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Viscosity approximation of solutions of a split feasibility problem in Hilbert spaces

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Abstract

In this paper, we study two viscosity approximation iterative methods for solving solutions of a split feasibility problem. Strong convergence theorems are established in the framework of infinite dimensional Hilbert spaces. ©2017 All rights reserved.

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1. Introduction and Preliminaries

Let C and D be two nonempty, closed, and convex subsets of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Recall that a mapping $f : D \to D$ is said to be contractive if and only if there exists a real constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leqslant \alpha \|x - y\|, \quad \forall x, y \in D.$$

$f : D \to D$ is said to be a Meir-Keeler contraction if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x - y\| \leqslant \epsilon + \delta \quad \text{implies} \quad \|f(x) - f(y)\| \leqslant \epsilon$$

for all $x, y \in C$. It is known that every Meir-Keeler contraction is a generalization of contractions and also has a unique fixed point; see [12] and the references therein.

$f : D \to D$ is said to be nonexpansive if and only if

$$\|f(x) - f(y)\| \leqslant \|x - y\|, \quad \forall x, y \in D.$$

For every point $x \in H$, there exists a unique nearest point in D denoted by $P_D x$ such that

$$\| x - P_D x \| \leqslant \| x - y \|, \quad \forall y \in D.$$

P_D is called the metric projection of H onto D. It is well-known that P_D is nonexpansive mapping and

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satisfies
$$\langle x-y, P_D x - P_D y\rangle \geqslant \| P_D x - P_D y \|^2, \quad \forall x,y \in H.$$
Moreover, $P_D x$ is characterized by the fact $P_D x \in D$ and
$$\langle x - P_D x, y - P_D x\rangle \leqslant 0,$$
and
$$\| x-y \|^2 \geqslant \| x - P_D x \|^2 + \| y - P_D x \|^2, \quad \forall x \in H, y \in D.$$
In a real Hilbert space the following holds
$$\| \lambda x + (1-\lambda) y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x-y \|^2$$
for all $x,y \in H$ and $\lambda \in (0,1)$. It is well-known that every nonexpansive operator $f : H \to H$ satisfies for all $x,y \in H \times H$, the inequality
$$\langle (x-f(x)) - (y-f(y)), f(y)-f(x)\rangle \leqslant \frac{1}{2} \| (f(x)-x) - (f(y)-y) \|^2,$$
and therefore, we get for all $(x,y) \in H \times \mathrm{Fix}(f)$,
$$\langle x - f(x), y - f(y)\rangle \leqslant \frac{1}{2} \| f(x) - x \|^2 .$$
A mapping $f : H \to H$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e., $f := (1-\alpha)I + \alpha g$ where $\alpha \in (0,1)$ and $g : H \to H$ is nonexpansive and I is the identity operator on H. We note that averaged mappings are nonexpansive. Further, firmly nonexpansive mappings (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged. If $f = (1-\alpha)g + \alpha g'$, where $g : H \to H$ is averaged, $g' : H \to H$ is nonexpansive and $\alpha \in (0,1)$, then f is averaged. The composite of finitely many averaged mappings is still averaged.

If D is bounded closed and convex, then the set of fixed points of nonexpansive mapping f is not empty. The theory of nonexpansive mappings has been investigated for solving various convex optimization problems; see [8, 6, 15, 14] and the references therein. Halpern iterative algorithm is an efficient tool to study fixed points of nonexpansvie mappings in infinite dimensional Hilbert spaces. Halpern iterative algorithm generated a sequence in the following manner
$$x_1 \in D, x_{n+1} = \alpha_n u + (1-\alpha_n) f(x_n), \quad \forall n \geqslant 1,$$
where u is a fixed element in D and f is a nonexpansive mapping. It is known that $\{x_n\}$ converges to a special fixed point of f with some restrictions imposed on $\{\alpha_n\}$. For more convergence results in the framework of Hilbert spaces, one is referred to [1, 9, 10, 19, 20] and the references there. Moudafi viscosity iterative algorithm has recently extensively investigated for solving fixed points of the class of nonexpansive mappings; see [13] and the references therein. He proved that the special fixed point is also a solution to some monotone variational inequality; see, also [7, 16, 18] and the references therein. Recently, Suzuki [17] further improved the viscosity approximation method with the Meir-Keeler contraction.

