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Stability of impulsive differential systems with state-dependent impulses via the linear decomposition method

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Abstract

In this paper, we discuss the stability problem of the impulsive differential systems with state-dependent impulses. By using the linear decomposition methods, some sufficient conditions ensuring stability of the impulsive differential systems with state-dependent impulses are obtained and the estimate of the solution of such nonlinear systems is also acquired. Our results improve and generalize some of the known results given in earlier references. An example is given to demonstrate our results.

Keywords: Stability, impulsive differential systems, state-dependent impulses, linear decomposition methods.
2010 MSC: 34K20, 34K45.

1. Introduction

Impulsive differential systems provide basic mathematical models to describe the phenomenon of abrupt changes at certain instants in their states, and the corresponding theory of impulsive differential systems has been rapidly developed in the past decades, see [1, 4, 14, 15, 21]. In general, we can see an impulsive system as a hybrid one which is composed of discrete dynamics, continuous dynamics, criteria for deciding when the states of the system are to be reset. Therefore, an impulsive system has continuous and discrete dynamic behaviors. The complicated dynamic behaviors can be caused because of the interaction of continuous-time dynamics and discrete-time dynamics. Hence, the impulsive system is more efficient than a continuous system or discrete systems in some cases [21]. For example, the savings rates of central bank can not be changed everyday. A deep-space spacecraft can not keep its engine running continuously if it has only a limited supply of fuel. In recent years, the theory of impulsive differential systems has been applied to many aspects including of stability analysis, control design and

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synchronization, etc. A large amount of results on impulsive systems have appeared in the literature, see [2, 5, 8, 18, 22].

As we all know, many results for stability of impulsive differential systems with fixed moments of impulse effect have been derived and numerical interesting conclusions have been obtained, see [3, 9–11, 13, 17, 19] and references therein. By the method of Lyapunov functionals, the authors in [6, 17] considered the exponential stability of impulsive systems with time delays and an impulsive delay inequality, respectively. Reference [7] discussed further the global stability of periodic solution problem for a class of impulsive neural networks with time delays by using contraction mapping theorem. Some necessary and sufficient conditions were given in [3] for stability of linear impulsive systems with periodic impulses by using continuous-time time-varying discontinuous Lyapunov functions. Based on impulsive control theory and Lyapunov functional techniques, the authors in [9–11] studied the stability properties of nonlinear differential systems with state-dependent delayed impulses, nonlinear systems with delayed impulses and impulsive stochastic functional differential equations, respectively. Furthermore, the problem of stability for impulsive differential systems with state-dependent impulses has gained much research attention. For example, the stability of delay differential equations with state-dependent impulses was studied in [16]. However, there are very few published literatures in related to the stability of impulsive differential systems with state-dependent impulses. Therefore, it requires further studies.

The purpose of this paper is to improve the results in [16, 20] and establish some sufficient conditions guaranteeing the stability of impulsive differential systems with state-dependent impulses. Based on the linear decomposition methods, the stability problem of nonlinear systems with state-dependent impulses is converted to that of the linear systems with fixed impulses. The remainder of the paper is organized as follows. In Section 2, we present some basic notations and definitions. In Section 3, we provide the main results. An example is given in Section 4 to demonstrate the feasibility and advantage of the obtained results. In Section 5, conclusions are eventually given.

2. Preliminaries

Notations. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{R}^n$ the n-dimensional real space equipped with the-Euclidean norm $|\cdot|$, $\mathbb{R}^{n\times m}$ the $n\times m$-dimensional matrix, $\mathbb{Z}_+$ the set of positive integer numbers, I the n-dimensional identity matrix, $\lambda_{max}(A)$ the maximum eigenvalue of symmetric matrix A. For any sets $\mathbb{A},\mathbb{B}\subseteq\mathbb{R}^k (1\leqslant k\leqslant n)$, $C[\mathbb{A},\mathbb{B}]=\{\psi:\mathbb{A}\to\mathbb{B} \text{ is continuous}\}$. $N(t_0,t)$ denotes the number of impulse times in the semi-open interval $[t_0,t)$.

