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Positive solution for a coupled system of nonlinear fractional differential equations with fractional integral conditions

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Abstract

By studying the properties of Green's function, constructing a special cone and applying fixed point theorem of cone expansion and compression of norm type, this paper investigates the existence of at least one and two positive solutions for a coupled system of nonlinear fractional differential equations involving fractional integral conditions and derivatives of arbitrary order. Two examples are given to illustrate our results.

Keywords: Positive solution, fractional differential equations, fractional integral conditions, fixed point theorem.
2010 MSC: 26A33, 34B18.

1. Introduction

In this paper, we study the existence of positive solution for a coupled system of nonlinear fractional differential equations (FDEs) with fractional integral conditions

$$\begin{cases} {}^C D^{\alpha_1} u(t) = f_1(t, u(t), v(t), u^{(m_1)}(t), v^{(m_2)}(t)), \ \ 0 < t < 1, \ \ n-1 < \alpha_1 < n, \\ {}^C D^{\alpha_2} v(t) = f_2(t, u(t), v(t), u^{(m_1)}(t), v^{(m_2)}(t)), \ \ 0 < t < 1, \ \ n-1 < \alpha_2 < n, \\ u^{(k)}(0) = 0, \ \ 0 \leqslant k \leqslant n-2, \ \ u^{(n-1)}(0) = \rho_1 I^{\beta_1} u(1), \ \ \rho_1, \beta_1 > 0, \\ v^{(k)}(0) = 0, \ \ 0 \leqslant k \leqslant n-2, \ \ v^{(n-1)}(0) = \rho_2 I^{\beta_2} v(1), \ \ \rho_2, \beta_2 > 0, \end{cases} \tag{1.1}$$

where ${}^C D^{\alpha_i}$, $i = 1, 2$ denote the Caputo fractional derivatives of order α_i and I^{β_i}, $i = 1, 2$ denote the Riemann-Liouville fractional integrals of order $\beta_i, f_i \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}_+)$, $i = 1, 2$ and $m_i \in \{1, 2, \cdots, n-2\}$, $\Gamma(n + \beta_i) > \rho_i$, $i = 1, 2$.

The study of the coupled system of fractional order is very significant because this kind of system can often occur in various applications. Examples include distributed order dynamical [13], duffing system [4], Lozenz systems [11], anomalous diffusion [17, 21], synchronization of coupled fractional-order chaotic systems [9, 10]. There are also a large number of papers investigating the solvability of coupled system of nonlinear fractional differential equations. For details, see [1, 5, 6, 15, 18, 22–24, 26, 27, 29]. Some recent results on coupled systems of fractional-order different equations, including nonlocal and integral boundary conditions can be found in [2, 16, 25, 28] and the references cited therein. At the same time,

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some new algorithms for finding solution of fractional differential equations are constructed based on different boundary conditions in [3, 8, 19] and the references cited therein.

Inspired by the work of above mentioned papers, we investigate the existence of at least one and two positive solutions for a coupled system of nonlinear FDEs (1.1). Though we make use a well-known tool of fixed point theorem of cone expansion and compression of norm type, yet its exposition to the given problem is new, which involves Riemann-Liouville fractional integral boundary conditions and derivatives of arbitrary order. Further, we construct a special cone by studying properties of Green's function.

The paper is organized as follows. In Section 2, we present some basic concepts and lemmas. In Section 3, the main results are formulated. In Section 4, two examples are given.

2. Preliminaries

First of all, we present some definitions and lemmas about fractional calculus, more details can be found in [14, 20].

Definition 2.1. For at least n-times continuous differentiable function $f : [0, \infty] \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$$^{C}D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt, \quad n-1<\alpha<n, \quad n=[\alpha]+1,$$

$[\alpha]$ denotes the integer part of number α.

Definition 2.2. The Riemann-Liouville fractional integral of order α for a continuous function f is defined as

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad \alpha>0,$$

provided the integral exists.

Lemma 2.3. *The fractional differential equation* $^{C}D^{\alpha}u(t) = 0, \alpha > 0$ *has a general solution*

$$u(t) = c_1 + c_2t + c_3t^2 + \cdots + c_nt^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 1, 2, \cdots, n, \quad n = [\alpha] + 1.$$

Lemma 2.4. *For any* $t \in [a, b]$, *then*

$$I^pI^qg(t) = I^{p+q}g(t) = I^qI^pg(t), \ {}^{C}D^pI^pg(t) = g(t), \ {}^{C}D^qI^pg(t) = I^{p-q}g(t), \ p > q \geqslant 0.$$

In order to prove our main results, we need the following auxiliary lemma which is the key to define the solution for the FDEs (1.1).

Lemma 2.5. *Let* $x, y \in L[0, 1]$ *and* $u, v \in AC^n[0, 1]$, $\Gamma(n + \beta_i) > \rho_i$. *Then the unique solution of the fractional boundary value problem*

$$\begin{cases} ^{C}D^{\alpha_1}u(t) = x(t), \ \ t \in (0,1), \ \ n-1<\alpha_1<n, \\ ^{C}D^{\alpha_2}v(t) = y(t), \ \ t \in (0,1), \ \ n-1<\alpha_2<n, \\ u^{(k)}(0) = 0, \ \ 0 \leqslant k \leqslant n-2, \ \ u^{(n-1)}(0) = \rho_1 I^{\beta_1}u(1), \\ v^{(k)}(0) = 0, \ \ 0 \leqslant k \leqslant n-2, \ \ v^{(n-1)}(0) = \rho_2 I^{\beta_2}v(1), \end{cases} \tag{2.1}$$

is

$$\begin{cases} u(t) = \int_0^1 G_1(t,s)x(s)ds, \\ v(t) = \int_0^1 G_2(t,s)y(s)ds, \end{cases} \tag{2.2}$$

where $\Delta_i = \dfrac{\rho_i\Gamma(n+\beta_i)}{(\Gamma(n+\beta_i)-\rho_i)\Gamma(\alpha_i+\beta_i)}, i=1,2,$

$$G_i(t,s)=\begin{cases}\dfrac{t^{n-1}(1-s)^{\alpha_i+\beta_i-1}\Delta_i}{\Gamma(n)}+\dfrac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & 0\leqslant s\leqslant t\leqslant 1,\\ \dfrac{t^{n-1}(1-s)^{\alpha_i+\beta_i-1}\Delta_i}{\Gamma(n)}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}\quad i=1,2. \tag{2.3}$$

