\max,u},u) &= \left[\frac{(p^*-p)\|u\|_a^p}{(\gamma+p^*-1)\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx}\right]^{-\frac{(p-p^*)}{p-1+\gamma}}\|u\|_a^p\\
&\quad-\left[\frac{(p^*-p)\|u\|_a^p}{(\gamma+p^*-1)\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx}\right]^{-\frac{(-\gamma-p^*+1)}{p-1+\gamma}}\\
&\quad\times\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx-\mu\int_\Omega(|x|^{-b}|u|)^{p^*}\,dx\\
&=\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{-\frac{(p-p^*)}{p-1+\gamma}}\|u\|_a^{\frac{-p(p-p^*)}{p-1+\gamma}+p}\left[\frac{1}{\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx}\right]^{-\frac{(p-p^*)}{p-1+\gamma}}\\
&\quad-\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{\gamma+p^*-1}{p-1+\gamma}}\|u\|_a^{\frac{p(\gamma+p^*-1)}{p-1+\gamma}}\left[\frac{1}{\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx}\right]^{\frac{\gamma+p^*-1}{p-1+\gamma}}\\
&\quad\times\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx\\
&\quad-\mu\int_\Omega(|x|^{-b}|u|)^{p^*}\,dx\\
&=\left(\frac{\gamma-1+p}{\gamma+p^*-1}\right)\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{p^*-p}{p-1+\gamma}}\frac{\|u\|_a^{\frac{p(p^*-1+\gamma)}{p-1+\gamma}}}{\left(\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx\right)^{\frac{p^*-p}{p+\gamma-1}}}-\mu\int_\Omega(|x|^{-b}|u|)^{p^*}\,dx.
\end{aligned}$$

Since $\|u\|_a^p\geqslant S_a\|u\|_{p^*}^p$ for every $u\in W_a^{1,p}(\Omega)\backslash\{0\}$, in view of Hölder inequality, we have

$$\begin{aligned}
\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}\,dx &\leqslant\left(\int_\Omega(|x|^{-b}|u|)^{(1-\gamma)\frac{p^*}{1-\gamma}}\,dx\right)^{\frac{1-\gamma}{p^*}}\left(\int_\Omega h(x)^{\frac{p^*}{p^*-1+\gamma}}\,dx\right)^{\frac{p^*-1+\gamma}{p^*}}\\
&\leqslant\|u\|_{p^*}^{1-\gamma}\|h\|_\infty|\Omega|^{\frac{p^*-1+\gamma}{p^*}}\\
&\leqslant\left(\frac{\|u\|_a}{\sqrt[p]{S_a}}\right)^{1-\gamma}\|h\|_\infty|\Omega|^{\frac{p^*-1+\gamma}{p^*}}.
\end{aligned}\tag{2.2}$$

Therefore,

$$\begin{aligned}
\varphi(t_{\max,u},u)&\geqslant\left(\frac{\gamma-1+p}{\gamma+p^*-1}\right)\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{p^*-p}{p-1+\gamma}}\frac{\|u\|_a^{\frac{p(p^*-1+\gamma)}{p-1+\gamma}}}{\left[\left(\frac{\|u\|_a}{\sqrt[p]{S_a}}\right)^{1-\gamma}\|h\|_\infty|\Omega|^{\frac{p^*-1+\gamma}{p^*}}\right]^{\frac{p^*-p}{p-1+\gamma}}}-\mu\left(\frac{\|u\|_a}{\sqrt[p]{S_a}}\right)^{p^*}\\
&=\left[\left(\frac{\gamma-1+p}{\gamma+p^*-1}\right)\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{p^*-p}{p-1+\gamma}}\left(\frac{1}{\|h\|_\infty}\right)^{\frac{p^*-p}{p-1+\gamma}}\frac{(\sqrt[p]{S_a})^{\frac{(1-\gamma)(p^*-p)}{p-1+\gamma}}}{|\Omega|^{\frac{(p^*-1+\gamma)(p^*-p)}{p^*(p-1+\gamma)}}}-\frac{\mu}{(\sqrt[p]{S_a})^{p^*}}\right]\|u\|_a^{p^*}
\end{aligned}$$

$$:= E(\mu)\|u\|_a^{p^*}.$$

We can see that

$$E(\mu)=0 \Leftrightarrow \quad \mu = \left(\frac{\gamma-1+p}{\gamma+p^*-1}\right)\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{p^*-p}{p-1+\gamma}}\left(\frac{1}{\|h\|_\infty}\right)^{\frac{p^*-p}{p-1+\gamma}}\frac{(\sqrt[p]{S_a})^{\frac{(1-\gamma)(p^*-p)}{p-1+\gamma}+p^*}}{|\Omega|^{\frac{(p^*-1+\gamma)(p^*-p)}{p^*(p-1+\gamma)}}}$$
$$= \left(\frac{\gamma-1+p}{\gamma+p^*-1}\right)\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{p^*-p}{p-1+\gamma}}\left(\frac{1}{\|h\|_\infty}\right)^{\frac{p^*-p}{p-1+\gamma}}\left(\frac{S_a}{|\Omega|^{\frac{(1+a-b)p}{N}}}\right)^{\frac{\gamma+p^*-1}{p-1+\gamma}}$$
$$:= T.$$

So for $\mu \in (0,T)$, we have that $E(\mu)>0$. Thus, $\varphi(t,u)$ has exactly two zeros $0<t^-(u)<t_{\max,u}<t^+(u)$ such that $\varphi'(t^-(u),u)>0>\varphi'(t^+(u),u)$. Since $\varphi(t^-(u),u)=0$, $\varphi'(t^-(u),u)>0$. Then $t^-(u)u\in\mathcal{N}_\mu$ and

$$(p-p^*)[t^-(u)]^{p-p^*-1}\|u\|_a^p+(\gamma+p^*-1)[t^-(u)]^{-\gamma-p^*}\int_\Omega h(x)(|x|^{-b}|u|)^{1-\gamma}dx>0.$$

By (2.1), we obtain that

$$(p-p^*)\|t^-(u)u\|_a^p+(\gamma+p^*-1)\int_\Omega h(x)(|t^-(u)u||x|^{-b})^{1-\gamma}dx>0.$$

That is $t^-(u)u\in\mathcal{N}_u^+$. Similarly, we get that $t^+(u)u\in\mathcal{N}_\mu^-$.

(2) Now, we will prove that $\mathcal{N}_\mu^0=\{0\}$. Assume by contradiction that there exists $u_*\in W_a^{1,p}(\Omega)\backslash\{0\}$ such that $t(u_*)u_*\in\mathcal{N}_\mu^0$, $t(u_*)u_*\neq 0$. Then

$$[t(u_*)]^{p-p^*}\|u_*\|_a^p-[t(u_*)]^{-\gamma-p^*+1}\int_\Omega h(x)(|x|^{-b}|u_*|)^{1-\gamma}dx-\mu\int_\Omega |u_*|^{p^*}dx=0,$$

and

$$(p-p^*)\|t(u_*)u_*\|_a^p+(\gamma+p^*-1)\int_\Omega h(x)(|x|^{-b}|t(u_*)u_*|)^{1-\gamma}dx=0. \tag{2.3}$$

Hence, for $\mu\in(0,T)$ and $u_*\in W_a^{1,p}(\Omega)\backslash\{0\}$, it follows from (2.2) and (2.3) that

$$0<E(\mu)\|u_*\|_a^{p^*}$$
$$\leqslant \left(\frac{\gamma-1+p}{\gamma+p^*-1}\right)\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{p^*-p}{p-1+\gamma}}\frac{\|u_*\|_a^{\frac{p(p^*-1+\gamma)}{p-1+\gamma}}}{\left(\int_\Omega h(x)(|x|^{-b}|u_*|)^{1-\gamma}dx\right)^{\frac{p^*-p}{p-1+\gamma}}}-\mu\int_\Omega\frac{|u_*|^{p^*}}{|x|^{bp^*}}dx$$
$$=\left(\frac{\gamma-1+p}{\gamma+p^*-1}\right)\left(\frac{p^*-p}{\gamma+p^*-1}\right)^{\frac{p^*-p}{p-1+\gamma}}\frac{\|u_*\|_a^{\frac{p(p^*-1+\gamma)}{p-1+\gamma}}}{\left(\frac{-p+p^*}{\gamma+p^*-1}[t(u_*)]^{p-1+\gamma}\|u_*\|_a^p\right)^{\frac{p^*-p}{p-1+\gamma}}}$$
$$-\left(\frac{p-1+\gamma}{\gamma+p^*-1}\right)[t(u_*)]^{p-p^*}\|u_*\|_a^p=0,$$

this is a contradiction. Therefore, $t(u_*)u_*=0$.

