Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 5125–5131
Research Article

ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Fixed points for φ_E-Geraghty contractions

Andreea Fulga*, Alexandrina Maria Proca

Department of Mathematics and Computer Sciences, Transilvania University of Braşov, Iuliu Maniu, 50, 505801, Braşov, România.

Communicated by P. Kumam

Abstract

In this paper, we introduce the new concept of a generalization of contraction so-called φ_E-Geraghty contraction and we establish a fixed point theorem for such mappings in complete metric spaces.

Keywords: φ_E-Geraghty contractions mapping, fixed point, contraction.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

Banach's contraction principle (BCP) [1] is one of the initial and also fundamental results in theory of fixed point.

It is know that BCP has been extended in many various directions by several authors, see [1–18] and the references therein. The following generalization is due to Geraghty [7].

Theorem 1.1 ([7])**.** *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be an operator. If* T *satisfies the following inequality:*

$$d(Tx, Ty) \leqslant \varphi(d(x, y)) \cdot d(x, y), \quad \forall x, y \in X,$$

where $\varphi : [0, \infty) \to [0, 1)$ *is a function which satisfies the condition*

$$\lim_{n\to\infty} \varphi(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0,$$

then T *has a unique fixed point.*

In this paper, starting from [16], we introduce the notion of φ_E-Geraghty contraction and prove a fixed point theorem for φ_E-contractions, which generalizes Theorem 1.1. Examples are given to show that our result is a proper extension.

*Corresponding author
Email addresses: afulga@unitbv.ro (Andreea Fulga), alexproca@unitbv.ro (Alexandrina Maria Proca)

doi:10.22436/jnsa.010.09.48

Received 2017-02-27

2. Main results

Definition 2.1. Let ϕ denote the class of functions $\varphi : [0,\infty) \to [0,1)$ which satisfy the condition

$$\lim_{n\to\infty} \varphi(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0.$$

Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a φ_E-Geraghty contraction on (X,d) if there exists $\varphi \in \phi$ such that

$$d(Tx,Ty) \leqslant \varphi(E(x,y)) \cdot E(x,y), \quad \forall x,y \in X, \tag{2.1}$$

where

$$E(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty)|. \tag{2.2}$$

Remark 2.2. Due to the fact that $\varphi : [0,\infty) \to [0,1)$ we have

$$d(Tx,Ty) \leqslant \varphi(E(x,y)) \cdot E(x,y) < E(x,y)$$

for any $x,y \in X$, with $x \neq y$.

Theorem 2.3. *Let (X,d) be a complete metric space and $T : X \to X$ be a φ_E-Geraghty contraction. Then T has a unique fixed point x^* and for all $x_0 \in X$, the sequence $\{T^n x_0\}$ is convergent to x^*.*

Proof. Let $x_0 \in X$ be arbitrary and fixed. We define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Then $Tx_{n_0} = x_{n_0}$. This proves that x_{n_0} is a fixed point of T.

From now, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then, $d(x_{n+1},x_n) > 0$ and it follows from (2.2) that for each $n \in \mathbb{N}$

$$0 < d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leqslant \varphi(E(x_{n-1},x_n)) \cdot E(x_{n-1},x_n), \tag{2.3}$$

where

$$\begin{aligned} E(x_{n-1},x_n) &= d(x_{n-1},x_n) + |d(x_{n-1},Tx_{n-1}) - d(x_n,Tx_n)| \\ &= d(x_{n-1},x_n) + |d(x_{n-1},x_n) - d(x_n,x_{n+1})|. \end{aligned}$$

If we denote by

$$d_n = d(x_{n-1},x_n),$$

we have

$$d_{n+1} \leqslant \varphi(d_n + |d_n - d_{n+1}|) \cdot (d_n + |d_n - d_{n+1}|).$$

If there exists $n \in \mathbb{N}$ such that $d_n \leqslant d_{n+1}$, then (2.3) becomes

$$d_{n+1} \leqslant \varphi(d_{n+1}) \cdot d_{n+1} < d_{n+1}.$$

But, it is a contradiction. Therefore, $d_n > d_{n+1}$ for all $n \in \mathbb{N}$. Thus, we have from (2.3)

$$d_{n+1} \leqslant \varphi(2d_n - d_{n+1}) \cdot (2d_n - d_{n+1}) \tag{2.4}$$

for all $n \in \mathbb{N}$.