Proof. Applying Lemma 2.3, the general solutions of the fractional differential equation in (2.1) can be given by

$$u(t)=c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1}+I^{\alpha_1}x(t), \tag{2.4}$$

$$v(t)=d_1+d_2t+d_3t^2+\cdots+d_nt^{n-1}+I^{\alpha_2}y(t). \tag{2.5}$$

Applying the condition $u^{(k)}(0)=0,\ v^{(k)}(0)=0,\ 0\leqslant k\leqslant n-2$ in (2.1), it is easy to know that $c_i=d_i=0,\ 1\leqslant i\leqslant n-1$. In view of the conditions $u^{(n-1)}(0)=\rho_1 I^{\beta_1}u(1)$ and $v^{(n-1)}(0)=\rho_2 I^{\beta_2}v(1)$, applying Lemma 2.4, we get

$$c_n=\frac{\rho_1\Gamma(n+\beta_1)}{\Gamma(n)(\Gamma(n+\beta_1)-\rho_1)}I^{\alpha_1+\beta_1}x(1),$$

$$d_n=\frac{\rho_2\Gamma(n+\beta_2)}{\Gamma(n)(\Gamma(n+\beta_2)-\rho_2)}I^{\alpha_2+\beta_2}y(1).$$

Substituting the values of $c_i, d_i, 1\leqslant i\leqslant n$ to (2.4) and (2.5), we obtain (2.2). The proof is completed. □

Remark 2.6. If $0\leqslant j\leqslant n-2$, then Green function $G_i(t,s)$ defined in (2.3) satisfies

$$G_{it}^{(j)}(t,s)=\begin{cases}\dfrac{t^{n-1-j}(1-s)^{\alpha_i+\beta_i-1}\Delta_i}{\Gamma(n-j)}+\dfrac{(t-s)^{\alpha_i-1-j}}{\Gamma(\alpha_i-j)}, & 0\leqslant s\leqslant t\leqslant 1,\\ \dfrac{t^{n-1-j}(1-s)^{\alpha_i+\beta_i-1}\Delta_i}{\Gamma(n-j)}, & 0\leqslant t\leqslant s\leqslant 1.\end{cases} \tag{2.6}$$

Lemma 2.7. *Function* $G_{it}^{(j)}(t,s)$ *defined in* (2.6) *satisfies*

(i) $G_{it}^{(j)}(t,s)\in C^{n-2}([0,1]\times[0,1],R_+)$ *and* $G_{it}^{(j)}(t,s)\geqslant 0,\ t,s\in[0,1]$;

(ii) $G_{it}^{(j)}(t,s)\leqslant \Lambda_{ij}=\dfrac{\Delta_i}{\Gamma(n-j)}+\dfrac{1}{\Gamma(\alpha_i-j)},\ t,s\in[0,1]$;

(iii) $G_{it}^{(j)}(t,s)\geqslant \Upsilon_{ij}=\dfrac{\Delta_i\xi_i^n(1-\xi_i')^{\alpha_i+\beta_i}}{\Gamma(n-j)},\ t,s\in[\xi_i,\xi_i']\subset(0,1)$.

Proof. From the definition of function $G_{it}^{(j)}(t,s)$ in (2.6), the conclusion of (i) and (ii) are obvious, so we omit them. We prove only the conclusion of (iii).

As $0<\xi_i\leqslant s\leqslant t\leqslant\xi_i'<1$,

$$G_{it}^{(j)}(t,s)\geqslant\frac{\Delta_i t^n(1-s)^{\alpha_i+\beta_i}}{\Gamma(n-j)}>\frac{\Delta_i\xi_i^n(1-\xi_i')^{\alpha_i+\beta_i}}{\Gamma(n-j)}=\Upsilon_{ij},$$

as $0<\xi_i\leqslant t\leqslant s\leqslant\xi_i'<1$,

$$G_{it}^{(j)}(t,s)\geqslant\frac{\Delta_i t^n(1-s)^{\alpha_i+\beta_i}}{\Gamma(n-j)}\geqslant\frac{\Delta_i\xi_i^n(1-\xi_i')^{\alpha_i+\beta_i}}{\Gamma(n-j)}=\Upsilon_{ij},$$

the proof of (iii) is completed. □

We define the space $X = \{u(t)|u(t) \in C^{m_1}[0,1]\}$ with the norm

$$\|u\|_X = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u^{(m_1)}(t)|.$$

Obviously, $(X, \|\cdot\|_X)$ is a Banach space. Also we define the space and $Y = \{v(t)|v(t) \in C^{m_2}[0,1]\}$ with the norm

$$\|v\|_Y = \max_{t\in[0,1]} |v(t)| + \max_{t\in[0,1]} |v^{(m_2)}(t)|.$$

Again $(Y, \|\cdot\|_Y)$ is a Banach space. Then the product space $(X \times Y, \|(u,v)\|_{X\times Y})$ is also a Banach space with the norm $\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y$.

In view of Lemma 2.5, we define the operator $T : X \times Y \to X \times Y$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)),$$

where

$$T_i(u,v)(t) = \int_0^1 G_i(t,s)f_i(s,u(s),v(s),u^{(m_1)}(s),v^{(m_2)}(s))ds, \quad i = 1,2.$$

Remark 2.8. A pair of function (u,v) is said to be a positive solution of the FDEs (1.1), if $u(t) > 0$, $v(t) > 0$, for all $t \in (0,1)$ and (u,v) satisfies FDEs (1.1).

