



Curves and surfaces of spacelike curves according to Bishop frame and their singularities

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Abstract

Legendrian dualities between pseudo-spherical images of spacelike curves in Minkowski 3-space are investigated by using the theory of Legendrian duality. Moreover, the singularities of parallel lightcone developables, dual surface, Bishop pseudo-spherical Darboux images and Bishop pseudo-spherical images, which are generated by spacelike curves, are classified from the viewpoints of wave fronts and caustics, and we also give some more detail descriptions on the conditions of those singularities. Finally, some properties of parallel slant helix are given. ©2017 All rights reserved.

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1. Introduction

Bishop [2] introduced that there exists an orthonormal relatively parallel adapted frame, which we call Bishop frame, other than the Frenet frame and compared features of them with the Frenet frame. The Bishop frame has many properties that make it ideal for mathematical research and computer graphics [8–10, 20]. Inspired by the work of Bishop, in [25], the authors introduced a new version of Bishop frame by using a common vector field as binormal vector field of a regular curve and called this frame as “Type-2 Bishop frame”. In 2008, Özdemir and Ergin extended Bishop frame to the non-lightlike curves in Minkowski 3-space, where they also called relative parallel adopted frame [18]. We know that the properties of geometric objects are independent of the choice of the coordinate systems. But, the researchers found that, when they adopted this frame, there will be some new geometric objects such as parallel slant curves, Bishop spherical images, Bishop Darboux images, etc.. After that, many regular curves and surfaces related to the Bishop frame have been treated in the Euclidean space [5, 6, 13–17, 22, 24], Minkowski space [4, 21, 23, 26], dual space [11] and Heisenberg group Heis^3 [12]. The current study hopes to research those singular curves and surfaces associate to the Bishop frame instead of regular ones. We will classify the singularities of parallel lightcone developables, dual surface, Bishop pseudo-spherical Darboux images and Bishop pseudo-spherical images, which are generated by spacelike curves embedded

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in Minkowski 3-space according to the relatively parallel adapted frame. Adopting the relatively parallel adapted frame as the basic tool, we get the main results of this paper, Proposition 2.2 and Theorem 2.3, which use the relatively parallel adapted frame for classification of singularity types of these geometric objects from the viewpoints of wave fronts and caustics. It is worth mentioning that we also investigate Legendrian dualities between pseudo-spherical images of spacelike curves in Minkowski 3-space by using the theory of Legendrian dualities (cf. Proposition 2.1) and get some meaningful properties of parallel slant helix (cf. Proposition 5.1).

The rest of this paper is organized as follows. Firstly, we introduce some basic concepts and the main results in the next section. Then, we introduce five different families of functions that will be useful to the study of geometric invariants of regular spacelike curves. Afterwards, some general results on the singularity theory are used for families of function germs and the main results are proved.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2. Basic concepts and results

In this section, we introduce the basic notions and results in Lorentzian geometry. For more detail descriptions, see [18, 19]. Let \mathbb{R}_1^3 denote the 3-dimensional Minkowski space, that is to say, the manifolds \mathbb{R}^3 with a flat Lorentzian metric \langle, \rangle of signature $(-, +, +)$, for any vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in \mathbb{R}^3 , $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3$. We also define a vector

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of \mathbb{R}_1^3 . One can easily show that $\langle \mathbf{x} \wedge \mathbf{y}, \mathbf{z} \rangle = \det(\mathbf{x}, \mathbf{y}, \mathbf{z})$. We say that a nonzero vector $\mathbf{x} \in \mathbb{R}_1^3 \setminus \{0\}$ is spacelike, lightlike or timelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}_1^3$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$.

Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a smooth regular curve (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), where I is an open interval. If $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$ for any $t \in I$, we call such a curve a *spacelike curve*. The arc-length of a spacelike curve $\gamma(t)$, measured from $\gamma(t_0)$, is $s(t) = \int_{t_0}^t \|\dot{\gamma}(t)\| dt$. Then the parameter s is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = \frac{d\gamma}{ds}$. So we say that a spacelike curve γ is parameterized by arc-length if it satisfies that $\|\gamma'(s)\| = 1$. Throughout the remainder in this paper we denote the parameter s of γ as the arc-length parameter. We denote $\mathbf{t}(s) = \gamma'(s)$ and we call $\mathbf{t}(s)$ a unit tangent vector of γ at s . We know that there exists an accompanying three-frames called Frenet frame for spacelike Frenet curve $\gamma(s)$. Denote by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ the moving Frenet frame along the unit speed spacelike Frenet curve $\gamma(s)$. Then, the Frenet formulas are given by

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ \delta(\mathbf{B}(s))k(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}.$$

Here, $\delta(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$, $k(s)$ and $\tau(s)$ are called curvature and torsion, respectively, see [15]. The Bishop frame equations for a parametrized unit length curve $\gamma(s)$ are as follows:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'_1(s) \\ \mathbf{N}'_2(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -\delta(\mathbf{N}_1(s))k_1(s) & 0 & 0 \\ -\delta(\mathbf{N}_2(s))k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}.$$

Here, we will call the set $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s)\}$ as *Bishop frame* and $k_1(s) = \delta(\mathbf{N}_1(s))\langle \mathbf{T}'(s), \mathbf{N}_1(s) \rangle$ and $k_2(s) = \delta(\mathbf{N}_2(s))\langle \mathbf{T}'(s), \mathbf{N}_2(s) \rangle$ as *Bishop curvatures*. The relation matrix can be expressed as

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta(s) & \sinh \theta(s) \\ 0 & \sinh \theta(s) & \cosh \theta(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}.$$

One can show that

$$k(s) = \sqrt{|\delta(\mathbf{N}_1(s))k_1^2(s) + \delta(\mathbf{N}_2(s))k_2^2(s)|}, \quad \theta(s) = \operatorname{arctanh} \left(\frac{k_2(s)}{k_1(s)} \right),$$

where $k_1(s) \neq 0$, $\tau(s) = \delta(\mathbf{N}_1(s)) \frac{d\theta(s)}{ds}$. Here, Bishop curvatures are also defined by

$$\begin{cases} k_1(s) = k(s) \cosh \theta(s), \\ k_2(s) = k(s) \sinh \theta(s), \end{cases}$$

and $\theta = \delta(\mathbf{N}_1(s)) \int \tau(s) ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0 , which disappears from τ (and hence from the Frenet frame) due to the differentiation [18]. Also, we assume that is positively oriented and the vector products of these vectors are defined as follows: $\mathbf{T}(s) \wedge \mathbf{N}_1(s) = \delta(\mathbf{N}_2(s))\mathbf{N}_2(s)$, $\mathbf{N}_1(s) \wedge \mathbf{N}_2(s) = \mathbf{T}(s)$, $\mathbf{N}_2(s) \wedge \mathbf{T}(s) = \delta(\mathbf{N}_1(s))\mathbf{N}_1(s)$. We define the *pseudo-spherical* by

$$Q_\epsilon^2 = \begin{cases} S_1^2 = \{\mathbf{x} \in \mathbb{R}_1^3 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}, & \text{if } \epsilon = +, \\ H_0^2 = \{\mathbf{x} \in \mathbb{R}_1^3 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\}, & \text{if } \epsilon = -. \end{cases}$$

We call S_1^2 the *de Sitter space* and H_0^2 the *hyperbolic space*. If we translate the three vector fields of Bishop trihedra to the center O of one of unit pseudo-spheres then we can get the *tangent de Sitter indicatrix*, the *first Bishop pseudo-spherical indicatrix* and the *second Bishop pseudo-spherical indicatrix* in [23] and we denote them by $\mathcal{TD}(s) = \mathbf{T}(s)$, $\mathcal{FN}(s) = \mathbf{N}_1(s)$, and $\mathcal{SN}(s) = \mathbf{N}_2(s)$ separately. A *lightlike unit circle* is defined by

$$S_+^1 = \{\mathbf{x} \in \mathbb{R}_1^3 \mid \mathbf{x} = (1, x_2, x_3), x_2^2 + x_3^2 = 1\},$$

and an *open lightcone* at the vertex \mathbf{p} is defined by

$$LC_{\mathbf{p}} = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \mid -(x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 = 0\},$$

where $\mathbf{p} = (p_1, p_2, p_3)$. For any lightlike vector $\mathbf{x} = (x_1, x_2, x_3)$, we write $\tilde{\mathbf{x}} = (1, \frac{x_2}{x_1}, \frac{x_3}{x_1}) \in S_+^1$. Especially, we consider the following set on the lightcone and call it the *lightcone bundle* of curve $\gamma(s)$ through \mathbf{v}_0 , for any $\mathbf{v}_0 \in \mathbb{R}_1^3$, $\mathcal{LC}(\mathbf{v}_0) = \{\mathbf{u} \in \mathbb{R}_1^3 \mid \langle \mathbf{u} - \mathbf{v}_0, \mathbf{u} - \mathbf{v}_0 \rangle = 0\}$.