□

Lemma 2.2. *Assume that $\mu\in(0,T)$, then $\mathcal{N}_\mu$ has the following properties*

$$\|V\|_a>A(\mu)>A_0>\|v\|_a,\quad \forall\, V\in\mathcal{N}_\mu^-,\ v\in\mathcal{N}_\mu^+,$$

where

$$A_0 = \left[\left(\frac{\gamma+p^*-1}{p^*-p}\right)\left(\frac{1}{\sqrt[p]{S_a}}\right)^{1-\gamma}\|h\|_\infty|\Omega|^{\frac{p^*-1+\gamma}{p^*}}\right]^{\frac{1}{p-1+\gamma}},$$

$$A(\mu) = \left[\frac{(p+\gamma-1)(\sqrt[p]{S_a})^{p^*}}{\mu(\gamma+p^*-1)}\right]^{\frac{1}{p^*-p}}.$$

Proof. If $v \in \mathcal{N}_\mu^+$, then

$$(p^*-p)\|v\|_a^p < (\gamma+p^*-1)\int_\Omega h(x)(|x|^{-b}|v|)|^{1-\gamma}dx.$$

From (2.2), we deduce that

$$\begin{aligned}\|v\|_a^p &< \frac{\gamma+p^*-1}{p^*-p}\int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma}dx\\ &\leqslant \left(\frac{\gamma+p^*-1}{p^*-p}\right)\left(\frac{\|v\|_a}{\sqrt[p]{S_a}}\right)^{1-\gamma}\|h\|_\infty|\Omega|^{\frac{p^*-1+\gamma}{p^*}},\end{aligned}$$

which yields

$$\begin{aligned}\|v\|_a &< \left[\left(\frac{\gamma+p^*-1}{p^*-p}\right)\left(\frac{1}{\sqrt[p]{S_a}}\right)^{1-\gamma}\|h\|_\infty|\Omega|^{\frac{p^*-1+\gamma}{p^*}}\right]^{\frac{1}{p-1+\gamma}}\\ &:= A_0.\end{aligned}$$

If $V \in \mathcal{N}_\mu^-$, we have

$$\begin{aligned}\|V\|_a^p &< \left(\frac{\gamma+p^*-1}{p+\gamma-1}\right)\mu\int_\Omega(|x|^{-b}|V|)^{p^*}dx\\ &\leqslant \left(\frac{\gamma+p^*-1}{p+\gamma-1}\right)\mu\left(\frac{\|V\|_a}{\sqrt[p]{S_a}}\right)^{p^*},\end{aligned}$$

which implies

$$\|V\|_a > \left[\frac{(p+\gamma-1)(\sqrt[p]{S_a})^{p^*}}{\mu(\gamma+p^*-1)}\right]^{\frac{1}{p^*-p}} := A(\mu).$$

In the following, we show that

$$\mu = T \Leftrightarrow A(\mu) = A_0.$$

In fact

$$\begin{aligned}\mu = T \Leftrightarrow A(\mu) &= \left(\frac{\gamma+p^*-1}{\gamma-1+p}\right)^{\frac{1}{p^*-p}}\left(\frac{\gamma+p^*-1}{p^*-p}\right)^{\frac{1}{p-1+\gamma}}(\|h\|_\infty)^{\frac{1}{p-1+\gamma}}\\ &\quad\times\frac{|\Omega|^{\frac{p(1+a-b)}{N}\frac{(\gamma+p^*-1)}{(p-1+\gamma)(p^*-p)}}}{(S_a)^{\frac{\gamma+p^*-1}{(p-1+\gamma)(p^*-p)}}}\left(\frac{p+\gamma-1}{\gamma+p^*-1}\right)^{\frac{1}{p^*-p}}(\sqrt[p]{S_a})^{\frac{p^*}{p^*-p}}\\ &= \left(\frac{\gamma+p^*-1}{p^*-p}\right)^{\frac{1}{p-1+\gamma}}\|h\|_\infty^{\frac{1}{p-1+\gamma}}\frac{|\,\Omega\,|^{\frac{p(1+a-b)(\gamma+p^*-1)}{N(p-1+\gamma)(p^*-p)}}}{(\sqrt[p]{S_a})^{\frac{p(\gamma+p^*-1)}{(p-1+\gamma)(p^*-p)}-\frac{p^*}{p^*-p}}}\\ &= \left[\left(\frac{\gamma+p^*-1}{p^*-p}\right)\|h\|_\infty\frac{|\Omega|^{\frac{p^*+\gamma-1}{p^*}}}{(\sqrt[p]{S_a})^{1-\gamma}}\right]^{\frac{1}{p-1+\gamma}}\\ &:= A_0.\end{aligned}$$

□

Lemma 2.3. *Assume that $\mu \in (0, T)$, then $\mathcal{N}_\mu^-$ is a closed set in $W_a^{1,p}$-topology.*

Proof. The proof is identical to that of [21, Lemma 2], we omit it here. □

Lemma 2.4. *Given* $v \in \mathcal{N}_\mu^{\pm}$, *then for every* $\phi \in W_a^{1,p}(\Omega)$, *there exist* $\epsilon > 0$ *and a continuous function* $f(w) > 0$, $w \in W_a^{1,p}(\Omega)$, $\|w\| < \epsilon$ *such that*

$$f(0) = 1, \quad \textit{and} \quad \frac{v + w\phi}{f(w)} \in \mathcal{N}_\mu^{\pm}, \quad \forall\, w \in W_a^{1,p}(\Omega), \quad \|w\| < \epsilon.$$

Proof. The proof is identical to that of [20, Lemma 2.4], we omit it here. □

3. Solution of (1.1) for all $\mu \in (0, T)$

Theorem 3.1. *Suppose that* $\mu \in (0, T)$. *Then the problem* (1.1) *has a solution* $v_0 \in W_a^{1,p}(\Omega)$ *satisfying* $I_\mu(v_0) < 0$ *and* $\|v_0\|_a \leqslant A_0$.

Proof. For every $v \in \mathcal{N}_\mu$, we deduce from (2.2) that

$$\begin{aligned} I_\mu(v) &= \frac{1}{p}\int_\Omega |x|^{-ap}|\nabla v|^p\,dx - \frac{1}{1-\gamma}\int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma}\,dx - \frac{\mu}{p^*}\int_\Omega \frac{|v|^{p^*}}{|x|^{bp^*}}\,dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega |x|^{-ap}|\nabla v|^p\,dx - \left(\frac{1}{1-\gamma} - \frac{1}{p^*}\right)\int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma}\,dx \\ &\geqslant \left(\frac{1}{p} - \frac{1}{p^*}\right)\|v\|_a^p - \left(\frac{1}{1-\gamma} - \frac{1}{p^*}\right)\left(\frac{1}{\sqrt[p]{S_a}}\right)^{1-\gamma}\|h\|_\infty|\Omega|^{\frac{p^*-1+\gamma}{p^*}}\|v\|_a^{1-\gamma}. \end{aligned}$$

Therefore, I_μ is coercive and bounded below in $\mathcal{N}_\mu$.

By Lemma 2.3, we have that $\mathcal{N}_\mu^+ \cup \{0\}$ and $\mathcal{N}_\mu^-$ are two closed sets in $W_a^{1,p}(\Omega)$ when $\mu \in (0, T)$. In terms of Ekeland's variational principle [3], we can find a sequence $(v_n) \subset \mathcal{N}_\mu^+ \cup \{0\}$ such that

(i) $I_\mu(v_n) < \inf_{\mathcal{N}_\mu^+ \cup \{0\}} I_\mu + \frac{1}{n}$;

(ii) $I_\mu(v) \geqslant I_\mu(v_n) - \frac{1}{n}\|v - v_n\|, \quad \forall\, v \in \mathcal{N}_\mu^+ \cup \{0\}$.