Lemma 2.9. *Suppose* (u,v) *is a positive solution for FDEs* (1.1)*, then*

$$\min_{t\in[\xi,\xi']}(u,v) \overset{\text{def}}{=} \min_{t\in[\xi,\xi']}(u(t)+u^{(m_1)}(t)) + \min_{t\in[\xi,\xi']}(v(t)+v^{(m_2)}(t)) \geqslant \gamma\|(u,v)\|_{X\times Y},$$

where $\xi = \max\{\xi_1,\xi_2\}, \xi' = \min\{\xi_1',\xi_2'\}, \gamma = \min\{(\Upsilon_{10}+\Upsilon_{1m_1})/(\Lambda_{10}+\Lambda_{1m_1}), (\Upsilon_{20}+\Upsilon_{2m_2})/(\Lambda_{20}+\Lambda_{2m_2})\}$.

Proof. By Lemma 2.5, we can obtain immediately that (u,v) is a solution of FDEs (1.1) if and only if $(u,v) \in X \times Y$ is a solution of the operator equations $T(u,v) = (u,v)$. So we have

$$u^{(h)}(t) = \int_0^1 G_{1t}^{(h)}(t,s)f_1(s,u(s),v(s),u^{(m_1)}(s),v^{(m_2)}(s))ds, \quad h = 0, m_1.$$

Further, by Lemma 2.7, we have

$$\begin{aligned}\|u\|_X &= \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u^{(m_1)}(t)| \\ &\leqslant (\Lambda_{10}+\Lambda_{1m_1})\int_0^1 f_1(s,u(s),v(s),u^{(m_1)}(s),v^{(m_2)}(s))ds.\end{aligned} \tag{2.7}$$

On the other hand

$$\begin{aligned}\min_{t\in[\xi_1,\xi_1']} u(t) + \min_{t\in[\xi_1,\xi_1']} u^{(m_1)}(t) &\geqslant \int_0^1 \min_{t\in[\xi_1,\xi_1']} G_1(t,s)f_1(s,u(s),v(s),u^{(m_1)}(s),v^{(m_2)}(s))ds \\ &\quad + \int_0^1 \min_{t\in[\xi_1,\xi_1']} G_{1t}^{(m_1)}(t,s)f_1(s,u(s),v(s),u^{(m_1)}(s),v^{(m_2)}(s))ds \\ &\geqslant (\Upsilon_{10}+\Upsilon_{1m_1})\int_0^1 f_1(s,u(s),v(s),u^{(m_1)}(s),v^{(m_2)}(s))ds \\ &\geqslant (\Upsilon_{10}+\Upsilon_{1m_1})(\Lambda_{10}+\Lambda_{1m_1})^{-1}\|u\|_X \\ &\geqslant \gamma\|u\|_X.\end{aligned} \tag{2.8}$$

Similar to (2.7) and (2.8), we can obtain

$$\min_{t\in[\xi_2,\xi_2']} v(t) + \min_{t\in[\xi_2,\xi_2']} v^{(m_2)}(t) \geqslant (\Upsilon_{20}+\Upsilon_{2m_2})(\Lambda_{20}+\Lambda_{2m_2})^{-1}\|v\|_Y \geqslant \gamma\|v\|_Y. \tag{2.9}$$

From (2.8) and (2.9), we get

$$\begin{aligned}\min_{t\in[\xi,\xi']}(u(t)+u^{(m_1)}(t))+\min_{t\in[\xi,\xi']}(v(t)+v^{(m_2)}(t)) &\geqslant \min_{t\in[\xi_1,\xi_1']}(u(t)+u^{(m_1)}(t))\\ &\quad+\min_{t\in[\xi_2,\xi_2']}(v(t)+v^{(m_2)}(t))\\ &\geqslant \min_{t\in[\xi_1,\xi_1']}u(t)+\min_{t\in[\xi_1,\xi_1']}u^{(m_1)}(t)\\ &\quad+\min_{t\in[\xi_2,\xi_2']}v(t)+\min_{t\in[\xi_2,\xi_2']}v^{(m_2)}(t)\\ &\geqslant \gamma\|u\|_X+\gamma\|v\|_Y\\ &=\gamma\|(u,v)\|_{X\times Y}.\end{aligned}$$

The proof is completed. □

Let $K=\{(u,v)\in X\times Y: u(t)\geqslant 0,\ v(t)\geqslant 0,\ \forall t\in[0,1],\ \min_{t\in[\xi,\xi']}(u,v)\geqslant\gamma\|(u,v)\|_{X\times Y}\}$. So, we can obtain the following lemma.

Lemma 2.10. *The operator* $T:K\to K$ *is a completely continuous operator.*

Proof. We first show that operator $T:K\to K$. Since $G_i(t,s)\geqslant 0$, $f_i\geqslant 0$, for all $t,s\in[0,1]$, it is easy to know $T_i(u,v)(t)\geqslant 0$, for all $t\in[0,1]$. For all $(u,v)\in K$, similar to (2.8) and (2.9), we know

$$\min_{t\in[\xi_1,\xi_1']}T_1(u,v)(t)+\min_{t\in[\xi_1,\xi_1']}T_1^{(m_1)}(u,v)(t)\geqslant\gamma\|T_1(u,v)\|_X,$$

and

$$\min_{t\in[\xi_2,\xi_2']}T_2(u,v)(t)+\min_{t\in[\xi_2,\xi_2']}T_2^{(m_2)}(u,v)(t)\geqslant\gamma\|T_2(u,v)\|_Y.$$