Next, we will introduce a basic result on Legendrian dualities between pseudo-spheres [7] which has been proved to be a powerful tool for the study of surfaces in pseudo-spheres. One-forms on $\mathbb{R}_1^3 \times \mathbb{R}_1^3$ are defined by $\langle d\mathbf{v}, \mathbf{w} \rangle = -w_0 dv_0 + \sum_{i=1}^2 w_i dv_i$ and $\langle \mathbf{v}, d\mathbf{w} \rangle = -v_0 dw_0 + \sum_{i=1}^2 v_i dw_i$. Then, we consider the following two double fibrations:

- (1) (a) $H_0^2(-1) \times S_1^2 \supset \Delta_1 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$,
 (b) $\pi_{11} : \Delta_1 \rightarrow H_0^2(-1), \pi_{12} : \Delta_1 \rightarrow S_1^2$,
 (c) $\theta_{11} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_1}, \theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_1}$.
- (2) (a) $S_1^2 \times S_1^2 \supset \Delta_2 = \{(\mathbf{v}, \mathbf{w}) \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$,
 (b) $\pi_{21} : \Delta_2 \rightarrow S_1^2, \pi_{22} : \Delta_2 \rightarrow S_1^2$,
 (c) $\theta_{21} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_2}, \theta_{22} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_2}$.

Here $\pi_{i1}(\mathbf{v}, \mathbf{w}) = \mathbf{v}, \pi_{i2}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent plane field over Δ_i which is denoted by K_i ($i = 1, 2$). The basic duality theorem is that each (Δ_i, K_i) is a contact manifold and both of π_{ij} ($j = 1, 2$) are Legendrian fibrations. If there exists an isotropic mapping $i : L \rightarrow \Delta_i$, which means that $i^* \theta_{i1} = 0$, we say that $\pi_{i1}(i(L))$ and $\pi_{i2}(i(L))$ are Δ_i -dual to each other. It is easy to see that the condition $i^* \theta_{i1} = 0$ is equivalent to $i^* \theta_{i2} = 0$. Then we have the following proposition on the relationships between the first Bishop pseudo-spherical indicatrix and the second Bishop pseudo-spherical indicatrix with the help of the above Legendrian dualities.

Proposition 2.1. *Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed spacelike curve. We have the following.*

- (1) $\mathbf{N}_1(s)$ and $\mathbf{N}_2(s)$ are Δ_1 -dual to each other.
- (2) If $\delta(\mathbf{B}(s)) = -1$, then $\mathbf{T}(s)$ and $\mathbf{B}(s)$ are Δ_1 -dual to each other, otherwise $\mathbf{T}(s)$ and $\mathbf{B}(s)$ are Δ_2 -dual to each other.

Proof.

- (1) For the case that $\delta(\mathbf{N}_1(s)) = -1$, we consider the mapping

$$\mathfrak{L}_1(s) = (\mathbf{N}_1(s), \mathbf{N}_2(s)),$$

otherwise we change the positions of $\mathbf{N}_1(s)$ and $\mathbf{N}_2(s)$ in the mapping $\mathfrak{L}_1(s)$. Then we have $\langle \mathbf{N}_1(s), \mathbf{N}_2(s) \rangle = 0$ and

$$\mathfrak{L}_1^* \theta_{12}(s) = \langle \mathbf{N}_1(s), \mathbf{N}_2'(s) \rangle = \langle \mathbf{N}_1(s), -\delta(\mathbf{N}_2(s))k_2(s)\mathbf{T}(s) \rangle = 0.$$

The assertion (1) follows.

- (2) For the case that $\delta(\mathbf{B}(s)) = -1$, we consider the mapping

$$\mathfrak{L}_2(s) = (\mathbf{B}(s), \mathbf{T}(s)).$$

Then we have $\langle \mathbf{B}(s), \mathbf{T}(s) \rangle = 0$ and

$$\mathfrak{L}_2^* \theta_{11}(s) = \langle \mathbf{B}'(s), \mathbf{T}(s) \rangle = \langle \tau(s)\mathbf{N}(s), \mathbf{T}(s) \rangle = 0.$$

The first claim of assertion (2) follows. Using the same computation as the above proof, we can get the second claim. □

We define the *parallel lightcone developables* of $\gamma(s)$ as the map $\mathcal{P}\mathcal{L}\mathcal{D}_\gamma^\pm : I \times \mathbb{R} \rightarrow \mathbb{R}_1^3$ given by

$$\mathcal{P}\mathcal{L}\mathcal{D}_\gamma^\pm(s, u) = \gamma(s) + u(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)).$$

If $\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0$ and $k_1(s) \neq 0$, we can define

$$\mathbf{d}_\epsilon(s) = \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right).$$

We call every $\mathbf{d}_\pm(s)$ the *pseudo-spherical Darboux images* of $\gamma(s)$. The *dual surface* of γ is defined by

$$\mathcal{B}\mathcal{D}\mathcal{U}_\gamma(\mathbf{v}, \mathbf{u}) = \{(\mathbf{v}, \mathbf{u}) \in Q_\epsilon^2 \times \mathbb{R} \mid \mathbf{u} = \langle \gamma(s), \mathbf{v} \rangle, \langle \mathbf{T}(s), \mathbf{v} \rangle = 0\}.$$

We define a function $\rho(s) = k_1'(s)k_2(s) - k_1(s)k_2'(s)$ of a regular spacelike curve in \mathbb{R}_1^3 and we can describe some properties of parallel slant curve by $\rho(s)$. Let $F : \mathbb{R}_1^3 \rightarrow \mathbb{R}$ be a submersion and $\gamma : I \rightarrow \mathbb{R}_1^3$ be a regular unit speed spacelike Frenet curve. We say that $\gamma(s)$ and $F^{-1}(0)$ have k -point contact for $s = s_0$ if the function $g(s) = F \circ \gamma(s)$ satisfies

$$g(s_0) = g'(s_0) = g''(s_0) = \dots = g^{(k-1)}(s_0) = 0, g^{(k)}(s_0) \neq 0.$$

We also say that $\gamma(s)$ and $F^{-1}(0)$ have at least k -point contact for $s = s_0$ if the function $g(s) = F \circ \gamma(s)$ satisfies $g(s_0) = g'(s_0) = g''(s_0) = \dots = g^{(k-1)}(s_0) = 0$. Let function germ $F : (\mathbb{R} \times \mathbb{R}^r, (s, \mathbf{x})) \rightarrow \mathbb{R}$ be an r -parameter unfolding of $f(s)$, where $f(s) = F(s, \mathbf{x}_0)$. We say that $f(s)$ has A_k -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that $f(s)$ has A_k -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. We give a brief review on Legendrian singularity theory mainly due to Arnold [1]. The main

tool of Legendrian singularities theory is the notion of generating families. Let $G : (\mathbb{R} \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, 0)$ be a function germ. We say that G is a Morse family if the mapping

$$\Delta^*G = \left(G, \frac{\partial G}{\partial s} \right) : (\mathbb{R} \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}, \mathbf{0})$$

is non-singular, where $(s, \mathbf{v}) = (s, v_1, \dots, v_n) \in (\mathbb{R} \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $n - 1$ -dimensional submanifold,

$$\Sigma_*(G) = \left\{ (s, \mathbf{v}) \in (\mathbb{R} \times \mathbb{R}^n, \mathbf{0}) \mid G(s, \mathbf{v}) = \frac{\partial G}{\partial s}(s, \mathbf{v}) = 0 \right\}$$

and the map germ $\Phi_G : (\Sigma_*(G), \mathbf{0}) \longrightarrow \text{PT}^*\mathbb{R}^n$ defined by

$$\Phi_G(s, \mathbf{v}) = \left(\mathbf{v}, \left[\frac{\partial G}{\partial v_1}(s, \mathbf{v}) : \dots : \frac{\partial G}{\partial v_n}(s, \mathbf{v}) \right] \right)$$

is a Legendrian immersion germ. We call G a generating family of $\Phi_G(\Sigma_*G)$. Therefore the corresponding wave front is

$$W(\Phi_G) = \left\{ \mathbf{v} \in \mathbb{R}^n \mid \exists s \in \mathbb{R} \text{ such that } G(s, \mathbf{v}) = \frac{\partial G}{\partial s}(s, \mathbf{v}) = 0 \right\}.$$

We sometimes denote $D_G = W(\Phi_G)$ and call it the *discriminant set* of G .

Now, we can apply the above arguments to our situation and we get the following proposition which indicates that parallel lightcone developables, dual surface and Bishop pseudo-spherical images of space-like curves can be seen as wave fronts which have Legendrian singularities in the framework of the theory of Legendrian singularity and Bishop pseudo-spherical Darboux images can be seen as caustics which have Lagrangian singularities in the framework of the theory of Lagrangian singularity (see [1] for details). The main results of this paper are contained in the following proposition and theorem.

Proposition 2.2. *The parallel developables and dual surface are two dimensional wave fronts which have A_1 -singularity, A_2 -singularity, and A_3 -singularity. Bishop pseudo-spherical images are one dimensional wave fronts which have A_1 -singularity and A_2 -singularity. Bishop pseudo-spherical Darboux images can be seen as one dimensional caustics which have A_2 -singularity and A_3 -singularity.*

The following theorem gives some more detail descriptions on the conditions of those singularities.

Theorem 2.3. *Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed regular spacelike curve, we have the following.*

(A) *For $\mathbf{v}_0 = \mathfrak{B}\mathfrak{L}\mathfrak{D}^\pm(s_0, \mu_0)$ and the lightcone bundle $\mathfrak{L}\mathfrak{C}(\mathbf{v}_0)$ of the curve, suppose that*

$$\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0.$$

Then, one has the following claims.

