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Positive solutions to nonlinear fractional differential equations involving Stieltjes integrals conditions

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Abstract

In this paper, we consider the existence of positive solutions for a class of nonlinear fractional semipositone differential equations involving integral boundary conditions. Some existence results of positive solutions are obtained by means of Leray-Schauder's alternative and Krasnoselskii's fixed point theorem. An example is given to demonstrate the application of our main results.

Keywords: Integral boundary conditions, semipositone, fractional differential equation, positive solutions, fixed point theory.
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1. Introduction

This paper is concerned with the existence of positive solutions to the following boundary value problem (BVP) for fractional semipositone differential equation

$$\begin{cases} -D_{0+}^{\alpha}u(t) = \mu f(t, u(t), u'(t), \cdots, u^{(n-2)}(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u^{(n-2)}(1) = \lambda[u^{(n-2)}], \end{cases} \tag{1.1}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $n-1 < \alpha \leqslant n, n \geqslant 2$, $\mu > 0$ is a parameter, $f : (0, 1) \times \mathbb{R}_+^{n-1} \to \mathbb{R}$ is a continuous function and may be singular at $t = 0, 1$. $\lambda[v] = \int_0^1 v(t)dA(t)$ is a linear functional on $C[0, 1]$ given by a Stieltjes integral with A representing a suitable function of bounded variation. It is important to indicate that it is not assumed that $\lambda[v]$ is positive to all positive v. The measure dA can be a signed measure (see Remark 2.8, and Example 4.1). Here $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = [0, +\infty)$.

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (such as blood flow phenomena), economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and

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so on (see [25, 27]. The study of boundary value problems in the setting of fractional calculus has received a great attention in the last decade and a variety of results concerning the existence of solutions, based on various analytic techniques, can be found in the literature [1–8, 11, 12, 14, 15, 18–24, 28–36]. For example, by means of a mixed monotone method, Zhang [32] studied a unique positive solution for the singular boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+q(t)f(t,u(t),u'(t),\cdots,u^{(n-2)}(t))=0,\ 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=u^{(n-2)}(1)=0,\end{cases}$$

where $\alpha\in(n-1,n], n\geqslant 2$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f=g+h$ is nonlinear and g and h have different monotone properties.

Integral boundary conditions arise in thermal conduction problems [9], hydrodynamic problems [10] and semiconductor problems [17]. Recently, integral boundary value problems for fractional differential equations were investigated intensively [18, 20, 21, 26, 31, 33, 36]. In [33], authors using monotone iterative technique investigated the existence and uniqueness of the positive solutions of higher-order nonlocal fractional differential equations of the type:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\ 0<t<1, n-1<\alpha\leqslant n,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\quad u(1)=\lambda[u],\end{cases}\tag{1.2}$$

where $f\in C((0,1)\times\mathbb{R}_+,\mathbb{R}_+)$. By means of the Schauder's fixed point theorem, BVP (1.2) is also studied in [31]. In [30], the authors investigated problems (1.2) with $q(t)f(t,u(t))$ instead of $f(t,u(t))$, the existence and multiplicity of positive solutions are obtained by means of the fixed point index theory in cones.

However when f and boundary condition involve derivatives of the unknown function and the nonlinear term may take on negative values, to the best of our knowledge, there is no result established on fractional differential equations. The objective of the present study was to fill this gap. Inspired by the above works, in this paper, we consider the existence of positive solutions for fractional semipositone differential equation (1.1). Using Leray-Schauder's alternative and Krasnoselskii's fixed point theorem in cones and combining with an available transformation, we will obtain an interval of μ which ensures the existence of at least one positive solution for the BVP (1.1).

Our work presented in this paper has the following features. First of all, the nonlinear term f involves derivatives of unknown function. The second new feature is that the BVP (1.1) possesses singularity, that is, $f(t,z_1,\cdots,z_{n-1})$ may be singular at $t=0,1$. And the nonlinearity is allowed to be sign changing. Thirdly, we discuss the boundary conditions of BVP (1.1) are more general case, which include two point, three point, multi-point and some nonlocal conditions as special cases. In the end, the measure dA in the definition of λ can be a signed measure, see Remark 2.8 and Example 4.1 (point 2). The measure dA can be a signed measure as it is written in [30, 31, 33] but we did not see any example with such measure.

The rest of this paper is organized as follows. In Section 2, we present some lemmas that are used to prove our main results. In Section 3, the existence of positive solutions of the BVP (1.1) are established by using some fixed point theorems. In Section 4, we give an example to demonstrate the application of our theoretical results. The last section is devoted to a short conclusion.

2. Basic definitions and preliminaries

In this section, we introduce some definitions and notations of fractional calculus [25, 27] and present preliminary results needed in our proofs later.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha>0$ is given by

$$I^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds,$$

where $n-1<\alpha<n$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$ is defined as

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-s)^{n-\alpha-1}u(s)ds,$$

where $\mathbb{N}$ denotes the natural number set, the function $u(t)$ is n times continuously differentiable on $[0,+\infty)$.