Consider the following nonlinear impulsive differential system

$$\begin{cases} \dot{x}(t)=P(t)x+h(t,x), & t\neq\tau_k(x),\ t\geqslant t_0,\\ \Delta x=Q_kx+u_k(x), & t=\tau_k(x),\\ x(t_0)=x_0, & t_0\geqslant 0,\end{cases} \tag{2.1}$$

where $x(t)\in\mathbb{R}^n$ is the state variable, and $x(t)$ is right continuous, i.e., $x(t^+)=x(t)$. $P(t)\in\mathbb{R}^{n\times n}$ is bounded and continuous with $t\geqslant t_0$. $Q_k\in\mathbb{R}^{n\times n}$ is a constant matrix. $h(t,x):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$, is continuous in x, which satisfies $|x|\leqslant\eta$, $\eta>0$, and $h(t,x)$ is continuous or piecewise continuous for $t\geqslant t_0$. $u_k(x):\mathbb{R}^n\to\mathbb{R}^n, k\in\mathbb{Z}_+$ is continuous with $|x|\leqslant\eta$. $\tau_k(x)\in C[\mathbb{R}^n,\mathbb{R}_+]$, $\tau_k(x)<\tau_{k+1}(x)$ for any $k\in\mathbb{Z}_+$, and $\lim_{k\to\infty}\tau_k(x)=\infty$ is uniform in x. $S_k: t=\tau_k(x)$ denotes the impulsive surface for every $k\in\mathbb{Z}_+$.

The corresponding linear system is given as follows

$$\begin{cases} \dot{x}(t)=P(t)x, & t\neq t_k,\ t\geqslant t_0,\\ \Delta x=Q_kx, & t=t_k,\\ x(t_0)=x_0, & t_0\geqslant 0,\end{cases} \tag{2.2}$$

where t_k are some constants satisfying $|t_k-\tau_k(0)|\leqslant\varepsilon$ for $|x|\leqslant\eta$, where $\varepsilon=\varepsilon(\eta)>0$ is a given constant satisfying $\lim_{\eta\to 0}\varepsilon(\eta)=0$.

Some definitions and lemma used in this paper are given in the following.

Definition 2.1 ([12]). Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1). Then the zero solution of system (2.1) is said to be

(S1) stable, if for any $\epsilon > 0$ and $t_0 \geqslant 0$, there exists a constant $\delta = \delta(t_0, \epsilon) > 0$, such that $|x_0| < \delta$ implies $|x(t)| < \epsilon$, $t \geqslant t_0$;

(S2) attractive, if for any $\epsilon > 0$ and $t_0 \geqslant 0$, there exist constants $\delta = \delta(t_0) > 0$ and $T(\epsilon, t_0, x_0) > 0$ such that $|x_0| < \delta$ implies $|x(t)| < \epsilon$, $t \geqslant t_0 + T$;

(S3) asymptotically stable, if it is stable and attractive;

(S4) exponentially stable, if for any $\epsilon > 0$ and $t_0 \geqslant 0$, there exist constants $\delta(\epsilon) > 0$ and $\lambda > 0$, such that $|x_0| < \delta(\epsilon)$ implies $|x(t)| < \epsilon e^{-\lambda(t-t_0)}$, $t \geqslant t_0$.

Lemma 2.2 ([20]). *For $t \geqslant t_0$ let $u(t)$ be a nonnegative piecewise continuous function satisfying*

$$u(t) \leqslant c + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0 \leqslant \tau_i < t} b_i u(\tau_i),$$

where $c \geqslant 0$, $b_i \geqslant 0$, $v(s) > 0$, $u(t)$ has discontinuous points of the first kind at τ_i. Then we have

$$u(t) \leqslant c \prod_{t_0 < \tau_i < t} (1 + b_i) \exp\left(\int_{t_0}^{t} v(s)ds\right).$$

3. Main results

In this section, we shall establish some sufficient conditions ensuring the stability properties of system (2.1).

Theorem 3.1. *Suppose that*

(i) *for any $|x| \leqslant \eta$, and $k \in \mathbb{Z}_+$*

$$\tau_k\Big((I + Q_k)x + u_k(x)\Big) \leqslant \tau_k(x) \leqslant \tau_{k+1}\Big((I + Q_k)x + u_k(x)\Big);$$

(ii) *for any $t \geqslant t_0$, $|x| \leqslant \eta$, and $k \in \mathbb{Z}_+$, we have*

$$|h(t, x)| \leqslant a(t)|x|, \quad |u_k(x)| \leqslant b(t)|x|,$$

where $a(t) \in (\mathbb{R}_+, \mathbb{R}_+)$, $b(t) \in (\mathbb{R}_+, \mathbb{R}_+)$;

(iii) *the state transition matrix $\phi(t, s)$ of the linear system (2.2) satisfies*

$$|\phi(t, s)| \leqslant \frac{K}{1 + t - s}, \quad K > 0, \quad t \geqslant s \geqslant t_0;$$

(iv) *there exists a constant $M > 1$ such that*

$$\prod_{t_0 < t_k < t} \Big(1 + Kb(t_k)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right) < M < \infty, \quad t_k = \tau_k(x(t_k)).$$