We may assume $v_n \geqslant 0$ on $\Omega\backslash\{0\}$. Since I_μ is bounded below in $\mathcal{N}_\mu$, by above property (i), we know that (v_n) is bounded in $W_a^{1,p}(\Omega)$. Going if necessary to a subsequence,

$$\begin{aligned} &v_n \rightharpoonup v_0 \ \text{ weakly in } W_a^{1,p}(\Omega) \text{ and } L^{p^*}(\Omega), \\ &v_n \to v_0 \ \text{ a.e. in } \Omega, \\ &v_n \to v_0 \ \text{ strongly in } L^{1-\gamma}(\Omega). \end{aligned}$$

Let $v_n = v_0 + w_n$ with $w_n \rightharpoonup 0$ weakly in $W_a^{1,p}(\Omega)$. For every $v \in \mathcal{N}_\mu^+$, it follows from $p > 1$ that

$$\begin{aligned} I_\mu(v) &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\|v\|_a^p - \left(\frac{1}{1-\gamma} - \frac{1}{p^*}\right)\frac{p^* - p}{\gamma + p^* - 1}\|v\|_a^p \\ &= \frac{p^* - p}{p^*}\left(\frac{1}{p} - \frac{1}{1-\gamma}\right)\|v\|_a^p < 0, \end{aligned}$$

that is,

$$\inf_{\mathcal{N}_\mu^+ \cup \{0\}} I_\mu = \inf_{\mathcal{N}_\mu^+} I_\mu < 0.$$

Moreover, $I_\mu(v_0) \leqslant \liminf\limits_{n \longrightarrow \infty} I_\mu(v_n) = \inf_{\mathcal{N}_\mu^+ \cup \{0\}} I_\mu$. Hence, $v_0 \not\equiv 0$ and $(v_n) \subset \mathcal{N}_\mu^+$.

Claim 1. $v_0(x) > 0$ *a.e. in* Ω *and*

$$\int_\Omega h(x)(|x|^{-b})^{1-\gamma} v_0^{-\gamma}\phi dx \leqslant \int_\Omega |x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0 \nabla\phi dx - \mu\int_\Omega \frac{v_0^{p^*-1}}{|x|^{bp^*}}\phi dx, \ \ \forall\, \phi \in W_a^{1,p}(\Omega), \ \ \phi \geqslant 0.$$

For n sufficiently large and a suitable positive constant C_1, we have

$$(p^*-p)\|v_n\|_a^p - (\gamma+p^*-1)\int_\Omega h(x)(|x|^{-b}v_n)^{1-\gamma}dx \leqslant -C_1 < 0. \tag{3.1}$$

Let $\phi \in W_0^{1,p}(\Omega)$ with $\phi \geqslant 0$. By Lemma 2.4, we can find a continuous function $f_n(t)$ satisfying $f_n(0) = 1$ and $\frac{1}{f_n(t)}(v_n + t\phi) \in \mathcal{N}_\mu^+$ for every v_n. It follows from $v_n \in \mathcal{N}_\mu$ and $\frac{1}{f_n(t)}(v_n + t\phi) \in \mathcal{N}_\mu$ that

$$\|v_n\|_a^p - \int_\Omega h(x)(|x|^{-b}v_n)^{1-\gamma}dx - \mu\int_\Omega (|x|^{-b}v_n)^{p^*}dx = 0,$$

and

$$\frac{1}{f_n^p(t)}\|v_n+t\phi\|_a^p - \frac{1}{f_n^{1-\gamma}(t)}\int_\Omega h(x)[|x|^{-b}(v_n+t\phi)]^{1-\gamma}dx - \mu\frac{1}{f_n^{p^*}(t)}\int_\Omega [|x|^{-b}(v_n+t\phi)]^{p^*}dx = 0,$$

so we have that

$$\begin{aligned} 0 \ &= \left(\frac{1}{f_n^p(t)}-1\right)\|v_n+t\phi\|_a^p + (\|v_n+t\phi\|_a^p - \|v_n\|_a^p) \\ &\quad - \left(\frac{1}{f_n^{1-\gamma}(t)}-1\right)\int_\Omega h(x)[|x|^{-b}(v_n+t\phi)]^{1-\gamma}dx \\ &\quad - \int_\Omega h(x)[(|x|^{-b}(v_n+t\phi))^{1-\gamma} - (|x|^{-b}v_n)^{1-\gamma}]dx \\ &\quad - \mu\left(\frac{1}{f_n^{p^*}(t)}-1\right)\int_\Omega [|x|^{-b}(v_n+t\phi)]^{p^*}dx - \mu\int_\Omega [(|x|^{-b}(v_n+t\phi))^{p^*} - (|x|^{-b}v_n)^{p^*}]dx \\ &\leqslant \left(\frac{1}{f_n^p(t)}-1\right)\|v_n+t\phi\|_a^p + (\|v_n+t\phi\|_a^p - \|v_n\|_a^p) \\ &\quad - \left(\frac{1}{f_n^{1-\gamma}(t)}-1\right)\int_\Omega h(x)[|x|^{-b}(v_n+t\phi)]^{1-\gamma}dx - \mu\left(\frac{1}{f_n^{p^*}(t)}-1\right)\int_\Omega [|x|^{-b}(v_n+t\phi)]^{p^*}dx. \end{aligned}$$

Dividing by $t > 0$ and letting $t \to 0$, we conclude that

$$\begin{aligned} 0 \ &\leqslant -pf_n'(0)\|v_n\|_a^p + p\int_\Omega |x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n\nabla\phi dx \\ &\quad + (1-\gamma)f_n'(0)\int_\Omega h(x)(|x|^{-b}v_n)^{1-\gamma}dx + \mu p^* f_n'(0)\int_\Omega (|x|^{-b}v_n)^{p^*}dx \\ &= f_n'(0)\left[(p^*-p)\|v_n\|_a^p - (\gamma+p^*-1)\int_\Omega h(x)(|x|^{-b}v_n)^{1-\gamma}dx\right] \\ &\quad + p\int_\Omega |x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n\nabla\phi dx. \end{aligned} \tag{3.2}$$

It follows from (3.1) and (3.2) that $f_n'(0) \neq +\infty$. Now we show that $f_n'(0) \neq -\infty$. If no, we assume that $f_n'(0) = -\infty$. Since

$$\begin{aligned} |\tfrac{1}{f_n(t)} - 1|\|v_n\|_a + \tfrac{t}{f_n(t)}\|\phi\|_a \ &\geqslant \|(\tfrac{1}{f_n(t)}-1)v_n + \tfrac{t}{f_n(t)}\phi\|_a \\ &= \|\tfrac{1}{f_n(t)}(v_n+t\phi) - v_n\|_a, \end{aligned} \tag{3.3}$$

and $f_n(t) > f_n(0) = 1$ for n sufficiently large, using condition (ii) with $\nu = \frac{1}{f_n(t)}(\nu_n + t\phi) \in \mathcal{N}_\mu^+$, we obtain that

$$\begin{aligned}
&\left|\frac{1}{f_n(t)} - 1\right| \frac{\|\nu_n\|_a}{n} + \frac{t}{f_n(t)}\frac{\|\phi\|_a}{n} \\
&\quad \geqslant \frac{1}{n}\left\|\frac{1}{f_n(t)}(\nu_n + t\phi) - \nu_n\right\|_a \\
&\quad \geqslant I_\mu(\nu_n) - I_\mu\left[\tfrac{1}{f_n(t)}(\nu_n + t\phi)\right] \\
&\quad = \frac{1}{p}\|\nu_n\|_a^p - \frac{1}{1-\gamma}\int_\Omega h(x)(|x|^{-b}\nu_n)^{1-\gamma}dx - \frac{1}{p^*}\left[\|\nu_n\|_a^p - \int_\Omega h(x)(|x|^{-b}\nu_n)^{1-\gamma}dx\right] \\
&\qquad - \frac{1}{p}\frac{1}{f_n^p(t)}\|\nu_n + t\phi\|_a^p + \frac{1}{1-\gamma}\frac{1}{f_n^{1-\gamma}(t)}\int_\Omega h(x)[|x|^{-b}(\nu_n + t\phi)]^{1-\gamma}dx \\
&\qquad + \frac{1}{p^*}\left[\frac{1}{f_n^p(t)}\|\nu_n + t\phi\|_a^p - \frac{1}{f_n^{1-\gamma}(t)}\int_\Omega h(x)[|x|^{-b}(\nu_n + t\phi)]^{1-\gamma}dx\right] \\
&\quad = \left(\frac{1}{p} - \frac{1}{p^*}\right)\|\nu_n\|_a^p - \left(\frac{1}{p} - \frac{1}{p^*}\right)\frac{1}{f_n^p(t)}\|\nu_n + t\phi\|_a^p \\
&\qquad + \left(\frac{1}{p^*} - \frac{1}{1-\gamma}\right)\int_\Omega h(x)(|x|^{-b}\nu_n)^{1-\gamma}dx \\
&\qquad - \left(\frac{1}{p^*} - \frac{1}{1-\gamma}\right)\frac{1}{f_n^{1-\gamma}(t)}\int_\Omega h(x)[|x|^{-b}(\nu_n + t\phi)]^{1-\gamma}dx \\
&\quad = \left(\frac{1}{p^*} - \frac{1}{p}\right)[\|\nu_n + t\phi\|_a^p - \|\nu_n\|_a^p] + \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(1 - \frac{1}{f_n^p(t)}\right)\|\nu_n + t\phi\|_a^p \\
&\qquad - \left(\frac{1}{p^*} - \frac{1}{1-\gamma}\right)\frac{1}{f_n^{1-\gamma}(t)}\int_\Omega h(x)[(|x|^{-b}(\nu_n + t\phi))^{1-\gamma} - (|x|^{-b}\nu_n)^{1-\gamma}]dx \\
&\qquad + \left(\frac{1}{p^*} - \frac{1}{1-\gamma}\right)\left(1 - \frac{1}{f_n^{1-\gamma}(t)}\right)\int_\Omega h(x)(|x|^{-b}\nu_n)^{1-\gamma}dx.
\end{aligned}$$