- (1) *The curve $\gamma(s)$ and $\mathfrak{L}\mathfrak{C}(\mathbf{v}_0)$ have at least 2-point contact for s_0 .*
- (2) *The curve $\gamma(s)$ and $\mathfrak{L}\mathfrak{C}(\mathbf{v}_0)$ have at least 3-point contact for s_0 if and only if*

$$\mathbf{v}_0 = \gamma(s_0) + \frac{1}{\delta(\mathbf{N}_1(s_0))k_1(s_0) \pm \delta(\mathbf{N}_2(s_0))k_2(s_0)} (\mathbf{N}_1(s_0) \pm \mathbf{N}_2(s_0))$$

and $\delta(\mathbf{N}_1(s_0))k_1'(s_0) \pm \delta(\mathbf{N}_2(s_0))k_2'(s_0) \neq 0$. Under this condition, the germ of image $\mathfrak{B}\mathfrak{L}\mathfrak{D}^\pm(s, \mu)$ at $\mathfrak{B}\mathfrak{L}\mathfrak{D}^\pm(s_0, \mu_0)$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ and the locus of its singularity points $\mathfrak{B}\mathfrak{L}\mathfrak{D}^\pm(s_0, \mu_0)$ is locally diffeomorphic to the line (cf., Figure 1), where

$$\mu_0 = \frac{-1}{\delta(\mathbf{N}_1(s_0))k_1(s_0) \pm \delta(\mathbf{N}_2(s_0))k_2(s_0)}.$$

- (3) *The curve $\gamma(s)$ and $\mathfrak{L}\mathfrak{C}(\mathbf{v}_0)$ have at least 4-point contact for s_0 if and only if*

$$\mathbf{v}_0 = \gamma(s_0) + \frac{1}{\delta(\mathbf{N}_1(s_0))k_1(s_0) \pm \delta(\mathbf{N}_2(s_0))k_2(s_0)} (\mathbf{N}_1(s_0) \pm \mathbf{N}_2(s_0)),$$

$\delta(\mathbf{N}_1(s_0))k_1'(s_0) \pm \delta(\mathbf{N}_2(s_0))k_2'(s_0) = 0$ and

$$(k_1(s_0) \mp k_2(s_0))^2 + (k_1(s_0) \mp k_2(s_0))(k_2''(s_0) - k_1''(s_0)) - \delta(\mathbf{N}_1(s_0))k_1''(s_0) \mp \delta(\mathbf{N}_2(s_0))k_2''(s_0) \neq 0.$$

Under this condition, the germ of image $\mathfrak{B}\mathcal{L}\mathcal{D}^\pm(s, \mu)$ at $\mathfrak{B}\mathcal{L}\mathcal{D}^\pm(s_0, \mu_0)$ is locally diffeomorphic to the swallowtail and the locus of its singularity points $\mathfrak{B}\mathcal{L}\mathcal{D}^\pm(s_0, \frac{-1}{\delta(\mathbf{N}_1(s_0))k_1(s_0) \pm \delta(\mathbf{N}_2(s_0))k_2(s_0)})$ is locally diffeomorphic to the (2, 3, 4)-cusp (cf., Figure 2).

(B) Suppose that $\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0$ and $k_1(s) \neq 0$. Then, one have the following claims.

(1) The Bishop dual $\mathcal{B}\mathcal{D}\mathcal{U}_\gamma$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at s_0 if

$$\mathbf{v} = \pm \frac{|k_1(s_0)|}{\sqrt{|k_1^2(s_0) - k_2^2(s_0)|}} \left(\frac{k_2(s_0)}{k_1(s_0)} \mathbf{N}_1(s_0) + \mathbf{N}_2(s_0) \right),$$

$\mathbf{u} = \langle \boldsymbol{\gamma}(s_0), \mathbf{v} \rangle$ and $\rho(s_0) \neq 0$. Under this condition, the pseudo-spherical Darboux image of spacelike curve is locally diffeomorphic to a line $\{0\} \times \mathbb{R}$ at s_0 (cf., Figure 1).

(2) The Bishop dual $\mathcal{B}\mathcal{D}\mathcal{U}_\gamma$ is locally diffeomorphic to the SW at s_0 if

$$\mathbf{v} = \pm \frac{|k_1(s_0)|}{\sqrt{|k_1^2(s_0) - k_2^2(s_0)|}} \left(\frac{k_2(s_0)}{k_1(s_0)} \mathbf{N}_1(s_0) + \mathbf{N}_2(s_0) \right),$$

$\mathbf{u} = \langle \boldsymbol{\gamma}(s_0), \mathbf{v} \rangle$, $\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$. Under this condition, the pseudo-spherical Darboux image of spacelike curve is locally diffeomorphic to the cusp C at s_0 (cf., Figures 2 and 3).

(C) Suppose that $k_2(s) \neq 0$. Then, one has the following claims.

- (1) The first Bishop pseudo-spherical image $\mathbf{N}_1(s)$ is locally diffeomorphic to a line $\{0\} \times \mathbb{R}$ at s_0 if $k_1(s_0) \neq 0$.
- (2) The first Bishop pseudo-spherical image $\mathbf{N}_1(s)$ is locally diffeomorphic to the cusp C at s_0 if $k_1(s_0) = 0$ and $k_1'(s_0) \neq 0$ (cf., Figure 3).

(D) Suppose that $k_1(s) \neq 0$. Then, one has the following claims.

- (1) The second Bishop pseudo-spherical image $\mathbf{N}_2(s)$ is locally diffeomorphic to a line $\{0\} \times \mathbb{R}$ at s_0 if $k_2(s_0) \neq 0$.
- (2) The second Bishop pseudo-spherical image $\mathbf{N}_2(s)$ is locally diffeomorphic to the cusp C at s_0 if $k_2(s_0) = 0$ and $k_2'(s_0) \neq 0$ (cf., Figure 3).

Here, cusp is $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$, cuspidal edge is $C \times \mathbb{R} = \{(x_1, x_2) \mid x_1^2 = x_2^3\} \times \mathbb{R}$, swallowtail is

$$SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\},$$

and (2, 3, 4)-cusp is $C(2, 3, 4) = \{(t^2, t^3, t^4) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$. The pictures of cuspidal edge, swallowtail, and cusp will be seen in Figures 1, 2, and 3, respectively.

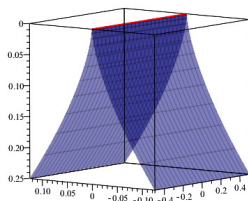


Figure 1: Cuspidal edge.

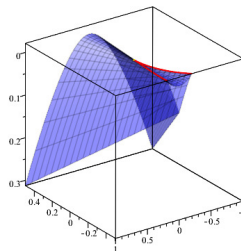


Figure 2: Swallowtail.

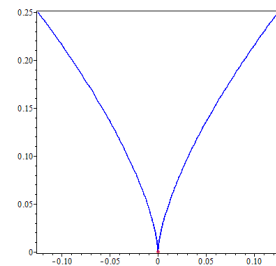


Figure 3: Cusp.

3. Functions on spacelike curve which can be seen as generating families

The purpose of this section is to construct some functions which can be seen as generating families and these functions will be useful to prove the main results.

We need to use some general results on the singularity theory for families of function germs. Detailed descriptions can be found in the book [3]. Let $F(s, \mathbf{x})$ be an unfolding of $f(s)$ and $f(s)$ has A_k -singularity ($k \geq 1$) at s_0 . We denote the $(k - 1)$ -jet of the partial derivative $\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0)$ at s_0 by

$$j^{(k-1)} \left(\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0) \right) (s_0) = \sum_{j=1}^{k-1} a_{ji} s^j$$

for $i = 1, \dots, r$. Then, $F(s, \mathbf{x})$ is called an \mathcal{R} -versal unfolding if the $k \times r$ matrix of coefficients (a_{0i}, a_{ji}) has rank k , ($k \leq r$), where $a_{0i} = \frac{\partial F}{\partial x_i}(s_0, \mathbf{x}_0)$. $F(s, \mathbf{x})$ is called an \mathcal{R}^+ -versal unfolding, if the $(k - 1) \times r$ matrix of coefficients (a_{ji}) has rank $k - 1$, ($k - 1 \leq r$). We now introduce the other important set concerning unfoldings. The bifurcation set of an \mathcal{R}^+ -versal unfolding $F(s, \mathbf{x})$ is the set

$$\mathfrak{B}_F = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \frac{\partial F}{\partial s}(s, \mathbf{x}) = \frac{\partial^2 F}{\partial s^2}(s, \mathbf{x}) = 0 \right\}$$

and it can be seen as caustic from the viewpoint of Lagrangian singularity theory. To prove the main results in this paper, we need the following well-known result ([3]).