Let now $d = \lim_{n\to\infty} d_n$ and we suppose that $d > 0$. Taking the limit as $n \to \infty$ in (2.4) we get

$$d = \lim_{n\to\infty} d_{n+1} \leqslant \lim_{n\to\infty} [\varphi(2d_n - d_{n+1}) \cdot (2d_n - d_{n+1})] \leqslant \lim_{n\to\infty} (2d_n - d_{n+1}).$$

It follows that $\lim_{n\to\infty} \varphi(2d_n - d_{n+1}) = 1$. Owing to the fact that $\varphi \in \phi$ we have

$$d = \lim_{n\to\infty} (2d_n - d_{n+1}) = 0,$$

which is a contradiction. Therefore,

$$d = \lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{2.5}$$

We claim now that $\{x_n\}$ is a Cauchy sequence. Suppose, on the contrary, that there exist $\varepsilon > 0$ and sequences $\{n(k)\}, \{m(k)\}$ of positive integers such that $n(k) > m(k) > k$ and

$$d\left(x_{n(k)}, x_{m(k)}\right) \geqslant \varepsilon, \quad d\left(x_{n(k)-1}, x_{m(k)}\right) < \varepsilon, \quad \forall k \in \mathbb{N}.$$

Using the triangle inequality, we have

$$\begin{aligned} \varepsilon \leqslant d\left(x_{n(k)}, x_{m(k)}\right) &\leqslant d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{m(k)}\right) \\ &< d\left(x_{n(k)}, x_{n(k)-1}\right) + \varepsilon. \end{aligned}$$

Combining (2.5) and the above inequality we obtain

$$\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon.$$

But, using the triangle inequality,

$$\begin{aligned} d\left(x_{n(k)}, x_{m(k)}\right) \leqslant\; & d\left(x_{n(k)}, x_{n(k)-1}\right) \\ & + d\left(x_{n(k)-1}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{m(k)}\right), \end{aligned}$$

and so

$$\begin{aligned} d\left(x_{n(k)}, x_{m(k)}\right) - d\left(x_{n(k)-1}, x_{m(k)-1}\right) \leqslant\; & d\left(x_{n(k)}, x_{n(k)-1}\right) \\ & + d\left(x_{m(k)-1}, x_{m(k)}\right). \end{aligned}$$

Also

$$\begin{aligned} d\left(x_{n(k)-1}, x_{m(k)-1}\right) \leqslant\; & d\left(x_{n(k)-1}, x_{n(k)}\right) \\ & + d\left(x_{n(k)}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{m(k)-1}\right), \end{aligned}$$

and

$$\begin{aligned} d\left(x_{n(k)-1}, x_{m(k)-1}\right) - d\left(x_{n(k)}, x_{m(k)}\right) \leqslant\; & d\left(x_{n(k)}, x_{n(k)-1}\right) \\ & + d\left(x_{m(k)-1}, x_{m(k)}\right). \end{aligned}$$

Now, we have

$$\begin{aligned} \left|d\left(x_{n(k)-1}, x_{m(k)-1}\right) - d\left(x_{n(k)}, x_{m(k)}\right)\right| \leqslant\; & d\left(x_{n(k)}, x_{n(k)-1}\right) \\ & + d\left(x_{m(k)-1}, x_{m(k)}\right), \end{aligned}$$

and

$$\begin{aligned} & \lim_{k\to\infty} \left|d\left(x_{n(k)}, x_{m(k)}\right) - d\left(x_{n(k)-1}, x_{m(k)-1}\right)\right| \\ & \quad \leqslant \lim_{k\to\infty} \left(d\left(x_{n(k)}, x_{m(k)}\right) + d\left(x_{m(k)-1}, x_{m(k)}\right)\right) = 0. \end{aligned}$$

Hence,

$$\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) = \lim_{k\to\infty} d\left(x_{n(k)-1}, x_{m(k)-1}\right) = \varepsilon. \tag{2.6}$$