Dividing by $t > 0$, and letting $t \to 0$, we know that

$$\begin{aligned}
f_n'(0)\frac{\|\nu_n\|_a}{n} + \frac{\|\phi\|_a}{n} \quad &\geqslant \frac{f_n'(0)}{p^*}\left[(p^* - p)\|\nu_n\|_a^p + (1 - \gamma - p^*)\int_\Omega h(x)(|x|^{-b}\nu_n)^{1-\gamma}dx\right] \\
&\quad + \left(\frac{p - p^*}{p^*}\right)\int_\Omega |x|^{-ap}|\nabla\nu_n|^{p-2}\nabla\nu_n\nabla\phi dx,
\end{aligned}$$

that is

$$\begin{aligned}
\frac{\|\phi\|_a}{n} \geqslant \frac{f_n'(0)}{p^*}&\left[(p^* - p)\|\nu_n\|_a^p + (1 - \gamma - p^*)\int_\Omega h(x)[|x|^{-b}\nu_n]^{1-\gamma}dx - \frac{p^*}{n}\|\nu_n\|_a\right] \\
&+ \int_\Omega |x|^{-ap}|\nabla\nu_n|^{p-2}\nabla\nu_n\nabla\phi dx,
\end{aligned} \tag{3.4}$$

which contradicts with $f_n'(0) = -\infty$ and

$$(p^* - p)\|\nu_n\|_a^p + (1 - \gamma - p^*)\int_\Omega h(x)[|x|^{-b}\nu_n]^{1-\gamma}dx - \frac{p^*}{n}\|\nu_n\|_a \leqslant -C_2 < 0.$$

In conclusion, $|f_n'(0)| < +\infty$. It follows from (3.1), (3.2) and (3.4) that $|f_n'(0)| \leqslant C_3$ for n sufficiently large and a suitable positive constant C_3.

Now, by (3.3) and condition (ii), we get

$$\begin{aligned}
&\frac{1}{n}\left[\left|\frac{1}{f_n(t)}-1\right|\|v_n\|_a+t\frac{1}{f_n(t)}\|\phi\|_a\right]\\
&\geqslant I_\mu(v_n)-I_\mu\left[\frac{1}{f_n(t)}(v_n+t\phi)\right]\\
&=\frac{1}{p}\|v_n\|_a^p-\frac{1}{1-\gamma}\int_\Omega h(x)[|x|^{-b}v_n]^{1-\gamma}dx-\frac{\mu}{p^*}\int_\Omega[|x|^{-b}v_n]^{p^*}dx-\frac{1}{p}\frac{1}{f_n^p(t)}\|v_n+t\phi\|_a^p\\
&\quad+\frac{1}{(1-\gamma)f_n^{1-\gamma}(t)}\int_\Omega h(x)[|x|^{-b}|v_n+t\phi|]^{1-\gamma}dx+\frac{\mu}{p^*f_n^{p^*}(t)}\int_\Omega[|x|^{-b}|v_n+t\phi|]^{p^*}dx\\
&=-\frac{1}{pf_n^p(t)}[\|v_n+t\phi\|_a^p-\|v_n\|_a^p]+\left(\frac{f_n^p(t)-1}{pf_n^p(t)}\right)\|v_n\|_a^p\\
&\quad+\frac{1}{(1-\gamma)f_n^{1-\gamma}(t)}\int_\Omega h(x)[(|x|^{-b}(v_n+t\phi))^{1-\gamma}-(|x|^{-b}v_n)^{1-\gamma}]dx\\
&\quad+\left(\frac{1-f_n^{1-\gamma}(t)}{(1-\gamma)f_n^{1-\gamma}(t)}\right)\int_\Omega h(x)[|x|^{-b}v_n]^{1-\gamma}dx\\
&\quad+\frac{\mu}{p^*f_n^{p^*}(t)}\int_\Omega[(|x|^{-b}(v_n+t\phi))^{p^*}-(|x|^{-b}v_n)^{p^*}]dx+\left(\frac{\mu(1-f_n^{p^*}(t))}{p^*f_n^{p^*}(t)}\right)\int_\Omega(|x|^{-b}v_n)^{p^*}dx.
\end{aligned}$$

Dividing by $t>0$, and passing to the limit as $t\to 0^+$, we get

$$\begin{aligned}
&\frac{1}{n}[|f_n'(0)|\|v_n\|_a+\|\phi\|_a]\\
&\geqslant f_n'(0)\left[\|v_n\|_a^p-\int_\Omega h(x)[|x|^{-b}v_n]^{1-\gamma}dx-\mu\int_\Omega[|x|^{-b}v_n]^{p^*}dx\right]-\int_\Omega|x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n\nabla\phi dx\\
&\quad+\mu\int_\Omega(|x|^{-b})^{p^*}(v_n)^{p^*-1}\phi dx+\liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega\frac{h(x)[(|x|^{-b}(v_n+t\phi))^{1-\gamma}-(|x|^{-b}v_n)^{1-\gamma}]}{t}dx\\
&=-\int_\Omega|x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n\nabla\phi dx+\mu\int_\Omega(|x|^{-b})^{p^*}(v_n)^{p^*-1}\phi dx\\
&\quad+\liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega\frac{h(x)[(|x|^{-b}(v_n+t\phi))^{1-\gamma}-(|x|^{-b}v_n)^{1-\gamma}]}{t}dx,
\end{aligned}$$

which yields,

$$\begin{aligned}
&\liminf_{t\to0^+}\frac{1}{1-\gamma}\int_\Omega\frac{h(x)[(|x|^{-b}(v_n+t\phi))^{1-\gamma}-(|x|^{-b}v_n)^{1-\gamma}]}{t}dx\\
&\quad\leqslant\int_\Omega|x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n\nabla\phi dx-\mu\int_\Omega(|x|^{-b})^{p^*}(v_n)^{p^*-1}\phi dx+\frac{|f_n'(0)|\|v_n\|_a+\|\phi\|_a}{n}.
\end{aligned}$$

Applying Fatou's Lemma, we have

$$\begin{aligned}
&\int_\Omega\liminf_{t\to0^+}\left[\frac{1}{1-\gamma}\frac{h(x)[(|x|^{-b}(v_n+t\phi))^{1-\gamma}-(|x|^{-b}v_n)^{1-\gamma}]}{t}\right]dx\\
&\quad\leqslant\int_\Omega|x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n\nabla\phi dx-\mu\int_\Omega(|x|^{-b})^{p^*}(v_n)^{p^*-1}\phi dx+\frac{|f_n'(0)|\|v_n\|_a+\|\phi\|_a}{n}.
\end{aligned}\tag{3.5}$$