Theorem 3.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be an r -parameter unfolding of $f(s)$ which has the A_k singularity at s_0 .*

(A) *Suppose that $F(s, \mathbf{x})$ is an \mathcal{R} -versal unfolding, then we have the following:*

- (1) *if $k = 1$, then \mathfrak{D}_F is locally diffeomorphic to $0 \times \mathbb{R}^{r-1}$;*
- (2) *if $k = 2$, then \mathfrak{D}_F is locally diffeomorphic to $\mathbb{C} \times \mathbb{R}^{r-2}$;*
- (3) *if $k = 3$, then \mathfrak{D}_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

(B) *Suppose that $F(s, \mathbf{x})$ is an \mathcal{R}^+ -versal unfolding, then we have the following:*

- (1) *if $k = 2$, then \mathfrak{B}_F is locally diffeomorphic to $0 \times \mathbb{R}^{r-1}$;*
- (2) *if $k = 3$, then \mathfrak{B}_F is locally diffeomorphic to $\mathbb{C} \times \mathbb{R}^{r-2}$.*

We will apply the above unfolding theory of function germ to the following five different families of functions on a spacelike curve to prove Proposition 2.2 and Theorem 2.3.

Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed regular spacelike curve. We define a function

$$G : I \times \mathbb{R}_1^3 \rightarrow \mathbb{R}, \text{ by } G(s, \mathbf{v}) = \langle \gamma(s) - \mathbf{v}, \gamma(s) - \mathbf{v} \rangle.$$

We call G the Lorentzian distance-square function on a spacelike curve γ . For any fixed vector $\mathbf{v} \in \mathbb{R}_1^3$, we denote $g_{\mathbf{v}}(s) = G(s, \mathbf{v})$. We also define two families of smooth functions

$$H : I \times Q_{\epsilon}^2 \rightarrow \mathbb{R} \text{ by } H(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle$$

and

$$\tilde{H} : I \times Q_{\epsilon}^2 \times \mathbb{R} \rightarrow \mathbb{R} \text{ by } \tilde{H}(s, \mathbf{v}, u) = H(s, \mathbf{v}) - u = \langle \gamma(s), \mathbf{v} \rangle - u.$$

We call each of them height function and extended height function of γ , respectively. For any fixed $\mathbf{v} \in Q_{\epsilon}^2$, we denote $h_{\mathbf{v}}(s) = H(s, \mathbf{v})$ and $\tilde{h}_{\mathbf{v}}(s) = \tilde{H}(s, \mathbf{v})$. We define two families of smooth functions on I as follows:

$$H_i : I \times Q_{\epsilon}^2 \rightarrow \mathbb{R} \text{ by } H_i(s, \mathbf{v}) = \langle \mathbf{N}_i(s), \mathbf{v} \rangle,$$

where $i = 1, 2$. We call it the first (resp. second) Bishop normal indicatrix height function when $i = 1$ (resp. $i = 2$). For any fixed $\mathbf{v} \in Q_{\epsilon}^2$, we denote $h_{i\mathbf{v}}(s) = H_i(s, \mathbf{v})$. We have the following proposition.

Proposition 3.2. *Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed regular spacelike curve.*

(A) *Suppose $\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0$, then we have the following.*

(1) $g_v(s) = g'_v(s) = 0$ if and only if

$$\gamma(s) - \mathbf{v} = \lambda(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)), \lambda \in \mathbb{R} \setminus \{0\};$$

(2) $g_v(s) = g'_v(s) = g''_v(s) = 0$ if and only if

$$\gamma(s) - \mathbf{v} = \frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} (\mathbf{N}_1(s) \pm \mathbf{N}_2(s))$$

and $\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0$;

(3) $g_v(s) = g'_v(s) = g''_v(s) = g_v^{(3)}(s) = 0$ if and only if

$$\gamma(s) - \mathbf{v} = \frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} (\mathbf{N}_1(s) \pm \mathbf{N}_2(s))$$

and

$$\delta(\mathbf{N}_1(s))k'_1(s) \pm \delta(\mathbf{N}_2(s))k'_2(s) = 0;$$

(4) $g_v(s) = g'_v(s) = g''_v(s) = g_v^{(3)}(s) = g_v^{(4)}(s) = 0$ if and only if

$$\gamma(s) - \mathbf{v} = \frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} (\mathbf{N}_1(s) \pm \mathbf{N}_2(s)),$$

$\delta(\mathbf{N}_1(s))k'_1(s) \pm \delta(\mathbf{N}_2(s))k'_2(s) = 0$ and

$$(k_1(s) \mp k_2(s))^2 + (k_1(s) \mp k_2(s))(k_2^2(s) - k_1^2(s)) - \delta(\mathbf{N}_1(s))k_1''(s) \mp \delta(\mathbf{N}_2(s))k_2''(s) = 0.$$

(B) Suppose that $\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0$ and $k_1(s) \neq 0$. Then we have the following.

(1) $h'_v(s) = 0$ if and only if there are real numbers λ and μ such that $\mathbf{v} = \lambda\mathbf{N}_1(s) + \mu\mathbf{N}_2(s)$ and

$$\delta(\mathbf{N}_1(s))\lambda^2 + \delta(\mathbf{N}_2(s))\mu^2 = \pm 1;$$

(2) $h'_v(s) = h''_v(s) = 0$ if and only if

$$\mathbf{v} = \pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right);$$

(3) $h'_v(s) = h''_v(s) = h_v^{(3)}(s) = 0$ if and only if

$$\mathbf{v} = \pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right)$$

and

$$k'_1(s)k_2(s) - k_1(s)k'_2(s) = 0;$$

(4) $h'_v(s) = h''_v(s) = h_v^{(3)}(s) = h_v^{(4)}(s) = 0$ if and only if

$$\mathbf{v} = \pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right)$$

and

$$k'_1(s)k_2(s) - k_1(s)k'_2(s) = (k'_1(s)k_2(s) - k_1(s)k'_2(s))' = 0.$$

(C) Suppose that $\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0$ and $k_1(s) \neq 0$. Then we have the following.

- (1) $\tilde{h}_v(s) = 0$ if and only if $\mathbf{u} = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle$;
- (2) $\tilde{h}_v(s) = \tilde{h}'_v(s) = 0$ if and only if there are real numbers λ and μ such that $\mathbf{v} = \lambda \mathbf{N}_1(s) + \mu \mathbf{N}_2(s)$, $\delta(\mathbf{N}_1(s))\lambda^2 + \delta(\mathbf{N}_2(s))\mu^2 = \pm 1$ and $\mathbf{u} = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle$;
- (3) $\tilde{h}_v(s) = \tilde{h}'_v(s) = \tilde{h}''_v(s) = 0$ if and only if

$$\mathbf{v} = \pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right)$$

and $\mathbf{u} = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle$;

- (4) $\tilde{h}_v(s) = \tilde{h}'_v(s) = \tilde{h}''_v(s) = \tilde{h}^{(3)}_v(s) = 0$ if and only if

$$\mathbf{v} = \pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right),$$

$\mathbf{u} = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle$, and $k'_1(s)k_2(s) - k_1(s)k'_2(s) = 0$;

- (5) $\tilde{h}_v(s) = \tilde{h}'_v(s) = \tilde{h}''_v(s) = \tilde{h}^{(3)}_v(s) = \tilde{h}^{(4)}_v(s) = 0$ if and only if

$$\mathbf{v} = \pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right),$$

$\mathbf{u} = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle$, and $k'_1(s)k_2(s) - k_1(s)k'_2(s) = (k'_1(s)k_2(s) - k_1(s)k'_2(s))' = 0$.

(D) Suppose that $k_1(s) \neq 0$. Then, we have the following claims.

- (1) $h_{1v}(s) = h'_{1v}(s) = 0$ if and only if $\mathbf{v} = \mp \mathbf{N}_2(s)$;
- (2) $h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = 0$ if and only if $\mathbf{v} = \mp \mathbf{N}_2(s)$ and $k_2(s) = 0$;
- (3) $h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = h^{(3)}_{1v}(s) = 0$ if and only if $\mathbf{v} = \mp \mathbf{N}_2(s)$ and $k_2(s) = k'_2(s) = 0$.

(E) Suppose that $k_2(s) \neq 0$. Then, one has the following claims.

- (1) $h_{2v}(s) = h'_{2v}(s) = 0$ if and only if $\mathbf{v} = \mp \mathbf{N}_1(s)$;
- (2) $h_{2v}(s) = h'_{2v}(s) = h''_{2v}(s) = 0$ if and only if $\mathbf{v} = \mp \mathbf{N}_1(s)$ and $k_1(s) = 0$;
- (3) $h_{2v}(s) = h'_{2v}(s) = h''_{2v}(s) = h^{(3)}_{2v}(s) = 0$ if and only if $\mathbf{v} = \mp \mathbf{N}_1(s)$ and $k_1(s) = k'_1(s) = 0$.

Proof.