Lemma 2.3. *If* $u, v : (0,+\infty) \to (-\infty,+\infty)$ *with* $\alpha > 0$, *then*

$$D_{0+}^{\alpha}(u(t)+v(t)) = D_{0+}^{\alpha}u(t) + D_{0+}^{\alpha}v(t).$$

Lemma 2.4.

(1) *If* $\alpha > 0$, $\delta > 0$, *then* $D_{0+}^{\delta}t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta)}t^{\alpha-\delta-1}$.

(2) *If* $u \in L(0,1)$, $\alpha > \delta > 0$, *then* $I^{\alpha}I^{\delta}u(t) = I^{\alpha+\delta}u(t)$, $D_{0+}^{\delta}I^{\alpha}u(t) = I^{\alpha-\delta}u(t)$, $D_{0+}^{\alpha}I^{\alpha}u(t) = u(t)$.

Lemma 2.5. *Let* $\alpha > 0$, *then the following equality holds for* $u \in L(0,1)$, $D_{0+}^{\alpha} \in L(0,1)$,

$$I^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

where $n-1 < \alpha \leqslant n$, $c_i \in \mathbb{R}$ $(i = 1,2,\cdots,n)$.

Set $\phi(t) = t(1-t)^{\alpha-n+1}$, $t \in [0,1]$ and

$$G(t,s) = \frac{1}{\Gamma(\alpha-n+2)}\begin{cases} [t(1-s)]^{\alpha-n+1} - (t-s)^{\alpha-n+1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ [t(1-s)]^{\alpha-n+1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Lemma 2.6 ([32])**.** *Let* $y \in C_r[0,1]$ $(C_r[0,1] = \{y \in C[0,1], t^ry \in C[0,1], 0 \leqslant r < 1\})$. *Then the boundary value problem*

$$\begin{cases} -D_{0+}^{\alpha-n+2}v(t) = y(t), & 0 < t < 1, n-1 < \alpha \leqslant n,\ n \geqslant 2, \\ v(0) = 0, \quad v(1) = 0, \end{cases}$$

has a unique solution

$$v(t) = \int_0^1 G(t,s)y(s)ds.$$

By Lemma 2.5, the unique solution of the problem

$$\begin{cases} -D_{0+}^{\alpha-n+2}v(t) = 0, & 0 < t < 1, n-1 < \alpha \leqslant n,\ n \geqslant 2, \\ v(0) = 0, \quad v(1) = \lambda[v] \end{cases} \tag{2.1}$$

is $\gamma(t) = t^{\alpha-n+1}$, with $\lambda[v]$ replaced by 1. As in [33], the Green's function for boundary value problem (2.1) is given by

$$H(t,s) = \frac{\gamma(t)}{1-\lambda[\gamma]}\mathscr{G}(s) + G(t,s), \tag{2.2}$$

where $\mathscr{G}(s) := \int_0^1 G(t,s)dA(t)$. Throughout the paper we assume the following condition (H_0) holds.

(H_0) A is a function of bounded variation, $\mathscr{G}(s) \geqslant 0$ for $s \in [0,1]$ and $0 \leqslant \lambda[\gamma] < 1$.

Remark 2.7. Note that the inequalities $\lambda[\gamma] \geqslant 0, \mathscr{G}(s) \geqslant 0$ from assumption (H_0) are trivially satisfied when dA is a positive measure.

In the next remark we consider the case when the measure changes the sign.

Remark 2.8. Take $dA(t) = g(t)dt$ with $g(t) = a - 1 - at$ and $a > 1$. Note that the measure changes the sign. Then

$$\lambda[\gamma] = \int_0^1 \gamma(t)g(t)dt = \int_0^1 t^{\alpha-n+1}(a - 1 - at)dt = \frac{a - 1 - (\alpha - n + 2)}{(\alpha - n + 2)(\alpha - n + 3)}.$$

Note that $0 \leqslant \lambda[\gamma] < 1$ provided that $\sqrt{a} < \alpha - n + 3 \leqslant a$. For example if $\alpha = 3$, then $3 \leqslant a < 9$, while if $\alpha = \frac{7}{2}$, then $\frac{5}{2} \leqslant a < \frac{25}{4}$. Moreover,

$$\begin{aligned}\mathscr{G}(s) = \int_0^1 G(t,s)g(t)dt &= \int_0^1 G(t,s)(a - 1 - at)dt \\ &= \frac{1}{\Gamma(\alpha - n + 3)}(1 - s)^{\alpha-n+1}s\left(\frac{a(2 - s)}{\alpha - n + 3} - 1\right) \\ &\geqslant \frac{1}{\Gamma(\alpha - n + 3)}(1 - s)^{\alpha-n+1}s\left(\frac{a}{\alpha - n + 3} - 1\right) \geqslant 0.\end{aligned}$$

Lemma 2.9. *Suppose that* (H_0) *holds, then the functions* $G(t,s)$ *and* $H(t,s)$ *have the following properties:*

(1) $G(t,s)$ *and* $H(t,s)$ *are nonnegative and continuous for* $(t,s) \in [0,1] \times [0,1]$;