Further, we have

$$\begin{aligned}\min_{t\in[\xi,\xi']}T(u,v)(t)&=\min_{t\in[\xi,\xi']}(T_1(u,v)(t),T_2(u,v)(t))\\ &=\min_{t\in[\xi,\xi']}(T_1(u,v)(t)+T_1^{(m_1)}(u,v)(t))\\ &\quad+\min_{t\in[\xi,\xi']}(T_2(u,v)(t)+T_2^{(m_2)}(u,v)(t))\\ &\geqslant\min_{t\in[\xi_1,\xi_1']}T_1(u,v)(t)+\min_{t\in[\xi_1,\xi_1']}T_1^{(m_1)}(u,v)(t)\\ &\quad+\min_{t\in[\xi_2,\xi_2']}T_2(u,v)(t)+\min_{t\in[\xi_2,\xi_2']}T_2^{(m_2)}(u,v)(t)\\ &\geqslant\gamma\|T_1(u,v)\|_X+\gamma\|T_2(u,v)\|_Y\\ &=\gamma\|T(u,v)\|_{X\times Y},\end{aligned}$$

so the operator $T:K\to K$.

Next, we show that the operator $T:K\to K$ is completely continuous. Since G_i, f_i are continuous, the operator T is continuous. Let $B_r=\{(u,v)\in K:\|(u,v)\|_{X\times Y}\leqslant r\}$ be bounded set in K. Let

$$M_i=\max\{f_i(t,u(t),v(t),u^{(m_1)}(t),v^{(m_2)}(t)):0\leqslant t\leqslant 1,(u,v)\in B_r\},\quad i=1,2.$$

For $(u,v)\in B_r$, by Lemma 2.7, we get

$$\begin{aligned}\|T(u,v)\|_{X\times Y}&=\|T_1(u,v)\|_X+\|T_2(u,v)\|_Y\\ &=\max_{t\in[0,1]}|T_1(u,v)(t)|+\max_{t\in[0,1]}|T_1(u,v)^{(m_1)}(t)|\\ &\quad+\max_{t\in[0,1]}|T_2(u,v)(t)|+\max_{t\in[0,1]}|T_2(u,v)^{(m_2)}(t)|\end{aligned}$$

$$\begin{aligned}&\leqslant (\Lambda_{10}+\Lambda_{1m_1})\int_0^1 f_1(t,u(t),v(t),u^{(m_1)}(t),v^{(m_2)}(t))ds\\&\quad+(\Lambda_{20}+\Lambda_{2m_2})\int_0^1 f_2(t,u(t),v(t),u^{(m_1)}(t),v^{(m_2)}(t))ds\\&\leqslant M_1(\Lambda_{10}+\Lambda_{1m_1})+M_2(\Lambda_{20}+\Lambda_{2m_2}).\end{aligned}$$

Therefore, the operator T is uniformly bounded in B_r.

Then, we show that the operator T is equicontinuous. Let $0\leqslant t_1<t_2\leqslant 1$, we have

$$\begin{aligned}&|T_1(u,v)^{(h)}(t_2)-T_1(u,v)^{(h)}(t_1)|\\&\leqslant M_1\Big[\int_0^{t_1}|G_{1t}^{(h)}(t_2,s)-G_{1t}^{(h)}(t_1,s)|ds+\int_{t_1}^{t_2}|G_{1t}^{(h)}(t_2,s)-G_{1t}^{(h)}(t_1,s)|ds\\&\quad+\int_{t_2}^{1}|G_{1t}^{(h)}(t_2,s)-G_{1t}^{(h)}(t_1,s)|ds\Big]\\&\leqslant M_1\Big[\int_0^{t_1}\Big(\frac{(t_2-s)^{\alpha_1-1-h}-(t_1-s)^{\alpha_1-1-h}}{\Gamma(\alpha_1-h)}+\frac{(t_2^{n-1-h}-t_1^{n-1-h})(1-s)^{\alpha_1+\beta_1-1}\Delta_1}{\Gamma(n-h)}\Big)ds\\&\quad+\int_{t_1}^{t_2}\Big(\frac{(t_2-s)^{\alpha_1-1-h}}{\Gamma(\alpha_1-h)}+\frac{(t_2^{n-1-h}-t_1^{n-1-h})(1-s)^{\alpha_1+\beta_1-1}\Delta_1}{\Gamma(n-h)}\Big)ds\\&\quad+\int_{t_2}^{1}\frac{(t_2^{n-1-h}-t_1^{n-1-h})(1-s)^{\alpha_1+\beta_1-1}\Delta_1}{\Gamma(n-h)}ds\Big]\\&=\frac{M_1}{\Gamma(\alpha_1-h+1)}(t_2^{\alpha_1-h}-t_1^{\alpha_1-h})+\frac{\Delta_1M_1}{\Gamma(n-h)}(t_2^{n-1-h}-t_1^{n-1-h}),\ h=0,m_1,\end{aligned}$$

and

$$\|T_1(u,v)(t_2)-T_1(u,v)(t_1)\|_X=|T_1(u,v)(t_2)-T_1(u,v)(t_1)|+|T_1(u,v)^{(m_1)}(t_2)-T_1(u,v)^{(m_1)}(t_1)|.$$

Thus we know that $\|T_1(u,v)(t_2)-T(u,v)_1(t_1)\|_X\to 0$ independent of u and v as $t_2\to t_1$. Similarly, it is easy to know that $\|T_2(u,v)(t_2)-T(u,v)_2(t_1)\|_Y\to 0$ independent of u and v as $t_2\to t_1$. On the other hand, we notice

$$\|T(u,v)(t_2)-T(u,v)(t_1)\|_{X\times Y}=\|T_1(u,v)(t_2)-T(u,v)_1(t_1)\|_X+\|T_2(u,v)(t_2)-T_2(u,v)(t_1)\|_Y,$$

which implies that $\|T(u,v)(t_2)-T(u,v)(t_1)\|_{X\times Y}\to 0$ independent of u and v as $t_2\to t_1$. So the operator T is equicontinuous in B_r. From above the arguments, we know that the operator T is completely continuous by Ascoli-Arzelà theorem. □