(A) (1) Since $\frac{1}{2}g'_v(s) = \langle \boldsymbol{\gamma}(s) - \mathbf{v}, \mathbf{T}(s) \rangle = 0$, we have that there are real numbers λ and μ such that $\boldsymbol{\gamma}(s) - \mathbf{v} = \lambda \mathbf{N}_1(s) + \mu \mathbf{N}_2(s)$. Thus, $g_v(s) = g'_v(s) = 0$ if and only if $\boldsymbol{\gamma}(s) - \mathbf{v} = \lambda \mathbf{N}_1(s) + \mu \mathbf{N}_2(s)$ and $\langle \lambda \mathbf{N}_1(s) + \mu \mathbf{N}_2(s), \lambda \mathbf{N}_1(s) + \mu \mathbf{N}_2(s) \rangle = 0$. This is equivalent to the condition that

$$\boldsymbol{\gamma}(s) - \mathbf{v} = \lambda(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)).$$

(2) Since $\frac{1}{2}g''_v(s) = 1 + \langle \boldsymbol{\gamma}(s) - \mathbf{v}, k_1(s)\mathbf{N}_1(s) + k_2(s)\mathbf{N}_2(s) \rangle = 0$, we have

$$\lambda(\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)) + 1 = 0.$$

Since $k_1(s) \pm k_2(s) \neq 0$, this means that $\lambda = \frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)}$. Thus, $g_v(s) = g'_v(s) = g''_v(s) = 0$ if and only if

$$\boldsymbol{\gamma}(s) - \mathbf{v} = \frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} (\mathbf{N}_1(s) \pm \mathbf{N}_2(s)).$$

(3) Since

$$\frac{1}{2}g^{(3)}_v(s) = \langle \boldsymbol{\gamma}(s) - \mathbf{v}, k'_1(s)\mathbf{N}_1(s) + k'_2(s)\mathbf{N}_2(s) - (\delta(\mathbf{N}_1(s))k_1^2(s) + \delta(\mathbf{N}_2(s))k_2^2(s))\mathbf{T}(s) \rangle = 0,$$

we have

$$\frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} \delta(\mathbf{N}_1(s))k'_1(s) \pm (\delta(\mathbf{N}_2(s))k'_2(s)) = 0.$$

This means that $\delta(\mathbf{N}_1(s))k'_1(s) \pm \delta(\mathbf{N}_2(s))k'_2(s) = 0$. Thus, $g_v(s) = g'_v(s) = g''_v(s) = g^{(3)}_v(s) = 0$ if and only if

$$\gamma(s) - \mathbf{v} = \frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} (\mathbf{N}_1(s) \pm \mathbf{N}_2(s))$$

and

$$\delta(\mathbf{N}_1(s))k_1'(s) \pm \delta(\mathbf{N}_2(s))k_2'(s) = 0.$$

(4) Since

$$\frac{1}{2}g_v^{(4)}(s) = -(\delta(\mathbf{N}_1(s))k_1^2(s) + \delta(\mathbf{N}_2(s))k_2^2(s)) + \langle \gamma(s) - \mathbf{v}, \alpha_1\mathbf{N}_1(s) + \alpha_2\mathbf{N}_2(s) + \alpha_3\mathbf{T}(s) \rangle = 0,$$

where

$$\begin{cases} \alpha_1 = (k_1''(s) - (\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s))k_1(s)), \\ \alpha_2 = (k_2''(s) - (\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s))k_2(s)), \\ \alpha_3 = -3(\delta(\mathbf{N}_1(s))k_1(s)k_1'(s) + \delta(\mathbf{N}_2(s))k_2(s)k_2'(s)), \end{cases}$$

we have

$$-(\delta(\mathbf{N}_1(s))k_1^2(s) + \delta(\mathbf{N}_2(s))k_2^2(s)) + \frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} \alpha(s) = 0.$$

Here

$$\alpha(s) = ((k_1''(s) - (\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s))k_1(s)) \pm (k_2''(s) - ((\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s))k_2(s)))).$$

This means that $(k_1(s) \mp k_2(s))^2 + (k_1(s) \mp k_2(s))(k_2^2(s) - k_1^2(s)) - \delta(\mathbf{N}_1(s))k_1''(s) \mp \delta(\mathbf{N}_2(s))k_2''(s) = 0$. Thus, $g_v(s) = g_v'(s) = g_v''(s) = g_v'''(s) = g_v^{(4)}(s) = 0$ if and only if

$$\gamma(s) - \mathbf{v} = \frac{-1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} (\mathbf{N}_1(s) \pm \mathbf{N}_2(s)),$$

$\delta(\mathbf{N}_1(s))k_1'(s) \pm \delta(\mathbf{N}_2(s))k_2'(s) = 0$, and

$$(k_1(s) \mp k_2(s))^2 + (k_1(s) \mp k_2(s))(k_2^2(s) - k_1^2(s)) - \delta(\mathbf{N}_1(s))k_1''(s) \mp \delta(\mathbf{N}_2(s))k_2''(s) = 0.$$

(B) (1) Since $h_v'(s) = \langle \gamma'(s), \mathbf{v} \rangle = \langle \mathbf{T}(s), \mathbf{v} \rangle = 0$, we have that there are real numbers λ and μ such that $\mathbf{v} = \lambda\mathbf{N}_1(s) + \mu\mathbf{N}_2(s)$. By the condition that $\mathbf{v} \in Q_e^2$, we get $\delta(\mathbf{N}_1(s))\lambda^2 + \delta(\mathbf{N}_2(s))\mu^2 = \pm 1$. The converse direction also holds.

(2) Since $h_v''(s) = \langle k_1(s)\mathbf{N}_1(s) + k_2(s)\mathbf{N}_2(s), \mathbf{v} \rangle = 0$, $\delta(\mathbf{N}_1(s))k_1(s)\lambda + \delta(\mathbf{N}_2(s))k_2(s)\mu = 0$. It follows from the fact $\delta(\mathbf{N}_1(s))\lambda^2 + \delta(\mathbf{N}_2(s))\mu^2 = \pm 1$ that

$$\mu = \pm \frac{|k_1(s)|}{\sqrt{|\delta(\mathbf{N}_2(s))k_1^2(s) + \delta(\mathbf{N}_1(s))k_2^2(s)|}} = \pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}}.$$

Therefore, we have

$$\mathbf{v} = \pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right).$$

(3) Since $h_v^{(3)}(s) = \langle k_1'(s)\mathbf{N}_1(s) + k_2'(s)\mathbf{N}_2(s) - (\delta(\mathbf{N}_1(s))k_1^2(s) + \delta(\mathbf{N}_2(s))k_2^2(s))\mathbf{T}, \mathbf{v} \rangle = 0$, we have

$$\pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{\delta(\mathbf{N}_1(s))k_1'(s)k_2(s) + \delta(\mathbf{N}_2(s))k_1(s)k_2'(s)}{k_1(s)} \right) = 0.$$

By the condition of (3), this is equivalent to the condition $k_1'(s)k_2(s) - k_1(s)k_2'(s) = 0$.

(4) Since $h_v^{(4)}(s) = \langle \beta_1 \mathbf{N}_1(s) + \beta_2 \mathbf{N}_2(s) - \beta_3 \mathbf{T}(s), \mathbf{v} \rangle = 0$, where

$$\begin{cases} \beta_1 = (k_1''(s) - \delta(\mathbf{N}_1(s))k_1^3(s) - \delta(\mathbf{N}_2(s))k_1(s)k_2^2(s)), \\ \beta_2 = (k_2''(s) - \delta(\mathbf{N}_1(s))k_1^2(s)k_2(s) - \delta(\mathbf{N}_2(s))k_2^3(s)), \\ \beta_3 = 3(\delta(\mathbf{N}_1(s))k_1(s)k_1'(s) + \delta(\mathbf{N}_2(s))k_2(s)k_2'(s)), \end{cases}$$

we have

$$\pm \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{\delta(\mathbf{N}_1(s))k_1''(s)k_2(s) + \delta(\mathbf{N}_2(s))k_1(s)k_2''(s)}{k_1(s)} \right) = 0.$$

By the condition of (4), this is equivalent to the condition

$$k_1'(s)k_2(s) - k_1(s)k_2'(s) = (k_1'(s)k_2(s) - k_1(s)k_2'(s))' = 0.$$

(C) The proof of (C) follows from the proof of (B), so we omit it.

(D) (1) If

$$h_{1v}(s) = \langle \mathbf{N}_1(s), \mathbf{v} \rangle = 0,$$

then we have that there are real numbers λ and μ such that $\mathbf{v} = \lambda \mathbf{T}(s) + \mu \mathbf{N}_2(s)$. Moreover, in combination with $\mathbf{v} \in Q_\epsilon^2$, $\lambda^2 + \delta(\mathbf{N}_2(s))\mu^2 = \pm 1$. It follows that $h_{1v}(s) = 0$ if and only if $\mathbf{v} = \lambda \mathbf{T}(s) + \mu \mathbf{N}_2(s)$ and $\lambda^2 + \delta(\mathbf{N}_2(s))\mu^2 = \pm 1$. When $h_{1v}(s) = 0$, the assertion (1) follows from the fact that

$$h'_{1v}(s) = \langle -\delta(\mathbf{N}_1(s))k_1(s)\mathbf{T}(s), \mathbf{v} \rangle = -\delta(\mathbf{N}_1(s))k_1(s)\lambda$$

and $k_1(s) \neq 0$. Thus, we get that $h_{1v}(s) = h'_{1v}(s) = 0$ if and only if $\mathbf{v} = \mp \mathbf{N}_2(s)$.