We take $\phi=e_1$ as a test-function in (3.5), we know that $v_n(t)>0$ a.e. in Ω, then

$$\begin{aligned}
\int_\Omega h(x)(|x|^{-b})^{1-\gamma}(v_n)^{-\gamma}\phi dx\leqslant&\int_\Omega|x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n\nabla\phi dx\\
&-\mu\int_\Omega(|x|^{-b})^{p^*}(v_n)^{p^*-1}\phi dx+\frac{|f_n'(0)|\|v_n\|_a+\|\phi\|_a}{n},
\end{aligned}$$

and note that $f_n'(0)$ is uniformly bounded in n, we can deduce that $v_0(x) > 0$ a.e. in Ω, and

$$\begin{aligned}\int_\Omega h(x)(|x|^{-b})^{1-\gamma}(v_0)^{-\gamma}\phi dx \leqslant \int_\Omega |x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0 \nabla\phi dx \\ - \mu\int_\Omega (|x|^{-b})^{p^*}(v_0)^{p^*-1}\phi dx, \quad \forall\, \phi \in W_a^{1,p}(\Omega), \quad \phi \geqslant 0.\end{aligned} \tag{3.6}$$

For $c \in \Omega$ let $\eta \in C_0^\infty(\Omega)$ such that $0 \leqslant \eta(x) \leqslant 1$ in Ω and $\eta(x) = 1$, for all $x \in \bar{B}_r(c) \subset \Omega$ for a suitable $r > 0$. Set

$$U_{\varepsilon,c}(x) = \epsilon^{-\frac{N-p}{p}}\eta(x)U^*\left(\frac{|x-a|}{\epsilon}\right) \in W_a^{1,p}(\Omega).$$

It is well-known that

$$\|U_{\epsilon,c}\|_a^p = B + O(\epsilon^{N-p}), \quad \|U_{\epsilon,c}\|_{p^*}^{p^*} = A + O(\epsilon^N),$$

and $S_a = \frac{B}{A^{\frac{p}{p^*}}}$, where $B = \int_{R^N}|x|^{-ap}|\nabla U^*|^p dx$, $A = \int_{R^N}(x^{-b}|U^*|)^{p^*}dx$ ([5, 23, 24]).

Claim 2. $v_0 \in \mathcal{N}_\mu$ *with* $\mu \in (0, T)$.

Let

$$a_0 = \|v_0\|_a^p - \int_\Omega h(x)(|x|^{-b}|v_0|)^{1-\gamma}dx - \mu\int_\Omega (|x|^{-b}|v_0|)^{p^*}dx.$$

Take $\phi = v_0$ in (3.6), we have that $a_0 \geqslant 0$. If no, and suppose that $a_0 > 0$. There exists a unique $\tau_0 > 0$ such that $\tau_0^p B - \mu\tau_0^{p^*}A = -a_0$. Note that $I_\mu(v_n) \to v_0 := \inf_{\mathcal{N}_\mu^+\cup\{0\}} I_\mu = \inf_{\mathcal{N}_\mu^+} I_\mu$ with $v_n \in \mathcal{N}_\mu^+(\subset \mathcal{N}_\mu)$, it follows from Brezis-lieb Lemma [4] that

$$\begin{aligned} v_0 + o(1) &= I_\mu(v_n) \\ &= \left(\frac{1}{p} - \frac{1}{1-\gamma}\right)\int_\Omega h(x)(|x|^{-b}|v_n|)^{1-\gamma}dx + \mu\left(\frac{1}{p} - \frac{1}{p^*}\right)\|v_n\|_{p^*}^{p^*} \\ &= \left(\frac{1}{p} - \frac{1}{1-\gamma}\right)\int_\Omega h(x)(|x|^{-b}|v_0|)^{1-\gamma}dx + \mu\left(\frac{1}{p} - \frac{1}{p^*}\right)\|v_0\|_{p^*}^{p^*} + \mu\left(\frac{1}{p} - \frac{1}{p^*}\right)\|w_n\|_{p^*}^{p^*} + o(1),\end{aligned}$$

and

$$\begin{aligned} 0 &= \|v_n\|_a^p - \int_\Omega h(x)(|x|^{-b}|v_n|)^{1-\gamma}dx - \mu\int_\Omega (|x|^{-b}|v_n|)^{p^*}dx \\ &= a_0 + \|w_n\|_a^p - \mu\|w_n\|_{p^*}^{p^*} + o(1) \\ &\geqslant a_0 + S_a\|w_n\|_{p^*}^p - \mu\|w_n\|_{p^*}^{p^*} + o(1),\end{aligned}$$

which means $\lim\limits_{n\to\infty}\|w_n\|_{p^*}$ exists and $\lim\limits_{n\to\infty}\|w_n\|_{p^*} \geqslant \tau_0 A^{\frac{1}{p^*}}$. That is to say, we have

$$v_0 \geqslant \left(\frac{1}{p} - \frac{1}{1-\gamma}\right)\int_\Omega h(x)(|x|^{-b}|v_0|)^{1-\gamma}dx + \mu\left(\frac{1}{p} - \frac{1}{p^*}\right)\|v_0\|_{p^*}^{p^*} + \mu\left(\frac{1}{p} - \frac{1}{p^*}\right)\tau_0^{p^*}A. \tag{3.7}$$

For every $v \in W_a^{1,p}(\Omega)$ with $a_v = \|v\|_a^p - \int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma}dx - \mu\|v\|_{p^*}^{p^*} > 0$, there exists $R_v > 0$ such that $a_v + R_v^p B - \mu R_v^{p^*}A < 0$, and hence

$$\begin{aligned} \|v + R_v U_{\epsilon,c}\|_a^p &- \int_\Omega h(x)(|x|^{-b}|v + R_v U_{\epsilon,c}|)^{1-\gamma}dx - \mu\|v + R_v U_{\epsilon,c}\|_{p^*}^{p^*} \\ &= \|v\|_a^p + R_v^p\|U_{\epsilon,c}\|_a^p - \int_\Omega h(x)(|x|^{-b}|v + R_v U_{\epsilon,c}|)^{1-\gamma}dx - \mu\left[\|v\|_{p^*}^{p^*} + R_v^{p^*}\|U_{\epsilon,c}\|_{p^*}^{p^*} + o(1)\right] \\ &= a_v + R_v^p B - \mu R_v^{p^*}A + o(1) < 0,\end{aligned}$$

for $\epsilon > 0$ small enough. Consequently, we can take $0 < d_{\epsilon,v} < R_v$ to hold

$$\|v + d_{\epsilon,v}U_{\epsilon,c}\|_a^p - \int_\Omega h(x)(|x|^{-b}|v + d_{\epsilon,v}U_{\epsilon,c}|)^{1-\gamma}dx - \mu\int_\Omega (|x|^{-b}|v + d_{\epsilon,v}U_{\epsilon,c}|)^{p^*}dx = 0. \tag{3.8}$$

Thus $v + d_{\varepsilon,v}U_{\varepsilon,c} \in \mathcal{N}_\mu$. In addition, note that $a_v > 0$, let $d_v > 0$ be the unique positive number satisfying

$d_v^p B - \mu d_v^{p^*} A = -a_v$. By (3.6), we get

$$a_v + d_{\epsilon,v}^p B - \mu d_{\epsilon,v}^{p^*} A + o(1) = 0,$$

and thus $d_{\epsilon,v} \to d_v$ as $\epsilon \to 0$ which leads to

$$\|v + d_{\epsilon,v} U_{\epsilon,c}\|_a^p = \|v\|_a^p + d_v^p B + o(1) > d_v^p B > (\tfrac{B}{\mu A})^{\frac{p}{p^*-p}} B = (\tfrac{1}{\mu})^{\frac{p}{p^*-p}} S_a^{\frac{N}{p}},$$

for $\epsilon > 0$ sufficiently small. Since

$$\|v + d_{\epsilon,v} U_{\epsilon,c}\|_a > \left(\frac{1}{\mu}\right)^{\frac{1}{p^*-p}} \sqrt[p]{S_a}^{\frac{N}{p}} > \left(\frac{p+\gamma-1}{\mu(\gamma+p^*-1)}\right)^{\frac{1}{p^*-p}} \sqrt[p]{S_a}^{\frac{N}{p}} = A(\mu).$$