A mapping $F : D \to H$ is said to be:

(i) monotone, if
$$\langle Fx - Fy, x-y\rangle \geqslant 0$$
for all $x,y \in D$;

(ii) ν-inverse strongly monotone, if
$$\langle Fx - Fy, x-y\rangle \geqslant \nu \|Fx - Fy\|^2$$
for all $x,y \in D$;

(iii) L-Lipschitzian, if

$$\|Fx - Fy\| \leqslant L\|x - y\|$$

for all $x, y \in D$, in particular, F is called nonexpansive when $L = 1$. It is known that if F is ν-inverse strongly monotone, then it is $\frac{1}{\nu}$-Lipschitzian and monotone.

Split feasibility problem was first introduced by Censor and Elfving [4] in 1994. Censor and Elfving first studied the split feasibility problem in a finite-dimensional Hilbert space for modeling inverse problems that arise from phase retrievals and in medical image reconstruction. Many image reconstruction problems can be formulated as the split feasibility problem; see, for example, [2] and the references therein. Recently, it is found that the SFP could also be applied to study the intensity-modulated radiation therapy; see, for example, [3, 5] and the references therein. Byrne [2] recently developed the split feasibility problem in the setting of infinite-dimensional Hilbert spaces.

Let C and Q be nonempty, closed, and convex subsets in Hilbert spaces H_1 and H_2, respectively. Then the split feasibility problem is formulated as finding a point $x \in C$ with the property:

$$x \in C, \quad Ax \in Q, \tag{1.1}$$

where $A : C \subset H_1 \to H_2$ is a bounded linear operator. We denote by Γ the solution set of the split feasibility problem, that is,

$$\Gamma = \{x \in H_1 : x \in C, \quad Ax \in Q\} = C \cap A^{-1}(Q).$$

It is clear that $A^{-1}(Q)$ is a closed convex subset of H_1, and hence Γ is also a closed convex subset of H_1. Let P_C and P_Q be metric projections onto sets C and Q, respectively. It is well-known that if $\Gamma \neq \emptyset$, then solving the SFP is equivalent to solving a fixed point equation

$$x = P_C(x - \gamma A^*(I - P_Q)Ax),$$

where A^* is the adjoint operator of A and $\gamma > 0$ is a parameter. If we define a mapping U_γ by

$$U_\gamma x = x - \gamma A^*(I - P_Q)Ax,$$

then we have $x = P_C U_\gamma x$. Assume that problem (1.1) is consistent, i.e., it has a solution, it is easy to see that $\mathrm{Fix}(U_\gamma) = A^{-1}(Q)$ and hence $\Gamma = C \cap \mathrm{Fix}(U_\gamma) = \mathrm{Fix}(P_C U_\gamma)$ for sufficiently small $\gamma > 0$. It is well-known that if $\gamma \in (0, 2/\|A\|^2)$, then U_γ is averaged and hence $P_C U_\gamma$ is also averaged. We observe that the averaged nonexpansiveness of U_γ heavily depends on the choice of γ, that is, $\gamma \in (0, 2/\|A\|^2)$ is required, and hence the choice of γ is closely related to the norm $\|A\|$ of operator A.

The following lemmas are essential to prove our main results.