(2) When $h_{1v}(s) = h'_{1v}(s) = 0$, the assertion (2) follows from the fact that

$$h''_{1v}(s) = \langle -\delta(\mathbf{N}_1(s))k_1'(s)\mathbf{T}(s) - \delta(\mathbf{N}_1(s))k_1^2(s)\mathbf{N}_1(s) - \delta(\mathbf{N}_1(s))k_1(s)k_2(s)\mathbf{N}_2(s), \mp \mathbf{N}_2(s) \rangle = \mp k_1(s)k_2(s)$$

and $k_1(s) \neq 0$.

(3) Under the condition that $h_{1v}(s) = h'_{1v}(s) = h''_{1v}(s) = 0$, this derivative is computed as follows: $h_{1v}^{(3)}(s) = \langle (-\delta(\mathbf{N}_1(s))k_1''(s) + k_1^3(s) - k_1(s)k_2^2(s))\mathbf{T}(s) - 3\delta(\mathbf{N}_1(s))k_1(s)k_1'(s)\mathbf{N}_1(s) - \delta(\mathbf{N}_1(s))(2k_1'(s)k_2(s) + k_1(s)k_2'(s))\mathbf{N}_2(s), \mp \mathbf{N}_2(s) \rangle = \mp(2k_1'(s)k_2(s) + k_1(s)k_2'(s)) = \mp k_1(s)k_2'(s)$. Since $k_1(s) \neq 0$, we get that $h_{1v}^{(3)}(s) = 0$ is equivalent to the condition $k_2'(s) = 0$. The assertion (3) follows.

(E) Using the same computation as the proof of (D), we can get (E). □

Proposition 3.3. Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a unit speed regular spacelike curve, then we have the following results.

(1) Suppose $\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0$. Then $\delta(\mathbf{N}_1(s))k_1'(s) \pm \delta(\mathbf{N}_2(s))k_2'(s) = 0$ if and only if each of

$$p_\pm = \gamma(s) + \frac{1}{\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s)} (\mathbf{N}_1(s) \pm \mathbf{N}_2(s))$$

is a constant vector.

(2) Suppose $k_1(s) \neq 0$. Then $k_1(s)k_2'(s) - k_1'(s)k_2(s) = 0$ if and only if each of

$$\mathbf{v} = \mp \frac{|k_1(s)|}{\sqrt{|k_1^2(s) - k_2^2(s)|}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) + \mathbf{N}_2(s) \right)$$

is a constant vector.

(3) Suppose $k_1(s) \neq 0$. Then, $k_2(s) = 0$ if and only if each of $\mathbf{v} = \mp \mathbf{N}_2(s)$ is a constant vector.

(4) Suppose $k_2(s) \neq 0$. Then, $k_1(s) = 0$ if and only if each of $\mathbf{v} = \mp \mathbf{N}_1(s)$ is a constant vector.

Then, we have the following proposition.

Proposition 3.4. *We consider functions defined in Proposition 3.2. Then we have the following claims.*

- (1) *If $g_{v_0}(s)$ has A_k -singularity ($k = 1, 2, 3$) at s_0 , then $G(s, \mathbf{v})$ is an \mathcal{R} -versal unfoldings of $g_{v_0}(s)$.*
- (2) *If $h_{v_0}(s)$ has A_k -singularity ($k = 2, 3$) at s_0 , then $H(s, \mathbf{v})$ is an \mathcal{R}^+ -versal unfolding of $h_{v_0}(s)$.*
- (3) *If $\tilde{h}_{v_0}(s)$ has A_k -singularity ($k = 2, 3$) at s_0 , then $\tilde{H}(s, \mathbf{v})$ is an \mathcal{R} -versal unfolding of $\tilde{h}_{v_0}(s)$.*
- (4) *If $h_{1v_0}(s)$ has A_k -singularity ($k = 1, 2$) at s_0 , then $H_1(s, \mathbf{v})$ is an \mathcal{R} -versal unfolding of $h_{1v_0}(s)$.*
- (5) *If $h_{2v_0}(s)$ has A_k -singularity ($k = 1, 2$) at s_0 , then $H_2(s, \mathbf{v})$ is an \mathcal{R} -versal unfolding of $h_{2v_0}(s)$.*

Proof.

(1) We denote that $\gamma(s) = (x_1(s), x_2(s), x_3(s))$ and $\mathbf{v} = (v_1, v_2, v_3)$. Under this notation we have

$$G(s, \mathbf{v}) = -(x_1(s) - v_1)^2 + (x_2(s) - v_2)^2 + (x_3(s) - v_3)^2.$$

Thus we have

$$\begin{aligned} \frac{\partial G}{\partial v_1}(s, \mathbf{v}) &= 2(x_1(s) - v_1), & \frac{\partial G}{\partial v_i}(s, \mathbf{v}) &= -2(x_i(s) - v_i), \quad i = 2, 3, \\ \frac{\partial}{\partial s} \frac{\partial G}{\partial v_1}(s, \mathbf{v}) &= 2x_1'(s), & \frac{\partial}{\partial s} \frac{\partial G}{\partial v_i}(s, \mathbf{v}) &= -2x_i'(s), \quad i = 2, 3, \\ \frac{\partial^2}{\partial s^2} \frac{\partial G}{\partial v_1}(s, \mathbf{v}) &= 2x_1''(s), & \frac{\partial^2}{\partial s^2} \frac{\partial G}{\partial v_i}(s, \mathbf{v}) &= -2x_i''(s), \quad i = 2, 3. \end{aligned}$$

Therefore the 2-jet of $\frac{\partial G}{\partial v_i}(s, \mathbf{v})$ ($i = 1, 2, 3$) at s_0 is given by

$$j^2\left(\frac{\partial G}{\partial v_i}(s, \mathbf{v}_0)\right)(s_0) = \frac{\partial}{\partial s} \frac{\partial G}{\partial v_i}(s - s_0) + \frac{1}{2} \frac{\partial^2}{\partial s^2} \frac{\partial G}{\partial v_i}(s - s_0)^2 = a_{1i}(s - s_0) + \frac{1}{2} a_{2i}(s - s_0)^2.$$

It is enough to show that the rank of the matrix A is 3, where

$$A = \begin{pmatrix} 2(x_1(s_0) - v_1) & -2(x_2(s_0) - v_2) & -2(x_3(s_0) - v_3) \\ 2x_1'(s_0) & -2x_2'(s_0) & -2x_3'(s_0) \\ 2x_1''(s_0) & -2x_2''(s_0) & -2x_3''(s_0) \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det A &= 8 \det(\gamma(s_0) - \mathbf{v}, \mathbf{T}(s_0), \mathbf{T}'(s_0)) \\ &= 8 \det\left(\frac{-1}{\delta(\mathbf{N}_1(s_0))k_1(s_0) \pm \delta(\mathbf{N}_2(s_0))k_2(s_0)} (\mathbf{N}_1(s_0) \pm \mathbf{N}_2(s_0)), \mathbf{T}, k_1(s_0)\mathbf{N}_1 + k_2(s_0)\mathbf{N}_2\right) \\ &= \frac{8}{\delta(\mathbf{N}_1(s_0))k_1(s_0) \pm \delta(\mathbf{N}_2(s_0))k_2(s_0)} (k_2(s_0) \mp k_1(s_0)) \\ &= \pm 8 \neq 0, \end{aligned}$$

which implies that the rank of A is 3. If we consider the matrix which consists of the first and the second rows of the matrix A , so that the rank of this matrix is two. This completes the proof.

(2) We denote that $\gamma(s) = (x_1(s), x_2(s), x_3(s))$, and $\mathbf{v} = (v_1, v_2, \pm\sqrt{\pm 1 + v_1^2 - v_2^2})$. Under this notation we have

$$H(s, \mathbf{v}) = -x_1(s)v_1 + x_2(s)v_2 \pm x_3(s)\sqrt{\pm 1 + v_1^2 - v_2^2}.$$

Thus we have

$$\frac{\partial H}{\partial v_1} = -x_1(s) \pm \frac{v_1 x_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}, \quad \frac{\partial H}{\partial v_2} = x_2(s) \mp \frac{v_2 x_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}},$$

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial H}{\partial v_1} &= -x'_1(s) \pm \frac{v_1 x'_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}, & \frac{\partial}{\partial s} \frac{\partial H}{\partial v_2} &= x'_2(s) \mp \frac{v_2 x'_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}, \\ \frac{\partial^2}{\partial s^2} \frac{\partial H}{\partial v_1} &= -x''_1(s) \pm \frac{v_1 x''_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}, & \frac{\partial^2}{\partial s^2} \frac{\partial H}{\partial v_2} &= x''_2(s) \mp \frac{v_2 x''_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}. \end{aligned}$$

Therefore the 2-jet of $\frac{\partial H}{\partial v_i}(s, \mathbf{v})$ ($i = 1, 2$) at s_0 is given by

$$j^2\left(\frac{\partial H}{\partial v_i}(s, \mathbf{v}_0)\right)(s_0) = \frac{\partial}{\partial s} \frac{\partial H}{\partial v_i}(s - s_0) + \frac{1}{2} \frac{\partial^2}{\partial s^2} \frac{\partial H}{\partial v_i}(s - s_0)^2 = a_{1i}(s - s_0) + \frac{1}{2} a_{2i}(s - s_0)^2.$$