By Lemma 2.2, we have $v + d_{\epsilon,v} U_{\epsilon,c} \in \mathcal{N}_\mu^-$. Note that $\inf_{\mathcal{N}_\mu^+} I_\mu = \inf_{\mathcal{N}_\mu} I_\mu$, we obtain that

$$\begin{aligned} v_0 &\leqslant I_\mu(v + d_{\epsilon,v} U_{\epsilon,c}) \\ &= \left(\frac{1}{p} - \frac{1}{1-\gamma}\right) \int_\Omega h(x)(|x|^{-b}|v + d_{\epsilon,v} U_{\epsilon,c}|)^{1-\gamma} dx + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) \|v + d_{\epsilon,v} U_{\epsilon,c}\|_{p^*}^{p^*} \\ &= \left(\frac{1}{p} - \frac{1}{1-\gamma}\right) \int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma} dx + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) \|v\|_{p^*}^{p^*} + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) d_{\epsilon,v}^{p^*} A + o(1), \end{aligned}$$

i.e.,

$$v_0 \leqslant \left(\frac{1}{p} - \frac{1}{1-\gamma}\right) \int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma} dx + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) \|v\|_{p^*}^{p^*} + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) d_v^{p^*} A. \tag{3.9}$$

Combining (3.7) and (3.9), we obtain that

$$v_0 = \left(\frac{1}{p} - \frac{1}{1-\gamma}\right) \int_\Omega h(x)(|x|^{-b}|v_0|)^{1-\gamma} dx + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_\Omega (|x|^{-b}|v_0|)^{p^*} dx + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) \tau_0^{p^*} A. \tag{3.10}$$

Consequently, v_0 is a local minimizer for the following functional:

$$\left(\frac{1}{p} - \frac{1}{1-\gamma}\right) \int_\Omega h(x)(|x|^{-b}|v|)^{1-\gamma} dx + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) \|v\|_{p^*}^{p^*} + \mu \left(\frac{1}{p} - \frac{1}{p^*}\right) d_v^{p^*} A. \tag{3.11}$$

For d_v, let $\psi \in C_0^\infty(\Omega)$ and $g(t) := d_{v_0+t\psi}$, we have

$$[g(t)]^p B - \mu [g(t)]^{p^*} A = -\left[\|v_0 + t\psi\|_a^p - \int_\Omega h(x)(|x|^{-b}|v_0 + t\psi|)^{1-\gamma} dx - \mu \int_\Omega (|x|^{-b}|v_0 + t\psi|)^{p^*} dx\right].$$

Since $a_0 > 0$, we have that $g(t)$ exists and $g(0) = \tau_0$. Therefore,

$$\begin{aligned} &\frac{[g(t)]^p B - \mu[g(t)]^{p^*} A - [g(0)]^p B + \mu [g(0)]^{p^*} A}{t} \\ &= \frac{[g(t) - g(0)][g^{p-1}(t) + \cdots + g^{p-1}(0)]B - \mu A [g(t) - g(0)][g^{p^*-1}(t) + \cdots + g^{p^*-1}(0)]}{t} \\ &= -\frac{1}{t}\left\{\|v_0 + t\psi\|_a^p - \int_\Omega h(x)(|x|^{-b}|v_0 + t\psi|)^{1-\gamma} dx - \mu \int_\Omega (|x|^{-b}|v_0 + t\psi|)^{p^*} dx \right. \\ &\quad \left. -\|v_0\|_a^p + \int_\Omega h(x)(|x|^{-b}|v_0|)^{1-\gamma} dx + \mu \int_\Omega (|x|^{-b}|v_0|)^{p^*} dx \right\} \\ &\xrightarrow{t \to 0} -\left[p \int_\Omega |x|^{-ap} |\nabla v_0|^{p-1} \nabla \psi dx - (1-\gamma) \int_\Omega \frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma} \psi dx - \mu p^* \int_\Omega (|x|^{-b})^{p^*} v_0^{p^*-1} \psi dx\right]. \end{aligned}$$

It follows that $g'(0)$ exists and

$$g'(0)=\frac{-1}{p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A}\left[p\int_\Omega |x|^{-ap}|\nabla v_0|^{p-1}\nabla\psi dx-(1-\gamma)\int_\Omega \frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}\psi dx\right.$$
$$\left.-\mu p^*\int_\Omega (|x|^{-b})^{p^*}v_0^{p^*-1}\psi dx\right].$$

Noting that (3.11) yields

$$\frac{d}{dt}\left\{\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\int_\Omega h(x)(|x|^{-b}|v_0+t\psi|)^{1-\gamma}dx+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega (|x|^{-b}|v_0+t\psi|)^{p^*}dx\right.$$
$$\left.+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)[g(t)]^{p^*}A\right\}|_{t=0}=0,$$

we have

$$\begin{aligned}&\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)(1-\gamma)\int_\Omega \frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}\psi dx\\&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\int_\Omega (|x|^{-b})^{p^*}v_0^{p^*-1}\psi dx+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0^{p^*-1}A\\&\times\left\{\frac{-1}{p\tau_0^{p^*-1}B-\mu p^*\tau_0^{p^*-1}A}\left[p\int_\Omega |x|^{-ap}|\nabla v_0|^{p-1}\nabla\psi dx-(1-\gamma)\int_\Omega \frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}\psi dx\right.\right.\\&\left.\left.-\mu p^*\int_\Omega (|x|^{-b})^{p^*}v_0^{p^*-1}\psi dx\right]\right\}=0,\end{aligned}\tag{3.12}$$

for all $\psi\in C_0^\infty(\Omega)$. Denote $d_{\epsilon,v_0}=\tau_0+\delta_\epsilon$, since $d_{\epsilon,v}\longrightarrow d_v$, as $\epsilon\longrightarrow 0$, we have $\delta_\epsilon\to 0$. All the above estimates are substituted in (3.8), we have

$$\begin{aligned}0&=\|v_0+d_{\epsilon,v_0}U_{\epsilon,c}\|_a^p-\int_\Omega h(x)(|x|^{-b}|v_0+d_{\epsilon,v_0}U_{\epsilon,c}|)^{1-\gamma}dx-\mu\int_\Omega(|x|^{-b}|v_0+d_{\epsilon,v_0}U_{\epsilon,c}|)^{p^*}dx\\&=\|v_0\|_a^p+d_{\epsilon,v_0}^p\|U_{\epsilon,c}\|_a^p+\int_\Omega\left[pd_{\epsilon,v_0}\nabla U_{\epsilon,c}|x|^{-ap}|\nabla v_0|^{p-1}+pd_{\epsilon,v_0}^{p-1}|\nabla U_{\epsilon,c}|^{p-1}|x|^{-ap}|\nabla v_0|\right]dx\\&\quad-\int_\Omega h(x)(|x|^{-b}v_0)^{1-\gamma}dx-(1-\gamma)\tau_0\int_\Omega h(x)(|x|^{-b})^{1-\gamma}v_0^{-\gamma}U_{\epsilon,c}dx\\&\quad-\mu\left[\int_\Omega(|x|^{-b})^{p^*}\left(v_0^{p^*}+p^*c_{\epsilon,v_0}U_{\epsilon,c}v_0^{p^*-1}+\cdots+p^*(c_{\epsilon,v_0}U_{\epsilon,c})^{p^*-1}v_0+(c_{\epsilon,v_0}U_{\epsilon,c})^{p^*}\right)dx\right]+o(\epsilon^{\frac{N-p}{p}})\\&=-(\tau_0^pB-\mu\tau_0^{p^*}A)+c_{\epsilon,v_0}^pB-\mu c_{\epsilon,v_0}^{p^*}A\\&\quad+\int_\Omega|x|^{-ap}\left[pc_{\epsilon,v_0}\nabla U_{\epsilon,c}|\nabla v_0|^{p-1}+pc_{\epsilon,v_0}^{p-1}|\nabla U_{\epsilon,c}|^{p-1}|\nabla v_0|\right]dx\\&\quad-(1-\gamma)\tau_0\int_\Omega h(x)(|x|^{-b})^{1-\gamma}(v_0)^{-\gamma}U_{\epsilon,c}dx\\&\quad-\mu\int_\Omega(|x|^{-b})^{p^*}(p^*d_{\epsilon,v_0}U_{\epsilon,c}v_0^{p^*-1}+p^*d_{\epsilon,v_0}^{p^*-1}U_{\epsilon,c}^{p^*-1}v_0)dx+o\left(\epsilon^{\frac{N-p}{p}}\right),\end{aligned}$$

which yields

$$\begin{aligned}&\left[p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A+o(1)\right](-\delta_\epsilon)\\&=\int_\Omega|x|^{-ap}\left[p\tau_0\nabla U_{\epsilon,c}|\nabla v_0|^{p-1}+p\tau_0^{p-1}|\nabla U_{\epsilon,c}|^{p-1}|\nabla v_0|\right]dx\\&\quad-(1-\gamma)\tau_0\int_\Omega h(x)(|x|^{-b})^{1-\gamma}v_0^{-\gamma}U_{\epsilon,c}dx\\&\quad-\mu\int_\Omega(|x|^{-b})^{p^*}(p^*\tau_0U_{\epsilon,c}v_0^{p^*-1}+p^*\tau_0^{p^*-1}U_{\epsilon,c}^{p^*-1}v_0)dx+o(\epsilon^{\frac{N-p}{p}}).\end{aligned}$$