(2) $G(t,s)$ *satisfies*

$$\text{(i)} \quad \frac{\phi(1-t)\phi(s)}{\Gamma(\alpha - n + 1)} \leqslant G(t,s) \leqslant \frac{\phi(s)}{\Gamma(\alpha - n + 1)} \quad \textit{for } t,s \in [0,1].$$

$$\text{(ii)} \quad G(t,s) \leqslant \frac{\gamma(t)}{\Gamma(\alpha - n + 2)} \quad \textit{for } t,s \in [0,1].$$

(3) $H(t,s)$ *satisfies*

$$\nu\gamma(t)\phi(s) \leqslant H(t,s) \leqslant \kappa\phi(s), \quad H(t,s) \leqslant \rho\gamma(t) \quad \textit{for } t,s \in [0,1],$$

where

$$\kappa = \left(\frac{\lambda[1]}{1 - \lambda[\gamma]} + 1\right)\frac{1}{\Gamma(\alpha - n + 1)}, \quad \nu = \frac{1}{(1 - \lambda[\gamma])\Gamma(\alpha - n + 1)}\int_0^1 \phi(1-t)dA(t),$$
$$\rho = \frac{1}{(1 - \lambda[\gamma])\Gamma(\alpha - n + 2)}.$$

Proof. (1) is clear, the left side of the first inequality of (2) can be found in [23], the right side of the first inequality of (2) can be found in [36] and the second inequality of (2) can be found in [32]. We only prove (3). In fact, by (2.2) and (2), we have

$$\begin{aligned} H(t,s) &= \frac{\gamma(t)}{1 - \lambda[\gamma]}\int_0^1 G(t,s)dA(t) + G(t,s) \\ &\leqslant \frac{1}{1 - \lambda[\gamma]}\int_0^1 \frac{\phi(s)}{\Gamma(\alpha - n + 1)}dA(t) + \frac{\phi(s)}{\Gamma(\alpha - n + 1)} = \left(\frac{\lambda[1]}{1 - \lambda[\gamma]} + 1\right)\frac{1}{\Gamma(\alpha - n + 1)}\phi(s) = \kappa\phi(s), \\ H(t,s) &= \frac{\gamma(t)}{1 - \lambda[\gamma]}\int_0^1 G(t,s)dA(t) + G(t,s) \\ &\geqslant \frac{\gamma(t)}{1 - \lambda[\gamma]}\int_0^1 \frac{\phi(1-t)\phi(s)}{\Gamma(\alpha - n + 1)}dA(t) = \frac{\gamma(t)\phi(s)}{(1 - \lambda[\gamma])\Gamma(\alpha - n + 1)}\int_0^1 \phi(1-t)dA(t) = \nu\gamma(t)\phi(s), \\ H(t,s) &= \frac{\gamma(t)}{1 - \lambda[\gamma]}\int_0^1 G(t,s)dA(t) + G(t,s) \\ &\leqslant \frac{\gamma(t)}{1 - \lambda[\gamma]}\int_0^1 \frac{\gamma(t)}{\Gamma(\alpha - n + 2)}dA(t) + \frac{\gamma(t)}{\Gamma(\alpha - n + 2)} = \frac{\gamma(t)}{(1 - \lambda[\gamma])\Gamma(\alpha - n + 2)} = \rho\gamma(t). \end{aligned}$$

□

In what follows, we give the assumptions to be used throughout the rest of this paper.

(H_1) $f:[0,1]\times\mathbb{R}_+^{n-1}\to\mathbb{R}$ is a continuous function, $f(t,0,\cdots,0)>0, t\in[0,1]$ and

$$f(t,z_1,z_2,\cdots,z_{n-1})\geqslant -e(t),\quad (t,z_1,z_2,\cdots,z_{n-1})\in[0,1]\times\mathbb{R}_+^{n-1},$$

where $e:[0,1]\to\mathbb{R}_+$ is a continuous function and $e(t)\not\equiv 0$ on $(0,1)$.

(H_1^*) $f:(0,1)\times\mathbb{R}_+^{n-1}\to\mathbb{R}$ is a continuous function and

$$f(t,z_1,z_2,\cdots,z_{n-1})\geqslant -e(t),\quad (t,z_1,z_2,\cdots,z_{n-1})\in(0,1)\times\mathbb{R}_+^{n-1},$$

where $e\in C((0,1),\mathbb{R}_+)\cap L[0,1]$ and $e(t)\not\equiv 0$ on $(0,1)$.

(H_2) For any constant $r>0$, $\int_0^1\phi(s)\max_{z_1,z_2,\cdots,z_{n-1}\in[0,r]}f(s,z_1,z_2,\cdots,z_{n-1})ds<+\infty$, $\int_0^1\phi(s)e(s)ds<+\infty$.

In order to overcome the difficulty due to the dependence of f on derivatives, we consider the following modified problem

$$\begin{cases} -D_{0+}^{\alpha-n+2}v(t)=\mu f(t,I_{0+}^{n-2}v(t),I_{0+}^{n-3}v(t),\cdots,I_{0+}^{1}v(t),v(t)), & 0<t<1,\\ v(0)=0,\quad v(1)=\lambda[v], \end{cases}\tag{2.3}$$

where $n-1<\alpha\leqslant n, n\geqslant 2$.