On the other hand, from (2.2) we have

$$\begin{aligned} \varepsilon \leqslant d\left(x_{n(k)}, x_{m(k)}\right) &= d\left(Tx_{n(k)-1}, Tx_{m(k)-1}\right) \\ &\leqslant \varphi\left(E\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) \cdot E\left(x_{n(k)-1}, x_{m(k)-1}\right). \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} E\left(x_{n(k)-1}, x_{m(k)-1}\right) &= d\left(x_{n(k)-1}, x_{m(k)-1}\right) + \left|d\left(x_{n(k)-1}, Tx_{n(k)-1}\right) - d\left(x_{m(k)-1}, Tx_{m(k)-1}\right)\right| \\ &= d\left(x_{n(k)-1}, x_{m(k)-1}\right) + \left|d\left(x_{n(k)-1}, x_{n(k)}\right) - d\left(x_{m(k)-1}, x_{m(k)}\right)\right|, \end{aligned}$$

using (2.5) and (2.6) we obtain

$$\lim_{k\to\infty} E\left(x_{n(k)-1}, x_{m(k)-1}\right) = \varepsilon. \tag{2.8}$$

Combining (2.7), (2.8) with the property of φ, we get

$$\varepsilon \leqslant \lim_{k\to\infty} \varphi\left(E\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) \cdot \varepsilon \leqslant \varepsilon,$$

so,

$$\lim_{k\to\infty} \varphi\left[E\left(x_{n(k)-1}, x_{m(k)-1}\right)\right] = 1 \Longrightarrow \varepsilon = \lim_{k\to\infty} E\left(x_{n(k)-1}, x_{m(k)-1}\right) = 0.$$

It is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By completeness of $(X, d), \{x_n\}$ converges to some point $x^* \in X$,

$$\lim_{n\to\infty} d\left(x_n, x^*\right) = 0. \tag{2.9}$$

We shall prove that x^* is a fixed point of T.

If for each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_n} = Tx^*$ and $i_n > i_{n-1}$, then we have

$$x^* = \lim_{n\to\infty} x_{i_{n+1}} = Tx^*.$$

This proves that x^* is a fixed point of T.

Suppose now, there exists $N \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$. This implies

$$d\left(x_{n+1}, Tx^*\right) > 0, \quad \forall n > N.$$

It follows from (2.2) and the property of φ that

$$d\left(x_{n+1}, Tx^*\right) = d\left(Tx_n, Tx^*\right) \leqslant \varphi\left(E\left(x_n, x^*\right)\right) \cdot E\left(x_n, x^*\right) < E\left(x_n, x^*\right). \tag{2.10}$$

Since

$$\begin{aligned} E\left(x_n, x^*\right) &= d\left(x_n, x^*\right) + \left|d\left(x_n, Tx_n\right) - d\left(x^*, Tx^*\right)\right| \\ &= d\left(x_n, x^*\right) + \left|d\left(x_n, x_{n+1}\right) - d\left(x^*, Tx^*\right)\right|, \end{aligned}$$

combining (2.5) and (2.9) we have

$$\lim_{n\to\infty} E\left(x_n, x^*\right) = d\left(x^*, Tx^*\right).$$

Taking the limit as $n \to \infty$ in (2.10), since $d\left(x^*, Tx^*\right) > 0$, we have

$$d\left(x^*, Tx^*\right) \leqslant \lim_{n\to\infty} \varphi\left(E\left(x_n, x^*\right)\right) \cdot d\left(x^*, Tx^*\right) \leqslant d\left(x^*, Tx^*\right),$$

hence $\lim_{n\to\infty} \varphi(E(x_n, x^*)) = 1$. Thus, we obtain

$$d(x^*, Tx^*) = \lim_{n\to\infty} E(x_n, x^*) = 0.$$

It is a contradiction. Therefore $d(x^*, Tx^*) = 0$, that is, x^* is a fixed point of T.

Finally, we prove that the fixed point of T is unique. For this, let x^*, y^* be two fixed points of T, and suppose that

$$Tx^* = x^* \neq y^* = Ty^*.$$

Since

$$\begin{aligned} E(x^*, y^*) &= d(x^*, y^*) + |d(x^*, Tx^*) - d(y^*, Ty^*)| \\ &= d(x^*, y^*) + |d(x^*, x^*) - d(y^*, y^*)| \\ &= d(x^*, y^*), \end{aligned}$$

it follows from (2.1) that

$$\begin{aligned} 0 < d(x^*, y^*) = d(Tx^*, Ty^*) &\leqslant \varphi(E(x^*, y^*)) \cdot E(x^*, y^*) \\ = \varphi(d(x^*, y^*)) \cdot d(x^*, y^*) &< d(x^*, y^*). \end{aligned}$$

This is a contradiction. Then $d(x^*, y^*) = 0$, that is $x^* = y^*$. This proves that the fixed point of T is unique. □