It is enough to show that the rank of the matrix B is 2, where

$$B = \begin{pmatrix} -x'_1(s_0) \pm \frac{v_1 x'_3(s_0)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} & x'_2(s_0) \mp \frac{v_2 x'_3(s_0)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} \\ -x''_1(s_0) \pm \frac{v_1 x''_3(s_0)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} & x''_2(s_0) \mp \frac{v_2 x''_3(s_0)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} \end{pmatrix}.$$

We have

$$\begin{aligned} \det B &= \frac{v_1}{v_3} (x''_2 x'_3 - x'_2 x''_3)(s_0) + \frac{v_2}{v_3} (x'_1 x''_3 - x''_1 x'_3)(s_0) - (x'_1 x''_2 - x''_1 x'_2)(s_0) \\ &= -\frac{1}{v_3} \langle \mathbf{v}, \mathbf{T}(s_0) \wedge \mathbf{T}'(s_0) \rangle \\ &= -\frac{1}{v_3} \langle \mathbf{v}, \mathbf{T}(s_0) \wedge k_1(s_0) \mathbf{N}_1(s_0) + k_2(s_0) \mathbf{N}_2(s_0) \rangle \\ &= -\frac{1}{v_3} \langle \mathbf{v}, k_1(s_0) \delta(\mathbf{N}_2(s_0)) \mathbf{N}_2(s_0) - k_2(s_0) \delta(\mathbf{N}_1(s_0)) \mathbf{N}_1(s_0) \rangle \\ &= -\frac{1}{v_3} \left\langle \pm \frac{|k_1(s_0)|}{\sqrt{|k_1^2(s_0) - k_2^2(s_0)|}} \left(\frac{k_2(s_0)}{k_1(s_0)} \mathbf{N}_1 + \mathbf{N}_2 \right), k_1(s_0) \delta(\mathbf{N}_2) \mathbf{N}_2 - k_2(s_0) \delta(\mathbf{N}_1) \mathbf{N}_1 \right\rangle \\ &= \mp \operatorname{sgn}(k_1(s_0)) \frac{1}{v_3} \frac{k_1^2(s_0) - k_2^2(s_0)}{\sqrt{|k_1^2(s_0) - k_2^2(s_0)|}} \neq 0. \end{aligned}$$

Note that $\mathbf{v}_0 \in \mathfrak{B}_H$ is a singular point, where

$$\mathbf{v} = \pm \frac{|k_1(s_0)|}{\sqrt{|k_1^2(s_0) - k_2^2(s_0)|}} \left(\frac{k_2(s_0)}{k_1(s_0)} \mathbf{N}_1(s_0) + \mathbf{N}_2(s_0) \right).$$

This completes the proof.

(3) Under the same notations as (2), we have

$$\tilde{H}(s, \mathbf{v}) = -x_1(s)v_1 + x_2(s)v_2 \pm x_3(s)\sqrt{\pm 1 + v_1^2 - v_2^2} - u.$$

It is enough to show that the rank of the matrix C is 2, where

$$C = \begin{pmatrix} -x'_1(s) \pm \frac{v_1 x'_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} & x'_2(s) \mp \frac{v_2 x'_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} & -1 \\ -x''_1(s) \pm \frac{v_1 x''_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} & x''_2(s) \mp \frac{v_2 x''_3(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} & 0 \end{pmatrix}.$$

This follows from the proof of (2). Thus, we complete the proof of (3).

(4) We denote that $\mathbf{N}_1(s) = (n_{11}(s), n_{12}(s), n_{13}(s))$, $\mathbf{v} = (v_1, v_2, \pm\sqrt{\pm 1 + v_1^2 - v_2^2})$. Under this notation, we have that

$$H_1(s, \mathbf{v}) = -n_{11}(s)v_1 + n_{12}(s)v_2 \pm n_{13}(s)\sqrt{\pm 1 + v_1^2 - v_2^2}.$$

Thus, we have that

$$\begin{aligned} \frac{\partial H_1}{\partial v_1} &= -n_{11}(s) \pm \frac{v_1 n_{13}(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}, & \frac{\partial H_1}{\partial v_2} &= n_{12}(s) \mp \frac{v_2 n_{13}(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}, \\ \frac{\partial}{\partial s} \frac{\partial H_1}{\partial v_1} &= -n'_{11}(s) \pm \frac{v_1 n'_{13}(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}, & \frac{\partial}{\partial s} \frac{\partial H_1}{\partial v_2} &= n'_{12}(s) \mp \frac{v_2 n'_{13}(s)}{\sqrt{\pm 1 + v_1^2 - v_2^2}}. \end{aligned}$$

It is enough to show that the rank of the matrix D is 2, where

$$D = \begin{pmatrix} -n_{11}(s_0) \pm \frac{v_1 n_{13}(s_0)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} & n_{12}(s_0) \mp \frac{v_2 n_{13}(s_0)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} \\ -n'_{11}(s_0) \pm \frac{v_1 n'_{13}(s_0)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} & n'_{12}(s_0) \mp \frac{v_2 n'_{13}(s_0)}{\sqrt{\pm 1 + v_1^2 - v_2^2}} \end{pmatrix}.$$

Denote that $v_3 = \pm\sqrt{\pm 1 + v_1^2 - v_2^2}$. Then, we have that

$$\begin{aligned} \det D &= -(n'_{12}n_{11} - n_{12}n'_{11})(s_0) + \frac{v_1}{v_3}(n'_{12}n_{13} - n_{12}n'_{13})(s_0) + \frac{v_2}{v_3}(n'_{13}n_{11} - n_{13}n'_{11})(s_0) \\ &= -\frac{1}{v_3} \langle \mathbf{v}, \mathbf{N}_1(s_0) \wedge \mathbf{N}'_1(s_0) \rangle \\ &= -\frac{1}{v_3} \langle \mathbf{v}, \mathbf{N}_1(s_0) \wedge -\delta(\mathbf{N}_1(s_0))k_1(s_0)\mathbf{T}(s_0) \rangle \\ &= -\frac{1}{v_3} \langle \mathbf{v}, -k_1(s_0)\mathbf{N}_2(s_0) \rangle \\ &= -\frac{1}{v_3} \langle \mp \mathbf{N}_2(s_0), -k_1(s_0)\mathbf{N}_2(s_0) \rangle = \mp \delta(\mathbf{N}_2(s_0)) \frac{1}{v_3} k_1(s_0) \neq 0. \end{aligned}$$

Note that $\mathbf{v} \in \mathfrak{D}_{H_1}$ is a singular point, where

$$\mathbf{v} = \mp \mathbf{N}_2(s_0).$$

This completes the proof.

(5) Using the same computation as the proof of (4), we can get (5). □

4. Proof of the main results

Proposition 3.4 means that those functions are generating families and now we can apply the above arguments to our situation. By Proposition 3.2 and the definitions of wave fronts and caustics, we can prove Proposition 2.2.

Proof of Proposition 2.2. Taking $G(s, \mathbf{v}) = -(x_1(s) - v_1)^2 + (x_2(s) - v_2)^2 + (x_3(s) - v_3)^2$ in Proposition 3.4 for example, we get the Jacobi matrix of $\Delta^*G = (G, \frac{\partial G}{\partial s})$ as

$$J\Delta^*G = \begin{pmatrix} 2\langle \gamma(s_0) - \mathbf{v}, \mathbf{t}(s_0) \rangle & 2(x_1(s_0) - v_1) & -2(x_2(s_0) - v_2) & -2(x_3(s_0) - v_3) \\ 2 + 2\langle \gamma(s_0) - \mathbf{v}, \mathbf{t}'(s_0) \rangle & 2x'_1(s_0) & -2x'_2(s_0) & -2x'_3(s_0) \end{pmatrix}.$$

Since the rank of the matrix A in the proof of Proposition 3.4 (1) is 3, the rank of the matrix $J\Delta^*G = 2$.

This means that $G(s, \mathbf{v})$ is a Morse family. In this case we have a smooth 2-dimensional submanifold,

$$\begin{aligned} \Sigma_*(G) &= \{(s, \mathbf{v}) \in (\mathbb{R} \times \mathbb{R}^3, \mathbf{0}) \mid G(s, \mathbf{v}) = \frac{\partial G}{\partial s}(s, \mathbf{v}) = 0\} \\ &= \{(s, \mathbf{v}) \in (\mathbb{R} \times \mathbb{R}^3, \mathbf{0}) \mid \mathbf{v} = \gamma(s) + u(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)), s \in I\}. \end{aligned}$$

We denote $\pi : PT^*\mathbb{R}_1^3 \rightarrow \mathbb{R}_1^3$ by the canonical projection, then the map germ $\Phi_G : (\Sigma_*(G), \mathbf{0}) \rightarrow PT^*\mathbb{R}_1^3$ defined by

$$\Phi_G(s, \mathbf{v}) = (\mathbf{v}, [(x_1(s) - v_1) : -(x_2(s) - v_2) : -(x_3(s) - v_3)])$$

is a Legendrian immersion germ and G is a generating family of $\Phi_G(\Sigma_*G)$. So the map

$$\pi \circ \Phi_G(s, \mathbf{v}) : \Sigma_*(G) \rightarrow \mathbb{R}_1^3$$

given by $(s, \mathbf{v}) \rightarrow \mathbf{v}$ is a Legendrian map. The wave front

$$W(\Phi_G) = \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} = \gamma(s) + u(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)), s \in I\}$$

of $\gamma(s)$ is the set of critical values of the Legendrian map $\pi \circ \Phi_G(s, \mathbf{v})$, and is precisely D_G . As you can see that the parametrisations of the wave fronts are just the parallel developables given by

$$\mathcal{P}\mathcal{L}\mathcal{D}_\gamma^\pm(s, u) = \gamma(s) + u(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)).$$