Lemma 2.11 ([7, 12]). *Suppose* E *is a real Banach space and* P *is cone in* E, *and let* Ω_1,Ω_2 *be bounded open sets in* E *such that* $\theta\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$. *Let operator* $T:\ P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ *be completely continuous. Suppose that one of two conditions holds:*

(i) $\|Tu\|\leqslant\|u\|,\ \forall u\in P\cap\partial\Omega_1;\qquad \|Tu\|\geqslant\|u\|,\ \forall u\in P\cap\partial\Omega_2;$

(ii) $\|Tu\|\geqslant\|u\|,\ \forall u\in P\cap\partial\Omega_1;\qquad \|Tu\|\leqslant\|u\|,\ \forall u\in P\cap\partial\Omega_2.$

Then operator T has at least one fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3. Main results

In the following subsection, we establish our main results for FDEs (1.1) by using fixed point theory of cone expansion and compression of norm type. For convenience, we set

$$f^{i\beta}=\limsup_{u_0+u_1+v_0+v_1\to\beta}\ \max_{t\in[0,1]}\frac{f_i(t,u_0,v_0,u_1,v_1)}{u_0+u_1+v_0+v_1},$$

$$f_{i\beta} = \liminf_{u_0+u_1+v_0+v_1\to\beta} \min_{t\in[0,1]} \frac{f_i(t,u_0,v_0,u_1,v_1)}{u_0+u_1+v_0+v_1},$$

where $\beta = 0^+$ or $+\infty$. Let $r = \frac{1}{4}\min\{\Lambda_{10}^{-1},\Lambda_{1m_1}^{-1},\Lambda_{20}^{-1},\Lambda_{2m_2}^{-1}\}$, $R = \frac{1}{4\gamma(\xi'-\xi)}\max\{\Upsilon_{10}^{-1},\Upsilon_{1m_1}^{-1},\Upsilon_{20}^{-1},\Upsilon_{2m_2}^{-1}\}$.

Theorem 3.1. *If* $f^{10}, f^{20} \in [0,r)$ *and* $f_{1\infty}, f_{2\infty} \in (R,+\infty]$, *then there exists at least one positive solution for FDEs* (1.1) *in* K.

Proof. At first, it follows from the condition $f^{10}, f^{20} \in [0,r)$ that there exist $\mu_1 > 0$ and a sufficiently small $\varepsilon_1 > 0$ such that

$$f_i(t,u_0,v_0,u_1,v_1) \leqslant (f_{i0}+\varepsilon_1)(u_0+v_0+u_1+v_1),\ \forall t\in[0,1], (u_0+v_0+u_1+v_1)\leqslant \mu_1, \tag{3.1}$$

where $f_{i0}+\varepsilon_1 \leqslant r,\ i=1,2$.

Let $\Omega_1 = \{(u,v)\in K : \|(u,v)\|_{X\times Y} < \mu_1\}$. For all $(u,v)\in\partial\Omega_1\cap K$, using (3.1) and Lemma 2.7, we get

$$\begin{aligned}
|T_i^{(j)}(u,v)(t)| &\leqslant \Lambda_{ij}\int_0^1 f_i(s,u_0,v_0,u_1,v_1)ds\\
&\leqslant \Lambda_{ij}(f_{i0}+\varepsilon_1)\int_0^1 (u(s)+v(s)+u^{(m_1)}(s)+v^{(m_2)}(s))ds\\
&\leqslant \frac{1}{4}\|u\|_X + \frac{1}{4}\|v\|_Y = \frac{1}{4}\|(u,v)\|_{X\times Y},\quad i=1,2,\quad j=0,m_1,m_2.
\end{aligned}$$

Thus

$$\|T_1(u,v)\|_X = \max_{t\in[0,1]}|T_1(u,v)(t)| + \max_{t\in[0,1]}|T_1^{(m_1)}(u,v)(t)| \leqslant \frac{1}{2}\|(u,v)\|_{X\times Y},$$

and

$$\|T_2(u,v)\|_Y = \max_{t\in[0,1]}|T_2(u,v)(t)| + \max_{t\in[0,1]}|T_2^{(m_2)}(u,v)(t)| \leqslant \frac{1}{2}\|(u,v)\|_{X\times Y}.$$

So, we have

$$\|T(u,v)\|_{X\times Y} = \|T_1(u,v)\|_X + \|T_2(u,v)\|_Y \leqslant \|(u,v)\|_{X\times Y},\quad \forall (u,v)\in\partial\Omega_1\cap K. \tag{3.2}$$

On the other hand, it follows from the condition $f_{1\infty}, f_{2\infty} \in (R,+\infty]$ that there exist $l > \mu_1 > 0$ and a sufficiently small $\varepsilon_2 > 0$, such that

$$f_i(t,u_0,v_0,u_1,v_1) \geqslant (f_{i\infty}-\varepsilon_2)(u_0+v_0+u_1+v_1),\ \forall t\in[0,1], (u_0+v_0+u_1+v_1)\geqslant l, \tag{3.3}$$

where $f_{i0}-\varepsilon_2 \geqslant R, i=1,2$.