Lemma 2.10 ([21]). *The nonlocal fractional order boundary value problem (1.1) has a positive solution if and only if the nonlinear fractional integro-differential equation (2.3) has a positive solution.*

Lemma 2.11. *Assume that* (H_0) *holds, then the boundary value problems*

$$\begin{cases} -D_{0+}^{\alpha-n+2}\omega(t)=\mu e(t),\ \ 0<t<1,\\ \omega(0)=0,\quad \omega(1)=\lambda[\omega] \end{cases}$$

have unique solution

$$\omega(t)=\mu\int_0^1 H(t,s)e(s)ds,\tag{2.4}$$

which satisfies

$$\omega(t)\leqslant\mu\rho\gamma(t)\int_0^1 e(s)ds,\quad t\in[0,1].\tag{2.5}$$

Proof. It follows from (2.1), (2.2), Lemma 2.6 and (H_0) that (2.4)-(2.5) hold. □

Let $E=C[0,1]$, then E is a Banach space with the norm $\|u\|=\max_{0\leqslant t\leqslant 1}|u(t)|$ for any $u\in E$. Let $\Lambda=\frac{\nu}{\kappa}$ and

$$P=\{u\in E: u(t)\geqslant\Lambda\gamma(t)\|u\| \text{ for } t\in[0,1]\},\tag{2.6}$$

then P is a cone of E.

Define a modified function $[z(t)]^+$ for any $z\in C[0,1]$ by

$$[z(t)]^+=\begin{cases} z(t), & z(t)\geqslant 0,\\ 0, & z(t)<0. \end{cases}$$

Next we consider the following singular nonlinear approximate problem of (2.3)

$$\begin{cases} -D_{0+}^{\alpha-n+2}x(t)=\mu(F(t,[x(t)-\omega(t)]^+)+e(t)), & 0<t<1,\\ x(0)=0,\quad x(1)=\lambda[x], \end{cases}\tag{2.7}$$

with $F(t,[x(t)-\omega(t)]^+)=f(t,I_{0+}^{n-2}[x(t)-\omega(t)]^+,I_{0+}^{n-3}[x(t)-\omega(t)]^+,\cdots,I_{0+}^{1}[x(t)-\omega(t)]^+,[x(t)-\omega(t)]^+)$.

Lemma 2.12. *If x is a solution of the problem (2.7) with $x(t) > \omega(t)$ for $t \in (0,1]$, then $v(t) = x(t) - \omega(t)$ is a positive solution of the problem (2.3), and $u(t) = I_{0+}^{n-2}(x(t) - \omega(t))$ is a positive solution of the BVP (1.1).*

Proof. In fact, if $x \in C[0,1]$ is a positive solution of problem (2.7) such that $x(t) > \omega(t)$ for any $t \in (0,1]$, then from (2.7) and the definition of $[\cdot]^+$, we have

$$\begin{cases} -D_{0+}^{\alpha-n+2}x(t) = \mu(F(t, x(t) - \omega(t)) + e(t)), & 0 < t < 1, \\ x(0) = 0, \quad x(1) = \lambda[x]. \end{cases} \tag{2.8}$$

Let $v = x - \omega$, then $D_{0+}^{\alpha-n+2}v(t) = D_{0+}^{\alpha-n+2}x(t) - D_{0+}^{\alpha-n+2}\omega(t)$ for $t \in (0,1)$, which implies that

$$D_{0+}^{\alpha-n+2}x(t) = D_{0+}^{\alpha-n+2}v(t) + D_{0+}^{\alpha-n+2}\omega(t) = D_{0+}^{\alpha-n+2}v(t) - \mu e(t), \quad t \in (0,1).$$

Thus (2.8) becomes

$$\begin{cases} -D_{0+}^{\alpha-n+2}v(t) = \mu F(t, v(t)), & 0 < t < 1, \\ v(0) = 0, \quad v(1) = \lambda[v], \end{cases}$$

i.e.,

$$\begin{cases} -D_{0+}^{\alpha-n+2}v(t) = \mu f(t, I_{0+}^{n-2}v(t), I_{0+}^{n-3}v(t), \cdots, I_{0+}^{1}v(t), v(t)), & 0 < t < 1, \\ v(0) = 0, \quad v(1) = \lambda[v]. \end{cases}$$

So, $x - \omega$ is a positive solution of the problem (2.3). Together with Lemma 2.10, we obtain the conclusion of Lemma 2.12. The proof is completed. □

Employing (2.1), (2.2), and Lemma 2.6, the equation (2.7) can be expressed as

$$x(t) = \mu \int_0^1 H(t,s)(F(s, [x(s) - \omega(s)]^+) + e(s))ds. \tag{2.9}$$

Define an operator $T : P \to E$ by

$$Tx(t) = \mu \int_0^1 H(t,s)(F(s, [x(s) - \omega(s)]^+) + e(s))ds.$$

Clearly, if $x \in P$ is a fixed point of T, then x is a solution of the problem (2.7).

In order to prove our main results, we need the following lemmas.