Lemma 1.1 ([17]). *Let g be a Meir-Keeler on a convex subset C of a Banach space E. Then for each $\epsilon > 0$, there exists $\kappa \in (0, 1)$ such that $\|x - y\| \geqslant \epsilon$ implies $\|g(x) - g(y)\| \leqslant \kappa\|x - y\|$, for all $x, y \in C$.*

Lemma 1.2 ([11]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leqslant \alpha_n r_n + (1 - \alpha_n)a_n, \quad n \geqslant 1,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset \mathbb{R}$ satisfy

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} r_n < \infty$.

Then

$$\limsup_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} r_n.$$

The following two lemmas are not hard to derive.

Lemma 1.3. *Let* $P_C : H \to C$ *be the metric projection from* H *on a nonempty, closed, and convex subset* C. *Then the following conclusions hold true:*

(a) *Given* $x \in H$ *and* $z \in C$. *Then* $z = P_C x$ *if and only if there holds the inequality*

$$\langle x - z, y - z\rangle \leqslant 0, \ \forall y \in C.$$

(b) $\langle P_C x - P_C y, x - y\rangle \geqslant \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$

(c) $\langle (I - P_C)x - (I - P_C)y, x - y\rangle \geqslant \|(I - P_C)x - (I - P_C)y\|^2, \quad \forall x, y \in H.$

(d) $P_C = \frac{1}{2}I + \frac{1}{2}S$ *with* S *nonexpansive.*

(e) $\|P_C x - P_C y\|^2 \leqslant \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2$, $\forall x,\ y \in H$. *In particular, we have:*

(f) $\|P_C x - y\|^2 \leqslant \|x - y\|^2 - \|(I - P_C)x\|^2$, $\forall x \in H,\ y \in C$.

Lemma 1.4. *Let* H *be a real Hilbert space. Then the following equality holds*

$$\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 = \|x + y\|^2, \quad \forall x, y \in H.$$

2. Main results

Theorem 2.1. *Let* C *and* D *be two nonempty, closed, and convex subsets of a real Hilbert space* H *such that* $C \subset D$. *Let* $F : D \to H$ *be a* ν*-inverse strongly monotone operator such that* $C \cap F^{-1}(0) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $(0, 1]$ *that satisfy the following conditions:*

(i) $\alpha_n \to 0,\ \beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\alpha_n = o(\beta_n)$.

Let $f : C \to C$ *be an* α*-contractive mapping. Let* $\{x_n\}$ *be a sequence generated by the following iterative process*

$$x_1 \in D, \ \ x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)(x_n - \beta_n F x_n)], \ \ n \geqslant 1. \tag{2.1}$$

Then $\{x_n\}$ *converges in norm to* $x^* \in C \cap F^{-1}(0)$, *where* x^* *uniquely solves the following variational inequality*

$$\langle f(x^*) - x^*, x^* - x\rangle \geqslant 0, \quad \forall x \in C \cap F^{-1}(0).$$

Proof. Since F is continuous, we see that $F^{-1}(0)$ is closed. Next we show that $F^{-1}(0)$ is convex. Indeed, for any $x_1,\ x_2 \in F^{-1}(0)$, write $x_t = tx_1 + (1 - t)x_2$ for $t \in (0, 1)$. Then we have $x_t \in D$ and

$$\langle F x_t, x_t - x_1\rangle \geqslant \nu\|F x_t\|^2, \tag{2.2}$$

and

$$\langle F x_t, x_t - x_2\rangle \geqslant \nu\|F x_t\|^2. \tag{2.3}$$

Multiplying t and $(1 - t)$ on the both sides of (2.2) and (2.3), respectively, and adding up yields

$$0 = \langle F x_t, x_t - x_t\rangle \geqslant \nu\|F x_t\|^2,$$

which means that $F x_t = 0$ and $F^{-1}(0)$ is convex. Therefore $C \cap F^{-1}(0)$ is close and convex. So the metric projection onto $C \cap F^{-1}(0)$ is well-defined. Since $\mathrm{Proj}_{C\cap F^{-1}(0)} f$ is α-contractive, we see that $P_{C\cap F^{-1}(0)} f$ has a unique fixed point. Next, we use x^* to denote the unique fixed point, that is, $x^* = P_{C\cap F^{-1}(0)} f(x^*)$.