Let $\Omega_2 = \{(u,v)\in K : \|(u,v)\|_{X\times Y} < \mu_2\}$, where $\mu_2 = \max\{2\mu_1, l/\gamma\}$. For all $(u,v)\in\partial\Omega_2\cap K$, using Lemma 2.9, we get

$$\min_{t\in[\xi,\xi']}(u,v) = \min_{t\in[\xi,\xi']}(u(t)+u^{(m_1)}(t)) + \min_{t\in[\xi,\xi']}(v(t)+v^{(m_2)}(t)) \geqslant \gamma\|(u,v)\|_{X\times Y} = \gamma\mu_2 \geqslant l.$$

From Lemma 2.7 and (3.3), we can obtain

$$\begin{aligned}
\min_{t\in[\xi_i,\xi_i']} T_i^{(j)}(u,v)(t) &\geqslant \Upsilon_{ij}\int_0^1 f_i(s,u_0,v_0,u_1,v_1)ds\\
&\geqslant \Upsilon_{ij}(f_{i\infty}-\varepsilon_2)\int_\xi^{\xi'}(u(s)+v(s)+u^{(m_1)}(s)+v^{(m_2)}(s))ds\\
&\geqslant \Upsilon_{ij}(f_{i\infty}-\varepsilon_2)(\xi'-\xi)\gamma\|(u,v)\|_{X\times Y}\\
&\geqslant \frac{1}{4}\|(u,v)\|_{X\times Y},\quad i=1,2,\quad j=0,m_1,m_2.
\end{aligned}$$

Thus

$$\|T_1(u,v)\|_X \geqslant \min_{t\in[\xi_1,\xi_1']}|T_1(u,v)(t)| + \min_{t\in[\xi_1,\xi_1']}|T_1^{(m_1)}(u,v)(t)| \geqslant \frac{1}{2}\|(u,v)\|_{X\times Y},$$

and

$$\|T_2(u,v)\|_Y \geqslant \min_{t\in[\xi_2,\xi_2']} |T_2(u,v)(t)| + \min_{t\in[\xi_2,\xi_2']} |T_2^{(m_2)}(u,v)(t)| \geqslant \frac{1}{2}\|(u,v)\|_{X\times Y}.$$

So, we have

$$\|T(u,v)\|_{X\times Y} = \|T_1(u,v)\|_X + \|T_2(u,v)\|_Y \geqslant \|(u,v)\|_{X\times Y}, \quad \forall (u,v) \in \partial\Omega_2 \cap K. \tag{3.4}$$

Thus, from (3.2), (3.4), Lemma 2.10 and Lemma 2.11, the operator T has at least a fixed point (u,v) in $K\cap(\overline{\Omega}_2 \setminus \Omega_1)$. This means that FDEs (1.1) has at least one positive solution (u,v) satisfying $u(t) > 0, v(t) > 0$. □

Theorem 3.2. *If $f^{1\infty}, f^{2\infty} \in [0,r)$ and $f_{10}, f_{20} \in (R,+\infty]$, then there exists at least one positive solution for FDEs (1.1) in K.*

Proof. The proof of Theorem 3.2 is similar to that of Theorem 3.1, so we omit it. □

Theorem 3.3. *If $f_{10}, f_{2\infty} \in (2R,+\infty]$ and $f_i(t,u,v,u^{(m_1)},v^{(m_2)}) \in (0,mr)$, $i = 1,2$ for all $t \in [0,1]$ and $(u,v) \in \partial\Omega_3 \cap K$, where $\Omega_3 = \{(u,v) \in K : \|(u,v)\|_{X\times Y} < m\}$, then there exist at least two positive solutions for FDEs (1.1) in K.*

Proof. At first, it follows from the condition $f_{10} \in (2R,+\infty]$ that there exist $0 < m_1 < m$ and a sufficiently small $\varepsilon_3 > 0$, such that

$$f_1(t,u_0,v_0,u_1,v_1) \geqslant (f_{10}-\varepsilon_3)(u_0+v_0+u_1+v_1), \quad \forall t \in [0,1], \quad (u_0+v_0+u_1+v_1) \leqslant m_1, \tag{3.5}$$

where $f_{10}-\varepsilon_3 \geqslant 2R$.

Let $\Omega_4 = \{(u,v) \in K : \|(u,v)\|_{X\times Y} < m_1\}$. For all $(u,v) \in \partial\Omega_4 \cap K$, using (3.5) and Lemma 2.9, we get

$$\begin{aligned}\|T(u,v)\|_{X\times Y} &\geqslant \min_{t\in[\xi,\xi']} T(u,v)(t) \geqslant \min_{t\in[\xi,\xi']} T_1(u,v)(t) + \min_{t\in[\xi,\xi']} T_1^{(m_1)}(u,v)(t)\\ &\geqslant (f_{10}-\varepsilon_3)(\Upsilon_{10}+\Upsilon_{1m_1})\int_{\xi}^{\xi'} (u(s)+v(s)+u^{(m_1)}(s)+v^{(m_2)}(s))ds\\ &\geqslant (f_{10}-\varepsilon_3)(\Upsilon_{10}+\Upsilon_{1m_1})(\xi'-\xi)\gamma\|(u,v)\|_{X\times Y}\\ &\geqslant \|(u,v)\|_{X\times Y}.\end{aligned} \tag{3.6}$$

On the other hand, it follows from the condition $f_{2\infty} \in (2R,+\infty]$ that there exist $m_2 > m > 0$ and a sufficiently small $\varepsilon_4 > 0$, such that

$$f_2(t,u_0,v_0,u_1,v_1) \geqslant (f_{2\infty}-\varepsilon_4)(u_0+v_0+u_1+v_1), \quad \forall t \in [0,1], \quad u_0+v_0+u_1+v_1 \geqslant m_2, \tag{3.7}$$

where $f_{2\infty}-\varepsilon_4 \geqslant 2R$.