Lemma 2.13 ([13])**.** *Let X be a real Banach space, and Ω be a bounded open subset of X, where $\theta \in \Omega$, $T : \overline{\Omega} \to X$ is a completely continuous operator. Then, either there exist $x \in \partial\Omega, \sigma \in (0,1)$ such that $\sigma T(x) = x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

Lemma 2.14 ([16])**.** *Let X be a real Banach space, and P be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open sets of X with $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tx\| \leqslant \|x\|, \quad x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geqslant \|x\|, \quad x \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tx\| \geqslant \|x\|, \quad x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leqslant \|x\|, \quad x \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

In this section, we present and prove our main results.

Lemma 3.1. *Assume that (H_0) and (H_1) (or (H_0), (H_1^*), and (H_2)) hold. Then $T : P \to P$ is a completely continuous operator.*

Proof. By routine discussion, we get that $T : P \to E$ is well-defined. Now we prove $T(P) \subset P$. For any $x \in P$, Lemma 2.9 implies that

$$\|Tx\| = \max_{0\leqslant t\leqslant 1} \mu \int_0^1 H(t,s)(F(s,[x(s)-\omega(s)]^+)+e(s))ds \leqslant \kappa\mu \int_0^1 \phi(s)(F(s,[x(s)-\omega(s)]^+)+e(s))ds.$$

On the other hand, from Lemma 2.9, we also have

$$(Tx)(t) = \mu \int_0^1 H(t,s)(F(s,[x(s)-\omega(s)]^+)+e(s))ds \geqslant \mu\nu\gamma(t) \int_0^1 \phi(s)(F(s,[x(s)-\omega(s)]^+)+e(s))ds.$$

So

$$(Tx)(t) \geqslant \Lambda\gamma(t)\|(Tx)\|,\ t\in[0,1], \tag{3.1}$$

and (3.1) yields that $T(P) \subset P$.

According to the Ascoli-Arzelà theorem and the Lebesgue dominated convergence theorem, we can easily get that $T : P \to P$ is a completely continuous operator. The proof is completed. □

Theorem 3.2. *Assume that* (H_0) *and* (H_1) *hold. Then there exists* μ^* *such that for any* $0 < \mu \leqslant \mu^*$, *BVP* (1.1) *has at least one positive solution.*

Proof. By (H_1), there exists $0 < \varepsilon < 1$, such that

$$f(t,z_1,z_2,\cdots,z_{n-1}) > 0, \quad t\in[0,1], \quad 0\leqslant z_1,z_2,\cdots,z_{n-1}\leqslant\varepsilon. \tag{3.2}$$

Let

$$0<\mu<\mu^* = \frac{\varepsilon}{2\kappa\bar{f}(\varepsilon)\int_0^1\phi(s)ds}, \qquad \bar{f}(\varepsilon) = \max_{\substack{t\in[0,1]\\ 0\leqslant z_1,\cdots,z_{n-1}\leqslant\varepsilon}} \{f(t,z_1,\cdots,z_{n-1})+e(t)\}.$$

Since $\lim_{z\to0}\frac{\bar{f}(z)}{z} = +\infty$ and $\frac{\bar{f}(\varepsilon)}{\varepsilon} < \frac{1}{2\kappa\mu\int_0^1\phi(s)ds}$, then there exists an $R_0 > 0$ satisfying $R_0 \in (0,\varepsilon)$, such that $\frac{\bar{f}(R_0)}{R_0} = \frac{1}{2\kappa\mu\int_0^1\phi(s)ds}$. Let $\Omega = \{x\in P : \|x\| < R_0\}$. If there exist $x\in\partial\Omega, \sigma\in(0,1)$ such that

$$x = \sigma T x,$$

we can claim that $\|x\| \neq R_0$. In fact, for $x\in\partial\Omega$, $s\in[0,1]$,

$$\begin{aligned}
&[x(s)-\omega(s)]^+ \leqslant x(s) \leqslant \|x\| = R_0,\\
|I_{0+}^i[x(s)-\omega(s)]^+| &= \left|\frac{1}{\Gamma(i)}\int_0^s (s-\tau)^{i-1}[x(\tau)-\omega(\tau)]^+ d\tau\right|\\
&\leqslant \frac{1}{(i-1)!}\int_0^s (s-\tau)^{i-1}|x(\tau)|d\tau\\
&\leqslant \frac{1}{(i-1)!}\int_0^s (s-\tau)^{i-1}d\tau\|x\| = \frac{1}{i!}s^i\|x\| \leqslant \|x\| = R_0, \quad i=1,2,\cdots,n-2.
\end{aligned}$$