Next we show that $\{x_n\}$ is bounded. Write $v_n = x_n - \beta_n F x_n$. For all $z \in F^{-1}(0)$, by using Lemma 1.3, we have

$$\begin{aligned} \|v_n - z\|^2 &= \|x_n - z - \beta_n(Fx_n - Fz)\|^2 \\ &= \|x_n - z\|^2 - 2\beta_n\langle x_n - z, Fx_n - Fz\rangle + \beta_n^2\|Fx_n - Fz\|^2 \\ &\leqslant \|x_n - z\|^2 - 2\beta_n\nu\|Fx_n\|^2 + \beta_n^2\|Fx_n\|^2 \\ &= \|x_n - z\|^2 - \beta_n(2\nu - \beta_n)\|Fx_n\|^2 \end{aligned} \tag{2.4}$$

for all $n \geqslant 1$. Since $\beta_n \to 0$, without loss of generality, we can assume that $\beta_n \leqslant 2\nu$. It follows from (2.4) that

$$\|v_n - z\| \leqslant \|x_n - z\|$$

for all $z \in F^{-1}(0)$ and all $n \geqslant 1$. It follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leqslant \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(v_n - x^*)\| \\ &\leqslant \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|v_n - x^*\| \\ &\leqslant (1 - \alpha_n(1-\alpha))\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\ &= (1 - \alpha_n(1-\alpha))\|x_n - x^*\| + \alpha_n(1-\alpha)\frac{\|f(x^*) - x^*\|}{1-\alpha} \end{aligned}$$

for all $n \geqslant 1$. This implies that

$$\|x_{n+1} - x^*\| \leqslant \max\left\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\alpha}\right\}.$$

This shows that $\{x_n\}$ is bounded. Using Lemma 1.4, we find from (2.4) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|P_C[\alpha_n f(x_n) + (1-\alpha_n)v_n] - P_C x^*\|^2 \\ &\leqslant \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(v_n - x^*)\|^2 \\ &= (1-\alpha_n)^2\|v_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x_n) - x^*, v_n - x^*\rangle + \alpha_n^2\|f(x_n) - x^*\|^2 \\ &\leqslant (1-\alpha_n)^2\|x_n - x^*\|^2 - \beta_n(2\nu - \beta_n)(1-\alpha_n)^2\|Fx_n\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - f(x^*), v_n - x^*\rangle \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, v_n - x^*\rangle + \alpha_n^2\|f(x_n) - x^*\|^2 \\ &\leqslant (1-\alpha_n)\|x_n - x^*\|^2 - \beta_n(2\nu - \beta_n)(1-\alpha_n)^2\|Fx_n\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\alpha\|x_n - x^*\|\|v_n - x^*\| \\ &\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, v_n - x^*\rangle + \alpha_n^2\|f(x_n) - x^*\|^2 \end{aligned}$$

for all $n \geqslant 1$. Setting $a_n = \|x_n - x^*\|^2$ and

$$\begin{aligned} r_n = &\frac{\beta_n}{\alpha_n}(1-\alpha_n)^2(2\nu - \beta_n)\|Fx_n\|^2 - 2(1-\alpha_n)\alpha\|x_n - x^*\|\|v_n - x^*\| \\ &- 2(1-\alpha_n)\langle f(x^*) - x^*, v_n - x^*\rangle - \alpha_n\|f(x_n) - x^*\|^2, \end{aligned}$$

we arrive at

$$a_{n+1} \leqslant (1-\alpha_n)a_n + \alpha_n(-r_n)$$

for all $n \geqslant 1$. Noting that $\{r_n\}$ is bounded below, we see that $\{-r_n\}$ is bounded above. By using Lemma 1.2, we conclude that

$$\limsup_{n\to\infty} a_n \leqslant \limsup_{n\to\infty}(-r_n) = -\liminf_{n\to\infty} r_n < \infty. \tag{2.5}$$