Then the zero solution of system (2.1) is asymptotically stable and each solution satisfies

$$|x(t)| \leqslant \frac{K|x_0| \prod_{t_0 < t_k < t} \Big(1 + Kb(t_k)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right)}{1 + t - t_0}, \quad t \geqslant t_0.$$

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of system (2.1) starting at (t_0, x_0), where x_0 is in a small neighborhood of $x = 0$. There exist constants $\overline{\eta} < \eta$ and $T \leqslant \infty$ such that $|x_0| \leqslant \overline{\eta}$ implies $|x(t)| < \eta$ for $t \in (t_0, t_0 + T]$. Based on the condition (i), $x(t)$ meets each surface $t = \tau_k(x)$ only once for $t \in (t_0, t_0 + T]$, then the equation $t = \tau_k(x(t))$ has a unique solution denoted by t_k. Then $x(t)$, $t \in (t_0, t_0 + T]$ is also a solution of the system as follows

$$\begin{cases} \dot{x}(t) = P(t)x + h(t, x), & t \neq t_k,\ t \geqslant t_0, \\ \Delta x = Q_k x + u_k(x), & t = t_k, \\ x(t_0) = x_0, & t_0 \geqslant 0. \end{cases} \tag{3.1}$$

Hence, one can represent $x(t)$ as

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, s)h(s, x(s))ds + \sum_{t_0 < t_k < t} \phi(t, t_k)u_k(x(t_k)), \quad t \in (t_0, t_0 + T].$$

We can derive that

$$|x(t)| \leqslant |\phi(t, t_0)||x_0| + \int_{t_0}^{t} |\phi(t, s)||h(s, x(s))|ds + \sum_{t_0 < t_k < t} |\phi(t, t_k)||u_k(x(t_k))|.$$

It then follows from assumptions (ii) and (iii) that

$$|x(t)| \leqslant \frac{K}{1 + t - t_0}|x_0| + \int_{t_0}^{t} \frac{K}{1 + t - s}a(s)|x(s)|ds + \sum_{t_0 < t_k < t} \frac{K}{1 + t - t_k}b(t_k)|x(t_k)|,$$

i.e.,

$$(1 + t - t_0)|x(t)| \leqslant K|x_0| + \int_{t_0}^{t} \frac{K(1 + t - t_0)}{1 + t - s}a(s)|x(s)|ds + \sum_{t_0 < t_k < t} \frac{K(1 + t - t_0)}{1 + t - t_k}b(t_k)|x(t_k)|,$$

which leads to

$$(1 + t - t_0)|x(t)| \leqslant K|x_0| + \int_{t_0}^{t} Ka(s)(1 + s - t_0)|x(s)|ds + \sum_{t_0 < t_k < t} Kb(t_k)(1 + t_k - t_0)|x(t_k)|.$$

Based on Lemma 2.2, one can get

$$(1 + t - t_0)|x(t)| \leqslant K|x_0| \prod_{t_0 < t_k < t} \Big(1 + Kb(t_k)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right),$$

i.e.,

$$|x(t)| \leqslant \frac{K|x_0| \prod_{t_0 < t_k < t} \Big(1 + Kb(t_k)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right)}{1 + t - t_0}.$$

If x_0 satisfies $|x_0| < \frac{\eta}{KM}$ and assumption (iv) holds, one can prove that the zero solution of system (2.1) is asymptotically stable and each solution satisfies

$$|x(t)| \leqslant \frac{K|x_0| \prod_{t_0 < t_k < t} \Big(1 + Kb(t_k)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right)}{1 + t - t_0}, \quad t \geqslant t_0.$$

□

Theorem 3.2. *Suppose that*

(i) *for any* $|x| \leqslant \eta$, *and* $k \in \mathbb{Z}_+$

$$\tau_k\Big((I+Q_k)x+u_k(x)\Big) \leqslant \tau_k(x) \leqslant \tau_{k+1}\Big((I+Q_k)x+u_k(x)\Big);$$

(ii) *for any* $t \geqslant t_0$, $|x| \leqslant \eta$ *and* $k \in \mathbb{Z}_+$, *we have*

$$|h(t,x)| \leqslant a(t)|x|, \quad |u_k(x)| \leqslant b(t)|x|,$$

where $a(t) \in (\mathbb{R}_+, \mathbb{R}_+)$, $b(t) \in (\mathbb{R}_+, \mathbb{R}_+)$;

(iii) *for any* $t \geqslant t_0$ *and* $n \in \mathbb{Z}_+$, *there exist constants* μ *and* ν *such that*

$$\lambda_{\max}\big(\frac{1}{2}(P(t)+P^T(t))\big) = \lambda_n(t) \leqslant \mu,$$

and

$$\lambda_{\max}\big((I+Q_n^T)(I+Q_n)\big) = \varpi_n^2 \leqslant \nu^2;$$

(iv) *for any* $|x| \leqslant \eta$ *the following limit*

$$\lim_{T\to\infty} \frac{N(t,t+T)}{T} = p,$$

exists and is uniform for $t \geqslant t_0$;

(v) *there exist constants* $M > 1$ *and* $\delta > 0$ *such that*

$$\prod_{t_0<t_k<t} \Big(1+Kb(t_k)\Big)\exp\left(\int_{t_0}^t Ka(s)ds\right) < M\exp\Big(\delta(t-t_0)\Big), \quad K \geqslant 1, \quad t_k = \tau_k(x(t_k));$$

(vi) $\mu + p\ln\nu < 0$.