Example 2.4. Let $X = \{0, 1, 4\}$, $d(x, y) = |x - y|$ and $T : X \to X$ defined by

$$T1 = T4 = 1, \quad T0 = 4.$$

Then, since $d(T0, T1) = 3$ and $d(0, 1) = 1$ we can not find a function $\varphi \in \phi$ satisfying

$$d(T0, T1) \leqslant \varphi(d(0, 1)) \cdot d(0, 1).$$

Therefore, T is not a Geraghty contraction. Now consider a function $\varphi : [0, \infty) \to [0, 1)$, defined by

$$\varphi(t) = \begin{cases} \frac{1}{1+\frac{t}{15}}, & t > 0, \\ \frac{1}{2}, & t = 0, \end{cases}$$

then T is a φ_E-Geraghty contraction.

Indeed, since

$$\begin{aligned} &d(0, 1) = 1, \quad d(0, 4) = 4, \quad d(1, 4) = 3, \\ &d(0, T0) = |0 - 4| = 4, \quad d(1, T1) = |1 - 1| = 0, \quad d(4, T4) = |4 - 1| = 3, \\ &d(T0, T1) = |4 - 1| = 3, \quad d(T0, T4) = |4 - 1| = 3, \quad d(T1, T4) = 0, \end{aligned}$$

for $x = 0$ and $y = 1$

$$3 = d(T0, T1) \geqslant \frac{15}{16} = \frac{d(0, 1)}{1 + \frac{d(0,1)}{15}} = \varphi(d(0, 1)) \cdot d(0, 1).$$

On the other hand,

$$\begin{aligned} E(0, 1) &= d(0, 1) + |d(0, T0) - d(1, T1)| = 1 + |4 - 0| = 5, \\ E(0, 4) &= d(0, 4) + |d(0, T0) - d(4, T4)| = 4 + |4 - 3| = 5, \\ E(1, 4) &= d(1, 4) + |d(1, T1) - d(4, T4)| = 3 + |0 - 3| = 6, \end{aligned}$$

so, we have the following cases:

Case 1. Let $x = 0$ and $y = 1$. Then

$$d(T0, T1) \leqslant \varphi(E(0,1)) \cdot E(0,1) \quad \Longleftrightarrow 3 \leqslant \frac{5}{1+\frac{5}{15}} = \frac{5}{1+\frac{1}{3}} = \frac{15}{4}.$$

Case 2. Let $x = 0$ and $y = 4$. Then

$$3 = d(T0, T4) \leqslant \varphi(E(0,4)) \cdot E(0,4) = \frac{5}{1+\frac{5}{15}} = \frac{15}{4}.$$

Case 3. Let $x = 1$ and $y = 4$. Then

$$0 = d(T1, T4) \leqslant \varphi(E(1,4)) \cdot E(1,4) = \frac{6}{1+\frac{6}{15}}.$$

This proves that T is a φ_E-Geraghty contraction.

Example 2.5. Let $T : \left[-\frac{2}{3}, \frac{2}{3}\right] \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[-\frac{2}{3}, 0\right], \\ -x, & x \in \left(0, \frac{2}{3}\right], \end{cases}$$

and $d(x, y) = |x - y|$. Let us consider the mapping

$$\varphi(t) = \begin{cases} \frac{1}{1+t}, t > 0, \\ \frac{1}{2}, t = 0, \end{cases}$$

We obtain that T is a φ_E-Geraghty contraction.

To see this, let us consider the following calculations. First, we observe that for $x, y \in \left(0, \frac{2}{3}\right]$, with $x \neq y$ we have

$$d(Tx, Ty) = |x - y| \leqslant \frac{|x-y|}{1+|x-y|} = \frac{1}{1+d(x,y)} \cdot d(x,y)$$
$$\Longleftrightarrow 1 + |x - y| \leqslant 1 \Longleftrightarrow |x - y| \leqslant 0.$$

This is a contradiction, so, Geraghty's theorem cannot be used to prove the existence of a fixed point of T.