Moreover, by (3.12), we obtain

$$\begin{aligned}
-\delta_\epsilon =& \frac{\tau_0}{p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A}\left[p\int_\Omega |x|^{-ap}|\nabla v_0|^{p-1}\nabla U_{\epsilon,c}\,dx\right]\\
&-\frac{\tau_0}{p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A}(1-\gamma)\int_\Omega \frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}U_{\epsilon,c}\,dx\\
&-\frac{\tau_0}{p\tau_0^{p-1}B-\tau_0^{p^*-1}A}\mu p^*\int_\Omega (|x|^{-b})^{p^*}v_0^{p^*-1}U_{\epsilon,c}\,dx\\
&+\frac{\tau_0^{p-1}}{p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A}\left[p\int_\Omega |x|^{-ap}|\nabla U_{\epsilon,c}|^{p-1}|\nabla v_0|dx\right]\\
&-\frac{\tau_0^{p^*-1}}{p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A}\mu p^*\int_\Omega (|x|^{-b})^{p^*}U_{\epsilon,c}^{p^*-1}v_0\,dx+o(\epsilon^{\frac{N-p}{p}})\\
=&\,\tau_0\frac{\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)(1-\gamma)\int_\Omega \frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}U_{\epsilon,c}\,dx+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\int_\Omega(|x|^{-b})^{p^*}v_0^{p^*-1}U_{\epsilon,c}\,dx}{\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0^{p^*-1}A}\\
&+\frac{\tau_0^{p-1}}{p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A}\left[p\int_\Omega |x|^{-ap}|\nabla U_{\epsilon,c}|^{p-1}|\nabla v_0|dx\right]\\
&-\frac{\tau_0^{p-1}}{p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A}\mu p^*\int_\Omega (|x|^{-b})^{p^*}U_{\epsilon,c}^{p^*-1}v_0\,dx+o(\epsilon^{\frac{N-p}{p}}).
\end{aligned}\tag{3.13}$$

Since $a_0>0$, we have

$$p\tau_0^{p-1}B-\mu p^*\tau_0^{p^*-1}A=\frac{p}{\tau_0}\left(\tau_0^pB-\mu\frac{p^*}{p}\tau_0^{p^*}A\right)<\frac{p}{\tau_0}(\tau_0^pB-\mu\tau_0^{p^*}A)=-\frac{p}{\tau_0}a_0<0.$$

And then, it follows from $v_0+d_{\epsilon,v_0}U_{\epsilon,c}\in\mathcal{N}_\mu$, (3.10) and (3.13) that

$$\begin{aligned}
I_\mu(v_0+d_{\epsilon,v_0}U_{\epsilon,c}) =& \left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\int_\Omega h(x)(|x|^{-b}|v_0+d_{\epsilon,v_0}U_{\epsilon,c}|)^{1-\gamma}dx\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega (|x|^{-b}|v_0+d_{\epsilon,v_0}U_{\epsilon,c}|)^{p^*}dx\\
=&\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\int_\Omega h(x)(|x|^{-b}v_0)^{1-\gamma}dx\\
&+\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\left[(1-\gamma)\tau_0\int_\Omega h(x)(|x|^{-b})^{1-\gamma}v_0^{-\gamma}U_{\epsilon,c}\,dx\right]\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega (|x|^{-b}|v_0|)^{p^*}dx\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)d_{\epsilon,v_0}^{p^*}A+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\int_\Omega (|x|^{-b})^{p^*}v_0^{p^*-1}d_{\epsilon,v_0}U_{\epsilon,c}\,dx\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*d_{\epsilon,v_0}^{p^*-1}\int_\Omega v_0U_{\epsilon,c}^{p^*-1}dx+o(\epsilon^{\frac{N-p}{p}})\\
=&\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\int_\Omega h(x)(|x|^{-b}v_0)^{1-\gamma}dx\\
&+\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\left[(1-\gamma)\tau_0\int_\Omega h(x)(|x|^{-b})^{1-\gamma}v_0^{-\gamma}U_{\epsilon,c}\,dx\right]\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega (|x|^{-b}|v_0|)^{p^*}dx+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)\tau_0^{p^*}A+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0^{p^*-1}\delta_\epsilon A
\end{aligned}$$

$$
\begin{aligned}
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0\int_\Omega(|x|^{-b})^{p^*}v_0^{p^*-1}U_{\epsilon,c}\,dx\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0^{p^*-1}\int_\Omega(|x|^{-b})^{p^*}v_0U_{\epsilon,c}^{p^*-1}dx\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*(p^*-1)\tau_0^{p^*-2}\delta_\epsilon\int_\Omega(|x|^{-b})^{p^*}v_0U_{\epsilon,c}^{p^*-1}dx+o(\epsilon^{\frac{N-p}{p}})\\
=\;&v_0+\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\left[(1-\gamma)\tau_0\int_\Omega h(x)(|x|^{-b})^{1-\gamma}v_0^{-\gamma}U_{\epsilon,c}\,dx\right]\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0^{p^*-1}\delta_\epsilon A+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0\int_\Omega(|x|^{-b})^{p^*}v_0^{p^*-1}U_{\epsilon,c}\,dx\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0^{p^*-1}\int_\Omega(|x|^{-b})^{p^*}v_0U_{\epsilon,c}^{p^*-1}dx\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*(p^*-1)\tau_0^{p^*-2}\delta_\epsilon\int_\Omega(|x|^{-b})^{p^*}v_0U_{\epsilon,c}^{p^*-1}dx+o(\epsilon^{\frac{N-p}{p}})\\
=\;&v_0+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\tau_0^{p^*-1}\frac{p\tau_0^{p^*-1}B}{p\tau_0^{p^*-1}B-\mu p^*\tau_0^{p^*-1}A}\int_\Omega(|x|^{-b})^{p^*}v_0U_{\epsilon,c}^{p^*-1}dx\\
&-(p^*-1)\left[\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)(1-\gamma)\int_\Omega\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}U_{\epsilon,c}\,dx\right.\\
&\left.+\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*\int_\Omega(|x|^{-b})^{p^*}v_0^{p^*-1}U_{\epsilon,c}\,dx\right]\int_\Omega(|x|^{-b})^{p^*}v_0U_{\epsilon,c}^{p^*-1}dx\\
&+\frac{\tau_0^{p^*-1}}{p\tau_0^{p^*-1}B-\mu p^*\tau_0^{p^*-1}A}\mu p^*\left(\int_\Omega(|x|^{-b})^{p^*}U_{\epsilon,c}^{p^*-1}v_0\,dx\right)\\
&\times\mu\left(\frac{1}{p}-\frac{1}{p^*}\right)p^*(p^*-1)\tau_0^{p^*-2}\int_\Omega(|x|^{-b})^{p^*}v_0U_{\epsilon,c}^{p^*-1}dx+o(\epsilon^{\frac{N-p}{p}})<v_0,
\end{aligned}
$$

which is a contradiction. This concludes the proof of Claim 2.

Claim 3. *v_0 is a solution of* (1.1).