Assume that $\liminf_{n\to\infty} r_n = \lim_{k\to\infty} r_{n_k}$, then $\{r_{n_k}\}$ is a bounded subsequence of $\{r_n\}$. This implies that there exists a positive constant ζ such that

$$\frac{\beta_{n_k}}{\alpha_{n_k}}(1-\alpha_{n_k})^2(2\nu-\beta_{n_k})\|Fx_{n_k}\|^2 \leqslant \zeta \tag{2.6}$$

for all $k \geqslant 1$. It follows from (2.6) that

$$\|Fx_{n_k}\|^2 \leqslant \zeta\frac{\alpha_{n_k}}{\beta_{n_k}}\frac{1}{(1-\alpha_{n_k})^2(2\nu-\beta_{n_k})}$$

for all $k \geqslant 1$, which derives that $Fx_{n_k} \to 0$ as $k \to \infty$, in view of conditions (i) and (iii) on $\{\alpha_n\}$ and $\{\beta_n\}$. Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$, then $\bar{x} \in C$, since $\{x_n\} \subset C$ and C is weakly closed. Setting $E = I - F$, we see that E is nonexpansive. From the demiclosed principal of nonexpansive mapping, we find that $\bar{x} = E\bar{x}$. It follows that we have also $F\bar{x} = 0$. Thus we have $\bar{x} \in C \cap F^{-1}(0)$. It follows that

$$\langle f(x^*) - x^*, \bar{x} - x^*\rangle \leqslant 0. \tag{2.7}$$

Since $\nu_n - x_n = -\beta_n F x_n$, $\beta_n \to 0$ and $\{Fx_n\}$ is bounded, we see that $\nu_n - x_n \to 0$ as $n \to \infty$, and thus $\nu_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Consequently, from the definition of $\{r_n\}$ and (2.7), we have

$$\liminf_{n\to\infty} r_n = \lim_{k\to\infty} r_{n_k} \geqslant -2\langle f(x^*) - x^*, \bar{x} - x^*\rangle \geqslant 0. \tag{2.8}$$

Combining (2.5) and (2.8), we derive that $a_n \to 0$ as $n \to \infty$. This completes the proof. □

Remark 2.2. Choose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\alpha_n = \frac{1}{n^a}$ anc $\beta_n = \frac{1}{n^b}$, where $0 < b < a \leqslant 1$. Then it is clear that conditions (i)-(iii) in Theorem 2.1 are satisfied.

Corollary 2.3. *Let* C *and* D *be two nonempty, closed, and convex subsets of a real Hilbert space* H *such that* $C \subset D$. *Let* $F : D \to H$ *be a* ν*-inverse strongly monotone operator such that* $C \cap F^{-1}(0) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $(0, 1]$ *that satisfy the following conditions:*

(i) $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\alpha_n = o(\beta_n)$.

Let $f : C \to C$ *be an* α*-contractive mapping. Let* $\{x_n\}$ *be a sequence generated by* (2.1)*, where* u *is a fixed element in* D. *Then* $\{x_n\}$ *converges in norm to* $x^* = P_{C\cap F^{-1}(0)}u$.

Next, we give a viscosity convergence theorem with a contraction.

Theorem 2.4. *Let* C *and* Q *be two nonempty, closed, and convex subsets of real Hilbert spaces* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Suppose that the split feasibility problem* (1.1) *is consistent, i.e.,* $\Gamma \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $(0, 1]$ *that satisfy the following conditions:*

(i) $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\alpha_n = o(\beta_n)$.