It follows that for a generic curve, the wave front $W(\Phi_G)$ of $\gamma(s)$ is locally either a regular surface, or has cuspidal edge singularity, or swallowtail. The local models of the wave front at \mathbf{v} corresponding to $s \in I$ depend on the \mathcal{R} -singularity type of $g_v(s)$ at s . For a generic $\gamma(s)$, $g_v(s)$ has local singularities of types A_1, A_2 , or A_3 . The wave front is a regular surface at an A_1 -singularity of $g_v(s)$. It is a cupidal edge at an A_2 -singularity of $g_v(s)$ and has swallowtail singularity at an A_3 -singularity of $g_v(s)$. If we apply the same arguments as $G(s, \mathbf{v})$ to the functions \tilde{H}, H_1 , and H_2 , respectively, we can get that dual surfaces are two dimensional wave fronts which have A_1 -singularity, A_2 -singularity, and A_3 -singularity. Bishop pseudo-spherical images are one dimensional wave fronts which have A_1 -singularity and A_2 -singularity. Bishop pseudo-spherical Darboux images can be seen as one dimensional caustics which have A_2 -singularity and A_3 -singularity in the framework of the theory of Lagrangian singularity. \square

Proof of Theorem 2.3.

(1) First, we consider the assertion (A). Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a regular unit speed spacelike curve with $\delta(\mathbf{N}_1(s))k_1(s) \pm \delta(\mathbf{N}_2(s))k_2(s) \neq 0$. For $\mathbf{v}_0 = \gamma(s_0) + u_0(\mathbf{N}_1(s_0) \pm \mathbf{N}_2(s_0))$, we give a function $\mathfrak{G} : \mathbb{R}_1^3 \rightarrow \mathbb{R}$ by $\mathfrak{G}(\mathbf{u}) = \langle \mathbf{u} - \mathbf{v}_0, \mathbf{u} - \mathbf{v}_0 \rangle$, then we have $g_{\mathbf{v}_0}(s) = \mathfrak{G}(\gamma(s))$. For $\mathbf{v}_0 = \mathfrak{B}\mathcal{L}\mathcal{D}^\pm(s_0, u_0)$, since $\mathcal{L}\mathcal{C}(\mathbf{v}_0) = \mathfrak{G}^{-1}(0)$ and 0 is a regular value of \mathfrak{G} , $g_{\mathbf{v}_0}(s)$ has the A_k -singularity at s_0 if and only if γ and $\mathcal{L}\mathcal{C}(\mathbf{v}_0)$ have $(k + 1)$ -point contact for s_0 . On the other hand, by Proposition 3.3, the discriminant set \mathfrak{D}_G of G is

$$\mathfrak{D}_G = \{\mathbf{v} = \gamma(s) + u(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)) \mid s \in I\}.$$

The assertion (A)(1) of Theorem 2.3 follows from Proposition 3.2, Proposition 3.4, and Theorem 3.1. Since the locus of the singularities of CE is locally diffeomorphic to the line, the assertion (A)(2) holds. Since the locus of singularities of SW is $C(2, 3, 4)$, the assertion (A)(3) holds.

(2) The discriminant set $\mathfrak{D}_{\tilde{H}}$ of \tilde{H} is

$$\mathfrak{D}_{\tilde{H}} = \{(\mathbf{v}, u) \in Q_\epsilon^2 \times \mathbb{R} \mid u = \langle \gamma(s), \mathbf{v} \rangle, \mathbf{v} = \lambda \mathbf{N}_1(s) + \mu \mathbf{N}_2(s), \delta(\mathbf{N}_1(s))\lambda^2 + \delta(\mathbf{N}_2(s))\mu^2 = \pm 1\}.$$

The assertion (B) of Theorem 2.3 follows from Proposition 3.2, Proposition 3.4, and Theorem 3.1 similar to the assertion (1).

(3) By Proposition 3.2, the discriminant set \mathcal{D}_{H_2} of H_2 is

$$\mathcal{D}_{H_2} = \{\mathbf{v} = \mp \mathbf{N}_1(s) \mid s \in I\}.$$

The assertion (C) of Theorem 2.3 follows from Proposition 3.2, Proposition 3.4, and Theorem 3.1.

(4) By Proposition 3.2, the discriminant set \mathcal{D}_{H_1} of H_1 is

$$\mathcal{D}_{H_1} = \{\mathbf{v} = \mp \mathbf{N}_2(s) \mid s \in I\}.$$

The assertion (D) of Theorem 2.3 follows from Proposition 3.2, Proposition 3.4, and Theorem 3.1. □

5. Parallel slant helix

A regular unit speed spacelike curve $\gamma : I \rightarrow \mathbb{R}_1^3$ is called a *parallel slant helix* according to parallel frame provided that the unit vector $\mathbf{N}_1(s)$ of γ has constant angle θ with some fixed unit vector \mathbf{u} , that is $\langle \mathbf{N}_1(s), \mathbf{u} \rangle = \text{constant}$, this condition is equivalent to $\frac{k_2(s)}{k_1(s)} = \text{constant}$. We can get some detail properties of parallel slant helix by the function $\rho(s)$ defined in Section 2, and Corollary 1 in [23].

Proposition 5.1. *Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a regular unit speed spacelike curve. We suppose that $k_1(s) \neq 0$ and $\rho(s) = 0$, then we have the following claims.*

- (1) $\gamma(s)$ is a parallel slant helix.
- (2) The parallel pseudo-spherical Darboux image is a constant map.
- (3) The tangent de Sitter indicatrix is a circle in osculating plane.
- (4) The parallel rectifying developable of γ is a cylindrical surface given by $\gamma(s) + \mathbf{u}\mathbf{e}(s)$, where $\mathbf{e}(s) = \mathbf{d}_\epsilon(s)$.
- (5) The normal hyperbolic indicatrix is a circle on the pseudo-spheres $H^2(-1)$ and the direction of the circle is given by the constant vector $\mathbf{d}_\epsilon(s) \equiv \mathbf{e}(s)$.
- (6) The normal de sitter indicatrix is a circle on the unit spheres S_1^2 and the direction of the circle is also given by the constant vector $\mathbf{d}_\epsilon(s) \equiv \mathbf{e}(s)$.

Proof.

(1) Suppose $\rho(s) = k_1'(s)k_2(s) - k_1(s)k_2'(s) = 0$, by straightforward calculations, we have

$$\left(\frac{k_2(s)}{k_1(s)}\right)' = \frac{k_2'(s)k_1(s) - k_2(s)k_1'(s)}{k_1^2(s)} = \frac{-\rho(s)}{k_1^2(s)} = 0.$$

This means that $\frac{k_2(s)}{k_1(s)} = \text{constant}$. By the definition of parallel slant helix, we know that $\gamma(s)$ is a parallel slant helix.

- (2) By the claim (2) of Proposition 3.2, we can get claim (2) easily.
- (3) By the Corollary 1 in [23], we also can get claim (3).
- (4) The claim (4) is clear by definition.
- (5) Suppose $\gamma : I \rightarrow \mathbb{R}_1^3$ is a regular unit speed spacelike curve with timelike $\mathbf{N}_1(s)$ and $k_1(s) > 0$. Other cases follow by the similar computation, we omit them here. Since

$$\mathbf{e} = \frac{-1}{\sqrt{|k_1^2(s) - k_2^2(s)|}} (k_2(s)\mathbf{N}_1(s) + k_1(s)\mathbf{N}_2(s)),$$

we get

$$\langle \mathbf{e}, \mathbf{N}_1 \rangle = \left\langle \frac{-1}{\sqrt{|k_1^2(s) - k_2^2(s)|}} (k_2(s)\mathbf{N}_1(s) + k_1(s)\mathbf{N}_2(s)), \mathbf{N}_1(s) \right\rangle = \frac{\frac{k_2}{k_1}}{\sqrt{|-1 + (\frac{k_2}{k_1})^2|}} = \text{constant}.$$

So $\cosh(\alpha(s)) = \langle \mathbf{e}(s), \mathbf{N}_1(s) \rangle$ is constant. This means that the first Bishop pseudo-spherical indicatrix is a circle on the hyperbolic space $H^2(-1)$ and the direction of the circle is given by the constant vector $\mathbf{d}_e(s) \equiv \mathbf{e}$.

(6) Under the assumptions of (5), we get

$$\cos(\beta(s)) = \langle \mathbf{e}(s), \mathbf{N}_2(s) \rangle = \frac{-1}{\sqrt{|-1 + (\frac{k_2}{k_1})^2|}}$$

is constant. This means that the second Bishop pseudo-spherical indicatrix is a circle on the de Sitter space S_1^2 and the direction of the circle is also given by the constant vector $\mathbf{d}(s) \equiv \mathbf{e}$. Other cases follow by the similar computation. \square

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