Then the zero solution of system (2.1) *is exponentially stable and each solution satisfies*

$$|x(t)| \leqslant K|x_0| \prod_{t_0<t_k<t} \Big(1+Kb(t_k)\Big)\exp\left(\int_{t_0}^t Ka(s)ds\right)\exp\Big(-\alpha(t-t_0)\Big), \quad \alpha > 0, \quad t \geqslant t_0.$$

Proof. Let $x(t) = x(t,t_0,x_0)$ be any solution of system (2.1) starting at (t_0,x_0), where x_0 is in a small neighborhood of $x=0$. There exist constants $\overline{\eta} < \eta$ and $T \leqslant \infty$ such that $|x_0| \leqslant \overline{\eta}$ implies $|x(t)| < \eta$ for $t \in (t_0, t_0+T]$. Based on the condition (i), $x(t)$ meets each surface $t = \tau_k(x)$ only once for $t \in (t_0,t_0+T]$, then the equation $t = \tau_k(x(t))$ has a unique solution denoted by t_k. Thus, $x(t)$ is also a solution of the system (3.1). Then $x(t)$ can be represented as

$$x(t) = \phi(t,t_0)x_0 + \int_{t_0}^t \phi(t,s)h(s,x(s))ds + \sum_{t_0<t_k<t} \phi(t,t_k)u_k(x(t_k)), \quad t \in (t_0,t_0+T],$$

where $\phi(t,s)$ is the state transition matrix of linear system (2.2). Let $\widetilde{x}(t)$ be an arbitrary solution of system (2.2) starting at (t_0,x_0). Setting $V(t) = \widetilde{x}^2(t) = \widetilde{x}^T(t)\widetilde{x}(t)$, $t \in [t_k,t_{k+1})$, $k \in \mathbb{Z}_+$, it follows that

$$\dot{V}(t) = 2\widetilde{x}^T(t)P(t)\widetilde{x}(t) = \widetilde{x}^T(t)(P(t)+P^T(t))\widetilde{x}(t) \leqslant 2\lambda_n(t)(\widetilde{x}^T(t)\widetilde{x}(t)) = 2\lambda_n(t)V(t), t \in [t_k,t_{k+1}), k \in \mathbb{Z}_+.$$

Then we have

$$V(t) \leqslant V(t_0) \exp\left(2\int_{t_0}^{t} \lambda_n(s)ds\right) \leqslant V(t_0) \exp\Big(2\mu(t-t_0)\Big), \quad t \in [t_0, t_1),$$

i.e.,

$$|\widetilde{x}(t)| \leqslant |x_0| \exp\Big(\mu(t-t_0)\Big), \quad t \in [t_0, t_1).$$

Note that

$$V(t_1) = \widetilde{x}^T(t_1)\widetilde{x}(t_1) = \widetilde{x}^T(t_1^-)(I+Q_1)^T(I+Q_1)\widetilde{x}(t_1^-) \leqslant \nu^2 V(t_1^-) \leqslant \nu^2 V(t_0) \exp\Big(2\mu(t_1-t_0)\Big),$$

we can deduce that

$$\begin{aligned} V(t) &\leqslant V(t_1) \exp\Big(2\mu(t-t_1)\Big) \\ &\leqslant \nu^2 V(t_0) \exp\Big(2\mu(t_1-t_0)\Big) \exp\Big(2\mu(t-t_1)\Big) \\ &= \nu^2 V(t_0) \exp\Big(2\mu(t-t_0)\Big), \quad t \in [t_1, t_2), \end{aligned}$$

i.e.,

$$|\widetilde{x}(t)| \leqslant \nu |x_0| \exp\Big(\mu(t-t_0)\Big), \quad t \in [t_1, t_2).$$

Proceeding as before, we obtain

$$|\widetilde{x}(t)| \leqslant |x_0| \nu^{N(t_0,t)} \exp\Big(\mu(t-t_0)\Big), \quad t \in [t_{N(t_0,t)}, t_{N(t_0,t)+1}).$$