It follows that

$$\begin{aligned}
x(t) = \sigma Tx(t) &= \sigma\mu\int_0^1 H(t,s)(F(s,[x(s)-\omega(s)]^+)+e(s))ds\\
&\leqslant \sigma\kappa\mu\int_0^1 \phi(s)(F(s,[x(s)-\omega(s)]^+)+e(s))ds\\
&\leqslant \kappa\mu\int_0^1 \phi(s) \max_{\substack{s\in[0,1]\\ 0\leqslant z_1,\cdots,z_{n-1}\leqslant R_0}} \{f(s,z_1,\cdots,z_{n-1})+e(s)\}ds\\
&\leqslant \kappa\mu\int_0^1 \phi(s)\bar{f}(R_0)ds = \kappa\mu\bar{f}(R_0)\int_0^1\phi(s)ds,
\end{aligned}$$

that is, $\frac{\overline{f}(R_0)}{R_0} \geqslant \frac{1}{\kappa\mu\int_0^1 \phi(s)ds} > \frac{1}{2\kappa\mu\int_0^1 \phi(s)ds}$, which implies that $\|x\| \neq R_0$ and $x \notin \partial\Omega$. By the nonlinear alternative theorem of Leray-Schauder type (Lemma 2.13), T has a fixed point $x \in \overline{\Omega}$. Furthermore, by (2.9), (3.2), and the fact that $R_0 < \varepsilon$, we get

$$x(t) = \mu\int_0^1 H(t,s)(F(s,[x(s)-\omega(s)]^+) + e(s))ds > \mu\int_0^1 H(t,s)e(s)ds = \omega(t), \quad t \in (0,1].$$

Thus, x is a positive solution of problem (2.7) and $x(t) > \omega(t)$ for $t \in (0,1]$. By Lemma 2.12, $u(t) = I_{0+}^{n-2}(x(t)-\omega(t))$ is a positive solution of BVP (1.1). The proof is completed. □

Theorem 3.3. *Assume that* (H_0), (H_1^*), *and* (H_2) *hold and the following condition is satisfied.*

(H_3) *There exists* $[a,b] \subset (0,1)$ *such that*

$$\lim_{z_{n-1}\to+\infty} \min_{\substack{t\in[a,b]\\ z_1\geqslant 0,\cdots,z_{n-2}\geqslant 0}} \frac{f(t,z_1,z_2\cdots,z_{n-2},z_{n-1})}{z_{n-1}} = +\infty.$$

Then there exists $\overline{\mu} > 0$ *such that for any* $0 < \mu < \overline{\mu}$, *BVP* (1.1) *has at least one positive solution.*

Proof. Let

$$\Omega_1 = \{x \in P : \|x\| < R_1\}, \text{ where } R_1 > \max\left\{1, \frac{2\rho}{\Lambda}\int_0^1 e(s)ds, \frac{\rho(\int_0^1 e(s)ds + 1)}{\Lambda}\right\},$$

and Λ, ρ are defined by (2.6) and Lemma 2.9, respectively. Let

$$\overline{\mu} = \min\left\{1, R_1\left[\kappa\int_0^1 \phi(s)\left(\max_{0\leqslant z_1,z_2,\cdots,z_{n-1}\leqslant R_1} f(s,z_1,z_2,\cdots,z_{n-2},z_{n-1}) + e(s)\right)ds\right]^{-1}\right\},$$

where κ is defined by Lemma 2.9. Then for any $x \in \partial\Omega_1$, $s \in [0,1]$, we have

$$\begin{aligned}
&[x(s)-\omega(s)]^+ \leqslant x(s) \leqslant \|x\| = R_1,\\
|I_{0+}^i[x(s)-\omega(s)]^+| &= \left|\frac{1}{\Gamma(i)}\int_0^s (s-\tau)^{i-1}[x(\tau)-\omega(\tau)]^+ d\tau\right|\\
&\leqslant \frac{1}{(i-1)!}\int_0^s (s-\tau)^{i-1}|x(\tau)|d\tau\\
&\leqslant \frac{1}{(i-1)!}\int_0^s (s-\tau)^{i-1}d\tau\|x\| = \frac{1}{i!}s^i\|x\| \leqslant \|x\| = R_1, \quad i = 1,2,\cdots,n-2.
\end{aligned}$$

Thus, for any $x \in \partial\Omega_1$, we have

$$\begin{aligned}
Tx(t) &= \mu\int_0^1 H(t,s)(F(s,[x(s)-\omega(s)]^+) + e(s))ds\\
&\leqslant \mu\kappa\int_0^1 \phi(s)(F(s,[x(s)-\omega(s)]^+) + e(s))ds\\
&\leqslant \mu\kappa\int_0^1 \phi(s)\left(\max_{0\leqslant z_1,z_2,\cdots,z_{n-1}\leqslant R_1} f(s,z_1,z_2,\cdots,z_{n-2},z_{n-1}) + e(s)\right)ds \leqslant R_1 = \|x\|,
\end{aligned}$$

which implies that

$$\|Tx\| \leqslant \|x\|, \quad x \in \partial\Omega_1. \tag{3.3}$$

On the other hand, by (H_3), there exists $R' > 0$ such that

$$f(t, z_1, \cdots, z_{n-2}, z_{n-1}) \geqslant N z_{n-1}, \quad t \in [a, b], \quad z_1, z_2, \cdots, z_{n-2} \geqslant 0, \quad z_{n-1} \geqslant R', \tag{3.4}$$

where $N > 0$ is a constant satisfying

$$\frac{\mu\nu N a^{2(\alpha-n+1)}\Lambda}{2}\int_a^b \phi(s)ds \geqslant 1,$$

where Λ, ν are defined by (2.6) and Lemma 2.9, respectively. Choose

$$R_2 > \max\left\{R_1, \frac{2R'}{a^{\alpha-n+1}\Lambda}, \frac{2R'(\alpha-1)(\alpha-2)\cdots(\alpha-n+2)}{a^{\alpha-1}\Lambda}\right\},$$