Let $f : H_1 \to H_1$ *be contractive mapping. Let* $\{x_n\}$ *be a sequence generated by the following iterative process*

$$x_1 \in H_1, \quad x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)(x_n - \beta_n A^*(I - P_Q)Ax_n)], \quad n \geqslant 1. \tag{2.9}$$

Then $\{x_n\}$ *converges in norm to* $x^* = P_\Gamma f(x^*)$*, that is,* x^* *uniquely solves the following variational inequality*

$$\langle f(x^*) - x^*, x^* - x\rangle \geqslant 0, \quad \forall x \in \Gamma.$$

Proof. Define $F : H_1 \to H_1$ by $Fx = A^*(I - P_Q)Ax$, for all $x \in H_1$. Then (2.9) becomes (2.1). It is sufficient to prove that F is $\frac{1}{\|A\|^2}$-inverse strongly monotone such that $F^{-1}(0) = A^{-1}(Q)$. Indeed, by using Lemma 1.3, we have

$$\begin{aligned}\langle x-y, Fx-Fy\rangle &= \langle x-y, A^*(I-P_Q)Ax - A^*(I-P_Q)Ay\rangle \\ &= \langle (I-P_Q)Ax-(I-P_Q)Ay, Ax-Ay\rangle \\ &\geqslant \|(I-P_Q)Ax-(I-P_Q)Ay\|^2 \\ &\geqslant \frac{1}{\|A\|^2}\|A^*(I-P_Q)Ax-A^*(I-P_Q)Ay\|^2 \\ &= \frac{1}{\|A\|^2}\|Fx-Fy\|^2,\end{aligned} \tag{2.10}$$

which verifies that F is $\frac{1}{\|A\|^2}$-inverse strongly monotone. Assume that $x \in F^{-1}(0)$. We have $Fx = 0$. Since $\Gamma \neq \emptyset$, we can take a point $w \in \Gamma$. This implies that $Aw = P_Q Aw$, and hence $Fw = 0$. In view of (2.10), we have

$$0 = \langle Fx - Fw, x - w\rangle \geqslant \|(I - P_Q)Ax\|^2,$$

which implies that $x \in A^{-1}(Q)$. It is clear that $A^{-1}(Q) \subset F^{-1}(0)$. Then $A^{-1}(Q) = F^{-1}(0)$. This completes the proof. □

Using Theorem 2.4, we have the following result.

Corollary 2.5. *Let C and Q be two nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that the split feasibility problem (1.1) is consistent, i.e., $\Gamma \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1]$ that satisfy the following conditions:*

(i) $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\alpha_n = o(\beta_n)$.

Let $f : H_1 \to H_1$ be contractive mapping. Let a sequence $\{x_n\}$ be generated by the following iterative process

$$x_1 \in H_1,\ \ x_{n+1} = P_C[\alpha_n u + (1-\alpha_n)(x_n - \beta_n A^*(I-P_Q)Ax_n)],\ \ n \geqslant 1,$$

where u is a fixed element in H_1. Then $\{x_n\}$ converges in norm to $x^* = P_\Gamma u$.

Finally, we give another viscosity convergence theorem with a Meir-Keeler contraction.

Theorem 2.6. *Let C and Q be two nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that the split feasibility problem (1.1) is consistent, i.e., $\Gamma \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1]$ that satisfy the following conditions:*

(i) $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\alpha_n = o(\beta_n)$.

Let $g : H_1 \to H_1$ be a Meir-Keeler contraction. Let $\{x_n\}$ be a sequence generated by the following iterative process

$$x_1 \in H_1,\ \ x_{n+1} = P_C[\alpha_n g(x_n) + (1-\alpha_n)(x_n - \beta_n A^*(I-P_Q)Ax_n)],\ \ n \geqslant 1. \tag{2.11}$$

Then $\{x_n\}$ strongly converges to x^*, where $x^* = P_\Gamma g(x^*)$, that is, x^* uniquely solves the following variational inequality

$$\langle f(x^*) - x^*, x^* - x\rangle \geqslant 0, \quad \forall x \in \Gamma.$$