In view of assumption (iv), for any $\epsilon > 0$ we have

$$(p-\epsilon)(t-t_0) \leqslant N(t_0, t) \leqslant (p+\epsilon)(t-t_0).$$

Thus, we can get

$$|\widetilde{x}(t)| \leqslant \nu^{(p\pm\epsilon)(t-t_0)} |x_0| \exp\Big(\mu(t-t_0)\Big) = |x_0| \exp\left(\Big(\mu + (p\pm\epsilon)\ln\nu\Big)(t-t_0)\right).$$

Considering assumption (vi), we can obtain that there exist $K \geqslant 1$ and $\alpha > 0$ with $0 < \alpha < |\mu + p\ln\nu|$ such that for any $t_0 \leqslant s \leqslant t \leqslant t+T$,

$$|\phi(t,s)Z| \leqslant K \exp\Big(-\alpha(t-s)\Big)|Z|, \ \ |Z| \leqslant \eta.$$

Hence, we have

$$\begin{aligned} |x(t)| &\leqslant |\phi(t,t_0)||x_0| + \int_{t_0}^{t} |\phi(t,s)||h(s,x(s))|ds + \sum_{t_0<t_k<t} |\phi(t,t_k)||u_k(x(t_k))| \\ &\leqslant K \exp\Big(-\alpha(t-t_0)\Big)|x_0| + \int_{t_0}^{t} K \exp\Big(-\alpha(t-s)\Big) a(s)|x(s)|ds \\ &\quad + \sum_{t_0<t_k<t} K \exp\Big(-\alpha(t-t_k)\Big) b(t_k)|x(t_k)|, \end{aligned}$$

i.e.,

$$\exp\Big(\alpha(t-t_0)\Big)|x(t)| \leqslant K|x_0| + \int_{t_0}^{t} Ka(s)\exp\Big(\alpha(s-t_0)\Big)|x(s)|ds + \sum_{t_0<t_k<t} Kb(t_k)\exp\Big(\alpha(t_k-t_0)\Big)|x(t_k)|.$$

Based on Lemma 2.2, it is easy to derive that

$$|x(t)| \leqslant K|x_0| \prod_{t_0<t_k<t}\Big(1+Kb(t_k)\Big)\exp\left(\int_{t_0}^{t} Ka(s)ds\right)\exp\Big(-\alpha(t-t_0)\Big).$$

If $|x_0| < \frac{\eta}{KM}$, $\delta < \alpha$ and assumption (v) hold, we can derive that the zero solution of system (2.1) is exponentially stable and each solution satisfies

$$|x(t)| \leqslant K|x_0| \prod_{t_0<t_k<t}\Big(1+Kb(t_k)\Big)\exp\left(\int_{t_0}^{t} Ka(s)ds\right)\exp\Big(-\alpha(t-t_0)\Big), \quad t \geqslant t_0.$$ □

Theorem 3.3. *Suppose that*

(i) *for any* $|x| \leqslant \eta$, *and* $k \in \mathbb{Z}_+$

$$\tau_k\Big((I+Q_k)x+u_k(x)\Big) \leqslant \tau_k(x) \leqslant \tau_{k+1}\Big((I+Q_k)x+u_k(x)\Big);$$

(ii) *for any* $t \geqslant t_0$, $|x| \leqslant \eta$, *and* $k \in \mathbb{Z}_+$, *we have*

$$|h(t,x)| \leqslant a(t)|x|, \quad |u_k(x)| \leqslant b(t)|x|,$$

where $a(t) \in (\mathbb{R}_+, \mathbb{R}_+)$, $b(t) \in (\mathbb{R}_+, \mathbb{R}_+)$;

(iii) *for any* $t \geqslant t_0$ *and* $n \in \mathbb{Z}_+$, *there exist constants* μ *and* ν *such that*

$$\lambda_{\max}(\frac{1}{2}(P(t)+P^T(t))) = \lambda_n(t) \leqslant \mu,$$

and

$$\lambda_{\max}((I+Q_n^T)(I+Q_n)) = \varpi_n^2 \leqslant \nu^2;$$

(iv) *for any* $k \in \mathbb{Z}_+$ *there exist constants* ξ_1 *and* ξ_2 *such that*

$$0 < \xi_1 \leqslant \min_{|x|\leqslant\eta} \tau_{k+1}(x) - \max_{|x|\leqslant\eta} \tau_k(x) \leqslant \xi_2,$$

or else

$$0 < \xi_1 \leqslant \min_{|x|\leqslant\eta} \tau_1(x) - t_0 \leqslant \xi_2;$$

(v) $\mu + \frac{1}{\xi}\ln\nu < 0$, *where* $\xi = \xi_1$ *if* $\nu \geqslant 1$, *or else* $\xi = \xi_2$ *if* $0 < \nu < 1$;