let $\Omega_2 = \{x \in P : \|x\| < R_2\}$. Then for any $x \in \partial\Omega_2$, by (2.5) we have

$$\begin{aligned} x(s) - \omega(s) &= x(s) - \mu\int_0^1 H(t, s)e(s)ds \\ &\geqslant x(s) - \mu\rho\gamma(t)\int_0^1 e(s)ds \\ &\geqslant x(s) - \frac{\mu\rho x(s)}{\Lambda R_2}\int_0^1 e(s)ds \\ &\geqslant \left(1 - \frac{\rho}{\Lambda R_2}\int_0^1 e(s)ds\right)x(s) \geqslant \frac{1}{2}x(s) \geqslant 0, \quad s \in [0, 1]. \end{aligned}$$

Hence, we have

$$x(s) - \omega(s) \geqslant \frac{1}{2}x(s) \geqslant \frac{1}{2}\Lambda\gamma(s)\|x\| \geqslant \frac{\Lambda R_2}{2}\min_{s\in[a,b]}\gamma(s) \geqslant \frac{a^{\alpha-n+1}\Lambda R_2}{2} \geqslant R' > 0, \quad s \in [a, b], \tag{3.5}$$

$$\begin{aligned} I_{0+}^{i}[x(s) - \omega(s)]^{+} = I_{0+}^{i}(x(s) - \omega(s)) &\geqslant \frac{\Lambda R_2}{2(i-1)!}\int_0^s (s-\tau)^{i-1}\gamma(\tau)d\tau \\ &\geqslant \frac{\Lambda R_2}{2(i-1)!}s^{\alpha-n+i+1}\mathbf{B}(i, \alpha-n+2) \\ &\geqslant \frac{\Lambda R_2 a^{\alpha-n+i+1}}{2(\alpha-n+i+1)(\alpha-n+i)\cdots(\alpha-n+2)} \\ &\geqslant R' > 0, \quad i = 1, 2, \cdots, n-2, \quad s \in [a, b], \end{aligned} \tag{3.6}$$

where $\mathbf{B}(\cdot,\cdot)$ is the beta function. Then for any $x \in \partial\Omega_2$, by (3.4), (3.5), and (3.6), we have

$$\begin{aligned} F(s, [x(s) - \omega(s)]^{+}) &= f(s, I_{0+}^{n-2}[x(s) - \omega(s)]^{+}, I_{0+}^{n-3}[x(s) - \omega(s)]^{+}, \cdots, I_{0+}^{1}[x(s) - \omega(s)]^{+}, [x(s) - \omega(s)]^{+}) \\ &\geqslant N[x(s) - \omega(s)]^{+} \geqslant \frac{N a^{\alpha-n+1}\Lambda R_2}{2}, \quad s \in [a, b]. \end{aligned}$$

Thus for any $x \in \partial\Omega_2$, $t \in [a, b]$, we have

$$\begin{aligned} Tx(t) &= \mu\int_0^1 H(t, s)(F(s, [x(s) - \omega(s)]^{+}) + e(s))ds \\ &\geqslant \mu\nu\gamma(t)\int_0^1 \phi(s)(F(s, [x(s) - \omega(s)]^{+}) + e(s))ds \\ &\geqslant \mu\nu\min_{t\in[a,b]}\gamma(t)\int_a^b \phi(s)F(s, [x(s) - \omega(s)]^{+})ds \\ &\geqslant \frac{\mu\nu N a^{2(\alpha-n+1)}\Lambda R_2}{2}\int_a^b \phi(s)ds \geqslant R_2, \end{aligned}$$

which implies that

$$\|Tx\| \geqslant \|x\|, \quad x \in \partial\Omega_2. \tag{3.7}$$

From (3.3), (3.7), and Lemma 2.14, we obtain that T has a fixed point x with $R_1 \leqslant \|x\| \leqslant R_2$. Noticing that $\|x\| \geqslant R_1$, we have

$$\begin{aligned} x(t) - \omega(t) &\geqslant \Lambda\gamma(t)\|x\| - \mu\rho\gamma(t)\int_0^1 e(s)ds \\ &\geqslant \Lambda\gamma(t)\frac{\rho(\int_0^1 e(s)ds + 1)}{\Lambda} - \rho\gamma(t)\int_0^1 e(s)ds \geqslant \rho\gamma(t) > 0, \quad t \in (0,1]. \end{aligned}$$

Thus, x is a positive solution of problem (2.7) and $x(t) > \omega(t)$ for $t \in (0,1]$. By Lemma 2.12, $u(t) = I_{0+}^{n-2}(x(t) - \omega(t))$ is a positive solution of BVP (1.1). The proof is completed. □