Proof. Define a sequence $\{y_n\}$ by

$$y_{n+1} = P_C[\alpha_n g(x^*) + (1-\alpha_n)(y_n - \beta_n A^*(I-P_Q)Ay_n)].$$

From Corollary 2.5, we see that $\{y_n\}$ strongly converges to $x^* = P_\Gamma g(x^*)$. Next, we prove that $x_n - y_n \to 0$ as $n \to \infty$. Let $\limsup_{n\to\infty} \| x_n - y_n \| = \lambda > 0$. For all $\varepsilon \in (0, \lambda)$, we can choose $\eta > 0$ such that

$$\limsup_{n\to\infty} \| x_n - y_n \| > \varepsilon + \eta.$$

For above $\varepsilon > 0$, we see [17] that there exists $\kappa \in (0,1)$ such that

$$\kappa \| x - y \| \geqslant \| f(x) - f(y) \|$$

for all $x, y \in H_1$ with $\| x - y \| \geqslant \varepsilon$, which implies that

$$\max\{\kappa \| x - y \|, \varepsilon\} \geqslant \| f(x) - f(y) \|$$

for all $x, y \in H_1$. Since $y_n \to x^*$ as $n \to \infty$, we see that there exists some integer $n_0 \geqslant 1$ such that $\eta(1-\beta) \geqslant \| y_n - z \|$, for all $n \geqslant n_0$.

Now, we divide the following two cases:

There exists some $n_1 \geqslant n_0$ such that $\| x_{n_1} - y_{n_1} \| \leqslant \varepsilon + \eta$. It follows that

$$\begin{aligned}
\| x_{n_1+1} - y_{n_1+1} \| &\leqslant \alpha_{n_1} \| g(x_{n_1}) - g(x^*) \| + (1-\alpha_{n_1}) \| Fx_{n_1} - Fy_{n_1} \| \\
&\leqslant \alpha_{n_1} \| g(x_{n_1}) - g(y_{n_1}) \| + \alpha_{n_1} \| g(y_{n_1}) - g(x^*) \| \\
&\quad + (1-\alpha_{n_1}) \| x_{n_1} - y_{n_1} \| \\
&\leqslant \alpha_{n_1} \max\{\kappa \| x_{n_1} - y_{n_1} \|, \varepsilon\} \\
&\quad + \alpha_{n_1} \| g(y_{n_1}) - g(x^*) \| + (1-\alpha_{n_1}) \| x_{n_1} - y_{n_1} \| \\
&\leqslant \varepsilon + \eta.
\end{aligned}$$

Similarly, we can prove that $\| x_{n_1+2} - y_{n_1+2} \| \leqslant \varepsilon + \eta$. By induction, we have $\| x_{n_1+m} - y_{n_1+m} \| \leqslant \varepsilon + \eta$, for all $m \geqslant 1$, which implies that $\limsup_{n\to\infty} \| x_n - y_n \| \leqslant \varepsilon + \eta$. This is a contradiction. Hence $x_n \to x^*$.

Finally, we show that the other case $\| x_{n_1} - y_{n_1} \| > \varepsilon + \eta$, for all $n \geqslant n_1$ is impossible. Note that $\kappa \| x_n - y_n \| \geqslant \| f(x_n) - f(y_n) \|$, for all $n \geqslant n_1$. It follows that

$$\begin{aligned}
\| x_{n_1+1} - y_{n_1+1} \| &\leqslant \alpha_n \| g(x_n) - g(y_n) \| + \alpha_n \| g(y_n) - g(z) \| \\
&\quad + (1-\alpha_n) \| x_n - y_n \| \\
&\leqslant (1-(1-\kappa)\alpha_n) \| x_n - y_n \| + \alpha_n \| y_n - x^* \|,
\end{aligned}$$

which yields to $x_n - y_n \to 0$ as $n \to \infty$. Hence, $\varepsilon + \eta \leqslant 0$, which is a contradiction. This shows that the second case is impossible. The proof is completed. □

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