(vi) *there exist constants* $M > 1$ *and* $\delta > 0$ *such that*

$$\prod_{t_0<t_k<t}\Big(1+Kb(t_k)\Big)\exp\left(\int_{t_0}^{t} Ka(s)ds\right) < M\exp\Big(\delta(t-t_0)\Big), \quad K \geqslant 1, \quad t_k = \tau_k(x(t_k)).$$

Then the zero solution of system (2.1) *is exponentially stable and each solution satisfies*

$$|x(t)| \leqslant K|x_0| \prod_{t_0<t_k<t} \Big(1+Kb(t_k)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right) \exp\Big(-\alpha(t-t_0)\Big), \quad \alpha>0, \quad t \geqslant t_0.$$

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of system (2.1) starting at (t_0, x_0), where x_0 is in a small neighborhood of $x = 0$. There exist constants $\overline{\eta} < \eta$ and $T \leqslant \infty$ such that $|x_0| \leqslant \overline{\eta}$ implies $|x(t)| < \eta$ for $t \in (t_0, t_0 + T]$. Based on the condition (i), $x(t)$ meets each surface $t = \tau_k(x)$ only once for $t \in (t_0, t_0 + T]$, then the equation $t = \tau_k(x(t))$ has a unique solution denoted by t_k. Thus, $x(t)$ is also a solution of the system (3.1). Then $x(t)$ can be represented as

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t,s)h(s, x(s))ds + \sum_{t_0<t_k<t} \phi(t, t_k)u_k(x(t_k)), \quad t \in (t_0, t_0 + T],$$

where $\phi(t, s)$ is the state transition matrix of linear system (2.2). Let $\widetilde{x}(t)$ be an arbitrary solution of system (2.2) starting at (t_0, x_0). Arguing as Theorem 3.2, we obtain

$$|\widetilde{x}(t)| \leqslant |x_0|\nu^{N(t_0,t)} \exp\Big(\mu(t-t_0)\Big), \quad t \in [t_{N(t_0,t)}, t_{N(t_0,t)+1}).$$

In view of assumption (iv), there exists a constant $m \in \mathbb{Z}_+$ such that

$$\xi_1 N(t_0, t) \leqslant t - t_0 \leqslant \xi_2 \Big[N(t_0, t) + m\Big].$$

Hence, we have

$$|\widetilde{x}(t)| \leqslant |x_0| \exp\Big((\mu + \frac{1}{\xi_1} \ln \nu)(t - t_0)\Big), \quad \nu \geqslant 1.$$

Or else,

$$|\widetilde{x}(t)| \leqslant \frac{1}{\nu^m}|x_0| \exp\Big((\mu + \frac{1}{\xi_2} \ln \nu)(t - t_0)\Big), \quad 0 < \nu < 1.$$

Considering assumption (v), we obtain that there exist $K \geqslant 1$ and $\alpha > 0$ with $0 < \alpha < |\mu + \frac{1}{\xi} \ln \nu|$ such that for any $t_0 \leqslant s \leqslant t \leqslant t + T$,

$$|\phi(t, s)Z| \leqslant K \exp\Big(-\alpha(t-s)\Big)|Z|, \quad |Z| \leqslant \eta.$$

Then arguing as Theorem 3.2, we have

$$|x(t)| \leqslant K|x_0| \prod_{t_0<t_k<t} \Big(1+Kb(t_k)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right) \exp\Big(-\alpha(t-t_0)\Big).$$

If $|x_0| < \frac{\eta}{KM}$, $\delta < \alpha$ and assumption (vi) hold, one can derive that the zero solution of system (2.1) is exponentially stable and each solution satisfies

$$|x(t)| \leqslant K|x_0| \prod_{t_0<t_k<t} \Big(1+Kb(t_k)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right) \exp\Big(-\alpha(t-t_0)\Big), \quad t \geqslant t_0.$$ □

Next result is concerned with stability of the following system

$$\begin{cases} \dot{x}(t) = Px + h(t, x), & t \neq \tau_k(x), \ t \geqslant t_0, \\ \Delta x = Qx + u_k(x), & t = \tau_k(x), \\ x(t_0) = x_0, & t_0 \geqslant 0, \end{cases} \tag{3.2}$$

where $P, Q \in \mathbb{R}^{n\times n}$ are constant matrices.