Now, we consider the following cases:

Case 1: Let $x, y > 0$, with $x < y$. Then, $d(x, y) = |x - y| = y - x$ and

$$E(x,y) = d(x,y) + |d(x, Tx) - d(y, Ty)| = |x - y| + ||x + x| - |y + y|| = 3(y - x).$$

Thus,

$$y - x = d(Tx, Ty) \leqslant \frac{E(x,y)}{1+E(x,y)} = \frac{3(y-x)}{1+3(y-x)}$$
$$\Longleftrightarrow 1 + 3(y - x) \leqslant 3 \Longleftrightarrow y - x \leqslant \frac{2}{3}.$$

Case 2: Let $x, y < 0,\ x < y$. Then, $d(x, y) = |x - y|$, $d(Tx, Ty) = 0$ and

$$E(x,y) = |x - y| + ||x - 0| - |y - 0|| = 2|x - y|.$$

So,

$$0 = d(Tx, Ty) \leqslant \frac{2|x-y|}{1+2|x-y|} = \frac{E(x,y)}{1+E(x,y)},$$

is true.

Case 3: Let $x \leqslant 0,\ y > 0$. Then, $d(x,y) = y - x$, $d(Tx,Ty) = y$ and

$$E(x,y) = y - x + |-x - |2y|| = y - x + |-x - 2y|.$$

Because $x < 0$, let us denote $-x = a \geqslant 0$. We have now two subcases:

(i) If $a \leqslant 2y$, then

$$E(x,y) = y + a + |a - 2y| = y + a + 2y - a = 3y \quad \text{and}$$
$$y = d(Tx,Ty) \leqslant \frac{3y}{1+3y} \Longleftrightarrow y + 3y^2 \leqslant 3y \Leftrightarrow 0 \leqslant y \leqslant \frac{2}{3}.$$

(ii) If $a > 2y$, then

$$E(x,y) = y + a + a - 2y = 2a - y \quad \text{and}$$
$$y = d(Tx,Ty) \leqslant \frac{2a-y}{1+2a-y} \Longleftrightarrow y + 2ay - y^2 \leqslant 2a - y$$
$$\Leftrightarrow \frac{2y-y^2}{2-2y} \leqslant a.$$

This is true, because we have

$$\frac{2y-y^2}{2-2y} \leqslant 2y < a \Longleftrightarrow 2y - y^2 \leqslant 4y - 4y^2 \Longleftrightarrow 3y^2 \leqslant 2y \Leftrightarrow y \in \left[0, \frac{2}{3}\right].$$

Since the conditions of Theorem 2.3 are satisfied, then T has a unique fixed point.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1

[2] V. Berinde, *Approximating fixed points of implicit almost contractions*, Hacet. J. Math. Stat., **41** (2012), 93–102.

[3] V. Berinde, M. Păcurar, *Fixed point theorems for nonself single-valued almost contractions*, Fixed Point Theory, **14** (2013), 301–311.

[4] N. Bilgili, E. Karapınar, K. Sadarangani, *A generalization for the best proximity point of Geraghty-contractions*, J. Inequal. Appl., **2013** (2013), 9 pages.

[5] J. Caballero, J. Harjani, K. Sadarangani, *A best proximity point theorem for Geraghty-contractions*, Fixed Point Theory Appl., **2012** (2012), 9 pages.

[6] S.-H. Cho, J.-S. Bae, E. Karapınar, *Fixed point theorems for α-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl., **2013** (2013), 11 pages.

[7] M. A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608. 1, 1.1

[8] M. A. Geraghty, *An improved criterion for fixed points of contraction mappings*, J. Math. Anal. Appl., **48** (1974), 811–817.

[9] N. Hussain, V. Parvaneh, B. Samet, C. Vetro, *Some fixed point theorems for generalized contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2015** (2015), 17 pages.

[10] M. Jleli, B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl., **2014** (2014), 8 pages.

[11] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60** (1968), 71–76 .

[12] E. Karapınar, *Edelstein type fixed point theorems*, Fixed Point Theory Appl., **2012** (2012), 12 pages.

[13] E. Karapınar, *Weak φ-contraction on partial metric spaces*, J. Comput. Anal. Appl., **14** (2012), 206–210

[14] C. Mongkolkeha, Y. J. Cho, P. Kumam, *Best proximity points for Geraghty's proximal contraction mappings*, Fixed Point Theory Appl., **2013** (2013), 17 pages.

[15] O. Popescu, *Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl., **2014** (2014), 12 pages.

[16] O. Popescu, *A new type of contractive mappings in complete metric spaces*, (submitted). 1

[17] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for αψ-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165.

[18] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc., **136** (2008), 1861–1869. 1