Let $\Omega_5 = \{(u,v) \in K : \|(u,v)\| < m_3\}$, where $m_3 > m_2$. For all $(u,v) \in \partial\Omega_5 \cap K$, applying (3.7) and Lemma 2.9, we have

$$\begin{aligned}\|T(u,v)\|_{X\times Y} &\geqslant \min_{t\in[\xi,\xi']} T(u,v)(t) \geqslant \min_{t\in[\xi,\xi']} T_2(u,v)(t) + \min_{t\in[\xi,\xi']} T_2^{(m_2)}(u,v)(t)\\ &\geqslant (f_{2\infty}-\varepsilon_4)(\Upsilon_{20}+\Upsilon_{2m_2})\int_{\xi}^{\xi'} (u(s)+v(s)+u^{(m_1)}(s)+v^{(m_2)}(s))ds\\ &\geqslant (f_{2\infty}-\varepsilon_4)(\xi'-\xi)\gamma\|(u,v)\|_{X\times Y}\\ &\geqslant |(u,v)\|_{X\times Y}.\end{aligned} \tag{3.8}$$

Further, from the condition of Theorem 3.3, for all $(u,v) \in \partial\Omega_3 \cap K$, we know

$$\begin{aligned}\|T(u,v)\|_{X\times Y} &= \|T_1(u,v)\|_X + \|T_2(u,v)\|_Y\\ &\leqslant rm(\Lambda_{10}+\Lambda_{1m_1}+\Lambda_{20}+\Lambda_{2m_1})\\ &\leqslant m = \|(u,v)\|_{X\times Y}.\end{aligned} \tag{3.9}$$

Thus, from (3.6), (3.8) and (3.9), Lemma 2.10 and Lemma 2.11, the operator T has at least a fixed point (u_1, v_1) in $K \cap (\overline{\Omega}_3 \setminus \Omega_4)$ and at least a fixed point (u_2, v_2) in $K \cap (\overline{\Omega}_5 \setminus \Omega_3)$. This means that FDEs (1.1) has at least two positive solutions satisfying $m_1 \leqslant \|(u_1, v_1)\|_{X \times Y} < m < \|(u_2, v_2)\|_{X \times Y} \leqslant m_3$. □

Similar to that of Theorem 3.3, we can obtain the following results.

Theorem 3.4. *If $f_{20}, f_{1\infty} \in (2R, +\infty]$ and $f_i(t, u, v, u^{(m_1)}, v^{(m_2)}) \in (0, mr)$, $i = 1, 2$ for all $t \in [0, 1]$ and $(u, v) \in \partial\Omega_3 \cap K$, where $\Omega_3 = \{(u, v) \in K : \|(u, v)\|_{X \times Y} < m\}$, then there exist at least two positive solutions for FDEs (1.1) in K.*

Theorem 3.5. *If $f_{10}, f_{1\infty} \in (2R, +\infty]$ and $f_i(t, u, v, u^{(m_1)}, v^{(m_2)}) \in (0, mr)$, $i = 1, 2$ for all $t \in [0, 1]$ and $(u, v) \in \partial\Omega_3 \cap K$, where $\Omega_3 = \{(u, v) \in K : \|(u, v)\|_{X \times Y} < m\}$, then there exist at least two positive solutions for FDEs (1.1) in K.*

Theorem 3.6. *If $f_{20}, f_{2\infty} \in (2R, +\infty]$ and $f_i(t, u, v, u^{(m_1)}, v^{(m_2)}) \in (0, mr)$, $i = 1, 2$ for all $t \in [0, 1]$ and $(u, v) \in \partial\Omega_3 \cap K$, where $\Omega_3 = \{(u, v) \in K : \|(u, v)\|_{X \times Y} < m\}$, then there exist at least two positive solutions for FDEs (1.1) in K.*

4. Some examples

In order to illustrate our results, we consider the following two examples.

Example 4.1. Consider the following coupled system of nonlinear FDEs with fractional integral conditions

$$\begin{cases} {}^{C}D^{\frac{7}{2}}u(t) = \frac{(1+t)}{2}[(u+v+u''+v')^2 + \mu\sin(u+v+u''+v')], \quad t \in (0,1), \\ {}^{C}D^{\frac{11}{3}}v(t) = \frac{1}{4}(u+v+u''+v')^3, \quad t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0, \quad u'''(0) = 2I^{\frac{3}{2}}u(1), \\ v(0) = v'(0) = v''(0) = 0, \quad v'''(0) = \frac{1}{3}I^{\frac{5}{3}}v'(1), \end{cases} \tag{4.1}$$

where

$$\alpha_1 = \frac{7}{2}, \quad \alpha_2 = \frac{11}{3}, \quad n = 4, \quad m_1 = 2, \quad m_2 = 1, \quad \rho_1 = 2, \quad \rho_2 = \frac{1}{3}, \quad \beta_1 = \frac{3}{2}, \quad \beta_2 = \frac{5}{3},$$

$$f_1(t, u_0, v_0, u_1, v_1) = \frac{(1+t)}{2}\big[(u_0 + v_0 + u_1 + v_1)^2 + \mu\sin(u_0 + v_0 + u_1 + v_1)\big],$$

$$f_2(t, u_0, v_0, u_1, v_1) = \frac{1}{4}(u_0 + v_0 + u_1 + v_1)^3.$$

It is obvious that $\Gamma(n + \beta_i) > \rho_i$, $i = 1, 2$. By direct calculation, we can obtain that $\Delta_1 = 49.9069$, $\Delta_2 = 1.8218$, $r = 0.0049$ and

$$f^{10} = \mu, \quad f^{20} = 0, \quad f^{1\infty} = +\infty, \quad f^{2\infty} = +\infty.$$

Let $\mu \in [0, 0.0049)$, Theorem 3.1 implies that FDEs (4.1) has at least one positive solution in K.