For $\psi\in W_a^{1,p}(\Omega)$ and $\epsilon>0$, we define

$$\Psi:=(v_0+\epsilon\psi)^+\in W_0^{1,p}(\Omega). \tag{3.14}$$

Equation (3.14) is substituted in (3.6), in view of Claim 2, we find that

$$
\begin{aligned}
0=\;&\int_\Omega\left(|x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla\Psi-\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}\Psi-\mu(|x|^{-b}v_0)^{p^*-1}\Psi\right)dx\\
=\;&\int_{[v_0+\epsilon\psi>0]}\left[|x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla(v_0+\epsilon\psi)\right.\\
&\left.-\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}(v_0+\epsilon\psi)-\mu(|x|^{-b})^{p^*}v_0^{p^*-1}(v_0+\epsilon\psi)\right]dx\\
=\;&\int_\Omega\left[|x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla(v_0+\epsilon\psi)-\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}(v_0+\epsilon\psi)-\mu(|x|^{-b})^{p^*}v_0^{p^*-1}(v_0+\epsilon\psi)\right]dx\\
&-\int_{[v_0+\epsilon\psi\leqslant0]}\left[|x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla(v_0+\epsilon\psi)\right.\\
&\left.-\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}(v_0+\epsilon\psi)-\mu(|x|^{-b})^{p^*}v_0^{p^*-1}(v_0+\epsilon\psi)\right]dx\\
=\;&\|v_0\|_a^p-\int_\Omega h(x)(|x|^{-b})^{1-\gamma}v_0^{1-\gamma}dx-\mu\int_\Omega(|x|^{-b}v_0)^{p^*}dx
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon\left(\int_\Omega |x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla\psi-\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}\psi-\mu(|x|^{-b})^{p^*}v_0^{p^*-1}\psi\right)dx\\
&-\int_{[v_0+\epsilon\psi\leqslant 0]}\Big(|x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla(v_0+\epsilon\psi)\\
&-\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}(v_0+\epsilon\psi)-\mu(|x|^{-b})^{p^*}v_0^{p^*-1}(v_0+\epsilon\psi)\Big)dx\\
\leqslant{}&\epsilon\int_\Omega\left(|x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla\psi-\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}\psi-\mu(|x|^{-b})^{p^*}v_0^{p^*-1}\psi\right)dx\\
&-\epsilon\int_{[v_0+\epsilon\psi\leqslant 0]}\left(|x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla\psi\right)dx.
\end{aligned}
$$

Note that the measurement of $\{v_0+\epsilon\psi\leqslant 0\}\longrightarrow 0$ as $\epsilon\to 0$. Dividing by ϵ and passing to the limit as $\epsilon\longrightarrow 0$, and hence

$$\int_\Omega\left(|x|^{-ap}|\nabla v_0|^{p-2}\nabla v_0\nabla\psi-\frac{h(x)(|x|^{-b})^{1-\gamma}}{v_0^\gamma}\psi-\mu(|x|^{-b})^{p^*}v_0^{p^*-1}\psi\right)dx\geqslant 0.$$

Therefore, v_0 is a solution of problem (1.1). Note that $v_n\rightharpoonup v_0$ weakly in $W_a^{1,p}(\Omega)$, we know that $\|v_0\|\leqslant \liminf_{n\longrightarrow\infty}\|v_n\|_a\leqslant A_0$. By Claim 2 and Lemma 2.2, we see that $v_0\in\mathcal{N}_\mu^+$. Thus, it follows from $I_\mu(v_n)\longrightarrow \inf_{\mathcal{N}_\mu^+} I_\mu$ that

$$\inf_{\mathcal{N}_\mu^+} I_\mu\geqslant\left(\frac{1}{p}-\frac{1}{p^*}\right)\|v_0\|_a^p-\left(\frac{1}{1-\gamma}-\frac{1}{p^*}\right)\int_\Omega h(x)(|x|^{-b}v_0)^{1-\gamma}dx=I_\mu(v_0).$$

Consequently, $I_\mu(v_0)=\inf_{\mathcal{N}_\mu^+} I_\mu$.

□

Theorem 3.2. *Assume that* $\mu\in(0,T)$. *Then the problem* (1.1) *has a solution* $V_0\in W_a^{1,p}(\Omega)$ *satisfying*

$$\|V_0\|_a\geqslant A(\mu)>A_0.$$

Proof. The proof of Theorem 3.2 is similar to that of Theorem 3.1, we omit it here.

□

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References

[1] B. Abdellaoui, V. Felli, I. Peral, *Existence and nonexistence for quasilinear equations involving the p-Laplacian*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., **9** (2006), 445–484. 1

[2] C. O. Alves, A. El Hamidi, *Nehari manifold and existence of positive solutions to a class of quasilinear problems*, Nonlinear Anal., **60** (2005), 611–624. 1

[3] J.-P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, Inc., New York, (1984). 3

[4] H. Brézis, E. Lieb, *A relation between pointwise convergence of functional and convergence of functionals*, Proc. Amer. Math. Soc., **88** (1983), 486–490. 3

[5] F. Catrina, Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math., **54** (2001), 229–258. 3

[6] Y.-P. Chen, J.-Q. Chen, *Existence of multiple positive weak solutions and estimates for extremal values to a class of elliptic problems with Hardy term and singular nonlinearity*, J. Math. Anal. Appl., **429** (2015), 873–900. 1

[7] Y.-P. Chen, J.-Q. Chen, *Multiple positive solutions for a semilinear equation with critical exponent and prescribed singularity*, Nonlinear Anal., **130** (2016), 121–137. 1

[8] J.-Q. Chen, E. M. Rocha, *Positive solutions for elliptic problems with critical nonlinearity and combined singularity*, Math. Bohem., **135** (2010), 413–422. 1

[9] S.-Q. Cong, Y.-Z. Han, *Compatibility conditions for the existence of weak solutions to a singular elliptic equation*, Bound. Value Probl., **2015** (2015), 11 pages.

[10] F. J. S. A. Corrêa, A. S. S. Corrêa, G. M. Figueiredo, *Positive solution for a class of pq-singular elliptic equation*, Nonlinear Anal. Real World Appl., **16** (2014), 163–169. 1

[11] Z.-Y. Deng, Y.-S. Huang, *On G-symmetric solutions of the quasilinear elliptic equations with singular weights and critical exponents*, Scientia Sinica Mathematica., **42** (2012), 1053–1066. 1

[12] Z.-Y. Deng, Y.-S. Huang, *On positive G-symmetric solutions of a weighted quasilinear elliptic equation with critical Hardy-Sobolev exponent*, Acta Math. Sci. Ser. B Engl. Ed., **34** (2014), 1619–1633. 1

[13] N. Ghoussoub, C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc., **352** (2000), 5703–5743. 1

[14] J. Giacomoni, I. Schindler, P. Takáč, *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., **6** (2007), 117–158. 1

[15] P. Han, *Quasilinear elliptic problems with critical exponents and Hardy terms*, Nonlinear Anal., **61** (2005), 735–758.

[16] Y. Jalilian, *On the exitence and multiplicity of solutions for a class of singular elliptic problems*, Comput. Math. Appl., **68** (2014), 664–680.

[17] D.-S. Kang, *On the quasilinear elliptic problems with critical Sobolev-Hardy exponents and Hardy terms*, Nonlinear Anal., **68** (2008), 1973–1985. 1

[18] D.-S. Kang, *Positve solutions to the weighted critical quasilinear problems*, Appl. Math. Compt., **213** (2009), 432–439. 1, 1

[19] N. H. Loc, K. Schmitt, *Boundary value problems for singular elliptic equations*, Rocky Mt. J. Math., **41** (2011), 555–572. 1

[20] Y.-b. Sang, *An exact estimate result for a semilinear equation with critical exponent and prescribed singularity*, J. Math. Anal. Appl., **447** (2017), 128–153. 2

[21] Y.-J. Sun, S.-J. Li, *A nonlinear elliptic equation with critical exponent: estimates for extremal values*, Nonlinear Anal., **69** (2008), 1856–1869. 1, 2

[22] Y.-J. Sun, S.-P. Wu, *An exact estimate result for a class of singular equations with critical exponents*, J. Funct. Anal., **260** (2011), 1257–1284. 1

[23] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **9** (1992), 281–304. 3

[24] S. Terracini, *On positive entire solutions to a class of equations with singular coefficient and critical exponent*, Adv. Differential Equations, **1** (1996), 241–264. 3

[25] L. Wang, Q.-L. Wei, D.-S. Kang, *Multiple positive solutions for* p*-Laplace elliptic equations involving concave-convex nonlinearities and a Hardy-type term*, Nonliear Anal., **74** (2011), 626–638. 1

[26] B. J. Xuan, *The eightvalue problem for a singular quasilinear elliptic equation*, Electron. J. Differ. Equat., **16** (2004), 1–11. 1, 1