Theorem 3.4. *Suppose that*

(i) *for any* $|x| \leqslant \eta$, *and* $k \in \mathbb{Z}_+$

$$\tau_k\Big((I+Q)x+u_k(x)\Big) \leqslant \tau_k(x) \leqslant \tau_{k+1}\Big((I+Q)x+u_k(x)\Big);$$

(ii) *for any* $t \geqslant t_0$, $|x| \leqslant \eta$ *and* $k \in \mathbb{Z}_+$, *we have*

$$|h(t,x)| \leqslant a(t)|x|, \quad |u_k(x)| \leqslant b(t)|x|,$$

where $a(t) \in (\mathbb{R}_+,\mathbb{R}_+)$, $b(t) \in (\mathbb{R}_+,\mathbb{R}_+)$;

(iii) *for any* $|x| \leqslant \eta$ *the following limit*

$$\lim_{T\to\infty} \frac{N(t,t+T)}{T} = p,$$

exists and is uniform for $t \geqslant t_0$;

(iv) $\mu = \max_{i=1}^{n} \mathrm{Re}\lambda_i(P)$, $\nu^2 = \max_{i=1}^{n} \mathrm{Re}\lambda_i\big[(I+Q^T)(I+Q)\big]$;

(v) *there exist constants* $M > 1$ *and* $\delta > 0$ *such that*

$$\prod_{t_0<t_k<t}\Big(1+Kb(t_k)\Big)\exp\left(\int_{t_0}^{t} Ka(s)ds\right) < M\exp\Big(\delta(t-t_0)\Big), \;\; K \geqslant 1, \;\; t_k = \tau_k(x(t_k));$$

(vi) $\mu + p\ln\nu < 0$.

Then the zero solution of system (3.2) *is exponentially stable and each solution satisfies*

$$|x(t)| \leqslant K|x_0| \prod_{t_0<t_k<t}\Big(1+Kb(t_k)\Big)\exp\left(\int_{t_0}^{t} Ka(s)ds\right)\exp\Big(-\alpha(t-t_0)\Big), \;\; \alpha > 0, \;\; t \geqslant t_0.$$

Proof. Let $x(t) = x(t,t_0,x_0)$ be any solution of system (2.1) starting at (t_0,x_0), where x_0 is in a small neighborhood of $x = 0$. There exist constants $\overline{\eta} < \eta$ and $T \leqslant \infty$ such that $|x_0| \leqslant \overline{\eta}$ implies $|x(t)| < \eta$ for $t \in (t_0,t_0+T]$. Based on the condition (i), $x(t)$ meets each surface $t = \tau_k(x)$ only once for $t \in (t_0,t_0+T]$, then the equation $t = \tau_k(x(t))$ has a unique solution denoted by t_k. Then $x(t)$ is also a solution of system as follows

$$\begin{cases} \dot{x}(t) = Px + h(t,x), & t \neq t_k, \; t \geqslant t_0, \\ \Delta x = Qx + u_k(x), & t = t_k, \\ x(t_0) = x_0, & t_0 \geqslant 0. \end{cases}$$

Thus, $x(t)$ can be represented as

$$x(t) = \phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,s)h(s,x(s))ds + \sum_{t_0<t_k<t} \phi(t,t_k)u_k(x(t_k)), \;\; t \in (t_0,t_0+T],$$

where $\phi(t,s)$ is the state transition matrix of the following linear system

$$\begin{cases} \dot{x}(t) = Px, & t \neq t_k, \; t \geqslant t_0, \\ \Delta x = Qx, & t = t_k, \\ x(t_0) = x_0, & t_0 \geqslant 0. \end{cases} \tag{3.3}$$

Let $\tilde{x}(t)$ be an arbitrary solution of system (3.3) starting at (t_0, x_0). Setting $V(t) = \tilde{x}^2(t) = \tilde{x}^T(t)\tilde{x}(t)$, $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$, it follows that

$$\dot{V}(t) = 2\tilde{x}^T(t)P\tilde{x}(t) = \tilde{x}^T(t)(P + P^T)\tilde{x}(t) \leqslant 2\mu(\tilde{x}^T(t)\tilde{x}(t)) = 2\mu V(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_+.$$

Then arguing as Theorem 3.2, we have

$$|\tilde{x}(t)| \leqslant |x_0|\nu^{N(t_0,t)} \exp\Big(\mu(t - t_0)\Big), \quad t \in [t_{N(t_0,t)}, t_{N(t_0,t)+1}).$$

Proceeding as Theorem 3.2, the proof is complete. □

4. Example

Example 4.1. Consider the following impulsive system

$$\begin{cases} \dot{x}(t) = -\dfrac{1}{1+t}x + \dfrac{1}{1+t^2}x, & t \neq \tau_i(x),\ t \geqslant t_0, \\ \Delta x = -\dfrac{1}{2}x + (\dfrac{1}{3} - \dfrac{1}{3+t^2})x, & t = \tau_i(x), \\ x(t_0) = x_0, & t_0 \geqslant 0, \end{cases} \tag{4.1}$$

where $\tau_i(x) = |x| + i, i \in \mathbb{Z}_+$. Note that $x(t_i^+) = x(t_i) = (I + Q_i)x(t_i^-) + u_i(x(t_i^-))$, one can choose $Q_i = -\frac{1}{2}$, $u_i(x) = b(t)x$, $b(t) = \frac{1}{3} - \frac{1}{3+t^2}$ and $\eta = 2$.