Example 4.2. Consider the following coupled system of nonlinear FDEs with fractional integral conditions

$$\begin{cases} {}^{C}D^{\frac{7}{2}}u(t) = \frac{1+t}{1000}(u+v+u''+v')^{\frac{1}{3}}, \quad t \in (0,1), \\ {}^{C}D^{\frac{11}{3}}v(t) = \frac{1}{400}(u+v+u''+v')^3, \quad t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0, \quad u'''(0) = 2I^{\frac{3}{2}}u(1), \\ v(0) = v'(0) = v''(0) = 0, \quad v'''(0) = \frac{1}{3}I^{\frac{5}{3}}v'(1), \end{cases} \tag{4.2}$$

where

$$\alpha_1 = \frac{7}{2}, \quad \alpha_2 = \frac{11}{3}, \quad n = 4, \quad m_1 = 2, \quad m_2 = 1, \quad \rho_1 = 2, \quad \rho_2 = \frac{1}{3}, \quad \beta_1 = \frac{3}{2}, \quad \beta_2 = \frac{5}{3},$$

$$f_1(t,u_0,v_0,u_1,v_1) = \frac{1+t}{1000}(u_0+v_0+u_1+v_1)^{\frac{1}{3}},$$
$$f_2(t,u_0,v_0,u_1,v_1) = \frac{1}{400}(u_0+v_0+u_1+v_1)^{3}.$$

Similar to that of Example 4.1, we can obtain that $r = 0.0049$ and $f_{10} = +\infty$, $f_{2\infty} = +\infty$. Let $m = 1$, by direct calculation, we can obtain that $f_1 < 0.002 < mr$, $f_2 < 0.0025 < mr$. Thus, Theorem 3.3 implies that FDEs (4.2) has at least two positive solutions in K.

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References

[1] B. Ahmad, J. J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl., **58** (2009), 1838–1843. 1
[2] B. Ahmad, S. K. Ntouyas, A. Alsaedi, *On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions*, Chaos Solitons Fractals, **83** (2016), 234–241. 1
[3] A. K. Alomari, F. Awawdeh, N. Tahat, F. Bani Ahmad, W. Shatanawi, *Multiple solutions for fractional differential equations: Analytic approach*, Appl. Math. Comput., **219** (2013), 8893–8903. 1
[4] P. Arena, R. Caponetto, L. Fortuna, D. Porto, *Chaos in a fractional order Duffing system. In: Proceedings of the 1997 European conference on circuit theory and design*, Technical University of Budapest, Budapest, (1997). 1
[5] C.-Z. Bai, J.-X. Fang, *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations*, Appl. Math. Comput., **150** (2004), 611–621. 1
[6] Y. Chen, D. Chen, Z. Lv, *The existence results for a coupled system of nonlinear fractional differential equations with multi-point boundary conditions*, Bull. Iranian Math. Soc., **38** (2012), 607–624. 1
[7] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, (1985). 2.11
[8] K. Diethelm, N. J. Ford, A. D. Freed, *A predictor-corrector approach for the numerical solution of fractional differential equations*, Nonlinear Dynam., **29** (2002), 3–22. 1
[9] M. Faieghi, S. Kuntanapreeda, H. Delavari, D. Baleanu, *LMI-based stabi-lization of a class of fractional-order chaotic systems*, Nonlinear Dynam., **72** (2013), 301–309. 1
[10] Z.-M. Ge, W.-R. Jhuang, *Chaos, control and synchronization of a fractional order rotational mechanical system with a centrifugal governor*, Chaos Solitons. Fractals., **33** (2007), 270–289. 1
[11] I. Grigorenko, E. Grigorenko, *Chaotic dynamics of the fractional Lorenz system*, Phys. Rev. Lett., **91** (2003), 1–4. 1
[12] D. J. Guo, V. Lakshmikanthan, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, (1988). 2.11
[13] Z. Jiao, Y.-Q. Chen, I. Podlubny, *Distributed-order dynamic systems*, Springer, London, (2012). 1
[14] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, (2006). 2
[15] Y. Li, Z. Wei, *Positive solutions for a coupled systems of mixed higher-order nonlinear singular differential equations*, Fixed Point Theory, **15** (2014), 167–178. 1
[16] K. Ma, Z. Han, Y. Zhang, *Stability conditions of a coupled system of fractional q-difference Lotka-Volterra model*, Int. J. Dyn. Syst. Differ. Equ., **6** (2016), 305–317. 1
[17] R. Metzler, J. Klafter, *The random walk's guide to anomalous diffusion:a fractional dynamics approach*, Phys. Rep., **2000** (2000), 77 pages. 1
[18] S. K. Ntouyas, M. Obaid, *A coupled system of fractional differential equations with nonlocal integral boundary conditions*, Adv. Difference Equ., **2012** (2012), 8 pages. 1
[19] A. Pedas, E. Tamme, *Numerical solution of nonlinear fractional differential equations by spline collocation methods*, J. Comput. Appl. Math., **255** (2014), 216–230. 1
[20] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, (1999). 2
[21] I. M. Sokolov, J. Klafter, A. Blumen, *Fractional kinetics*, Phys. Today, **55** (2002), 48–54. 1
[22] X.-W. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Lett., **22** (2009), 64–69. 1
[23] C. Yuan, *Multiple positive solutions for (n-1,1) type semipositone conjugate boundary value problems for coupled systems of nonlinear fractional differential equations*, Electron. J. Qual. Theory Differ. Equ., **2011** (2011), 12 pages.

[24] Y. Zhang, Z. Bai, T. Feng, *Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance*, Comput. Math. Appl., **61** (2011), 1032–1047. 1

[25] H. Zhang, W. Gao, *Existence of Solutions to a Coupled System of Higher-order Nonlinear Fractional Differential Equations with Anti-periodic Boundary Conditions*, J. Comput. Anal. Appl., **22** (2017), 262–270. 1

[26] X. Zhang, C. Zhu, Z. Wu, *Solvability for a coupled system of fractional differential equations with impulses at resonance*, Bound. Value Probl., **2013** (2013), 23 pages. 1

[27] Y. Zhao, S. Sun, Z. Han, W. Feng, *Positive solutions for for a coupled system of nonlinear differential equations of mixed fractional orders*, Adv. Difference Equ., **2011** (2011), 13 pages. 1

[28] X. Zhou, C. Xu, *Numerical Solution of the Coupled System of Nonlinear Fractional Ordinary Differential Equations*, Adv. Appl. Math. Mech., **9** (2017), 574–595. 1

[29] C. Zhu, X. Zhang, Z. Wu, *Solvability for a coupled system of fractional differential equations with integral boundary conditions*, Taiwanese J. Math., **17** (2013), 2039–2054. 1