It then follows that

$$\begin{aligned} \tau_i(x(t_i^+)) &= \tau_i\Big((I + Q_i)x(t_i^-) + u_i(x(t_i^-))\Big) \\ &= |(1 - \frac{1}{2})x(t_i^-) + (\frac{1}{3} - \frac{1}{3+t^2})x(t_i^-)| + i \\ &= |(\frac{5}{6} - \frac{1}{3+t^2})||x(t_i^-)| + i \\ &\leqslant \frac{5}{6}|x(t_i^-)| + i \\ &\leqslant |x(t_i^-)| + i \\ &= \tau_i(x(t_i^-)), \end{aligned}$$

and

$$\begin{aligned} \tau_{i+1}&\Big((I + Q_i)x(t_i^-) + u_i(x(t_i^-))\Big) \\ &= |(1 - \frac{1}{2})x(t_i^-) + (\frac{1}{3} - \frac{1}{3+t^2})x(t_i^-)| + i + 1 \\ &= |(\frac{5}{6} - \frac{1}{3+t^2})||x(t_i^-)| + i + 1 \\ &\geqslant \frac{1}{2}|x(t_i^-)| + i + 1 \\ &\geqslant |x(t_i^-)| + i \\ &= \tau_i(x(t_i^-)). \end{aligned}$$

Choose $h(t, x) = a(t)x$, $a(t) = \frac{1}{1+t^2}$, then it is easy to get that

$$|h(t, x)| \leqslant \frac{1}{1+t^2}|x|.$$

Note that $u_i(x) = (\frac{1}{3} - \frac{1}{3+t^2})x$, then one can derive that

$$|u_i(x)| \leqslant (\frac{1}{3} - \frac{1}{3+t^2})|x|.$$

The corresponding linear system of (4.1) is given as follows

$$\begin{cases} \dot{x}(t) = -\frac{1}{1+t}x, & t \neq t_i,\ t \geqslant t_0, \\ \Delta x = -\frac{1}{2}x, & t = t_i, \\ x(t_0) = x_0, & t_0 \geqslant 0. \end{cases} \tag{4.2}$$

Let $\widetilde{x}(t)$ be the solution of linear system (4.2) satisfying

$$\widetilde{x}(t) = \left(\frac{1}{2}\right)^{N(t_0,t)} (1+t_0)\frac{x_0}{1+t}, \quad t \in [t_{N(t_0,t)}, t_{N(t_0,t)+1}).$$

We have

$$|\widetilde{x}(t)| \leqslant \frac{K}{1+t-s}, \quad t \geqslant s \geqslant t_0,$$

where $0 < \frac{1}{2}(1+t_0)|x_0| = K < \infty$.

Furthermore, notice that

$$\prod_{t_0<t_i<t} \Big(1 + Kb(t_i)\Big) = \prod_{t_0<t_i<t} \Big(1 + K(\frac{1}{3} - \frac{1}{3+t_i^2})\Big),$$

and

$$\exp\left(\int_{t_0}^{t} Ka(s)ds\right) = \exp\left(\int_{t_0}^{t} K\frac{1}{1+s^2}ds\right),$$

are convergent, which implies

$$\prod_{t_0<t_i<t} \Big(1 + Kb(t_i)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right) < M < \infty, \quad t_i = \tau_i(x(t_i)).$$

Then it is not difficult to check that conditions (i)–(iv) are satisfied. Thus, we can get from Theorem 3.1 that the zero solution of system (4.1) is asymptotically stable and each solution satisfies

$$|x(t)| \leqslant \frac{K|x_0| \prod_{t_0<t_i<t} \Big(1 + Kb(t_i)\Big) \exp\left(\int_{t_0}^{t} Ka(s)ds\right)}{1+t-t_0}, \quad t \geqslant t_0.$$

5. Conclusion

We have studied the stability properties of impulsive differential systems with state-dependent impulses. Based on the linear decomposition methods, some sufficient conditions guaranteeing asymptotical and exponential stabilities of the impulsive differential systems with state-dependent impulses have been presented. Our results are more general than those mentioned in the literature. An example was given to show the effectiveness of the obtained results. It would be interesting to further extend the approach in this paper to address some nonlinear systems.

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