Remark 3.4. From the proof of Theorem 3.3, we know that the conclusion of Theorem 3.3 is valid if the condition of (H_3) is replaced by one of the following limits:

$$\lim_{z_1 \to +\infty} \min_{\substack{t\in[a,b] \\ z_2\geqslant 0, z_3\geqslant 0,\cdots,z_{n-1}\geqslant 0}} \frac{f(t, z_1, z_2 \cdots, z_{n-2}, z_{n-1})}{z_1} = +\infty,$$

$$\lim_{z_2 \to +\infty} \min_{\substack{t\in[a,b] \\ z_1\geqslant 0, z_3\geqslant 0,\cdots,z_{n-1}\geqslant 0}} \frac{f(t, z_1, z_2 \cdots, z_{n-2}, z_{n-1})}{z_2} = +\infty,$$

$$\vdots$$

$$\lim_{z_{n-2} \to +\infty} \min_{\substack{t\in[a,b] \\ z_1\geqslant 0,\cdots,z_{n-3}\geqslant 0, z_{n-1}\geqslant 0}} \frac{f(t, z_1, z_2 \cdots, z_{n-2}, z_{n-1})}{z_{n-2}} = +\infty.$$

Then there exists $\overline{\mu} > 0$ such that for any $0 < \mu < \overline{\mu}$, BVP (1.1) has at least one positive solution.

4. An example

Example 4.1. Consider the following problem

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t) + \mu\left(\frac{u^2(t)}{\sqrt{t}} + \frac{u'(t) + (u''(t))^3}{\sqrt{1-t}} - \frac{\cos t}{\sqrt{t(1-t)}}\right) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, u''(1) = \lambda[u''], \end{cases} \tag{4.1}$$

where

$$\alpha = \frac{7}{2}, \quad \mu > 0, \quad f(t, z_1, z_2, z_3) = \frac{z_1^2}{\sqrt{t}} + \frac{z_2 + z_3^3}{\sqrt{1-t}} - \frac{\cos t}{\sqrt{t(1-t)}}, \quad (t, z_1, z_2, z_3) \in (0,1) \times \mathbb{R}_+^3.$$

Let $e(t) = \frac{2}{\sqrt{t(1-t)}}$. Then, for $t \in (0,1)$, we have

$$f(t, z_1, z_2, z_3) \geqslant -e(t), \quad \lim_{z_3 \to +\infty} \min_{\substack{t\in[\frac{1}{3},\frac{2}{3}] \\ z_1\geqslant 0, z_2\geqslant 0}} \frac{f(t, z_1, z_2, z_3)}{z_3} = +\infty,$$

thus conditions (H_1^*) and (H_3) hold for

$$G(t,s) = \begin{cases} G_1(t,s) = \frac{[t(1-s)]^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}, & 0 \leqslant t \leqslant s \leqslant 1, \\ G_2(t,s) = \frac{[t(1-s)]^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Since $\phi(s) = s(1-s)^{\frac{1}{2}}, s \in [0,1]$, so we have

$$\int_0^1 \phi(s)e(s)ds = \int_0^1 s(1-s)^{\frac{1}{2}} \frac{2}{\sqrt{s(1-s)}}ds = 2\int_0^1 s^{\frac{1}{2}}ds = \frac{4}{3} < +\infty.$$

And for any constant $r > 0$,

$$\begin{aligned}\int_0^1 \phi(s) \max_{z_1,z_2,z_3\in[0,r]} f(s,z_1,z_2,z_3)ds & \int_0^1 s(1-s)^{\frac{1}{2}} \max_{z_1,z_2,z_3\in[0,r]} \left(\frac{z_1^2}{\sqrt{s}} + \frac{z_2+z_3^3}{\sqrt{1-s}} - \frac{\cos s}{\sqrt{s(1-s)}}\right) ds \\ & \leqslant r^2 \int_0^1 s^{\frac{1}{2}}(1-s)^{\frac{1}{2}}ds + (r+r^3)\int_0^1 s ds + \int_0^1 s^{\frac{1}{2}}ds < +\infty.\end{aligned}$$

So, condition (H_2) holds. In the following we discuss condition (H_0) holds when $\lambda[\cdot]$ takes different cases.
1. Let $\lambda[v] = 0$. In this case, we have:

$$\lambda[\gamma] = 0, \quad \mathscr{G}(s) = 0.$$

2. Now, let $\lambda[u''] = \int_0^1 u''(t)(4-5t)dt$. Note that the function $g(t) = 4-5t$ changes the sign on the interval $[0,1]$. In this case, we have

$$\begin{aligned}\lambda[\gamma] &= \int_0^1 t^{\frac{1}{2}}dA(t) = \int_0^1 t^{\frac{1}{2}}(4-5t)dt = \frac{2}{3} < 1,\\ \mathscr{G}(s) &= \int_0^1 G(t,s)(4-5t)dt = \frac{1}{\Gamma(\frac{5}{2})}(1-s)^{\frac{1}{2}}s(3-2s) \geqslant 0, \quad s\in[0,1].\end{aligned}$$

3. Let $\lambda[u''] = \frac{1}{2}u''(\frac{1}{2})$. In this case, we have:

$$\lambda[\gamma] = \int_0^1 t^{\frac{1}{2}}dA(t) = \frac{\sqrt{2}}{4} \approx 0.353553 < 1, \qquad \mathscr{G}(s) = \frac{1}{2}G\left(\frac{1}{2},s\right) \geqslant 0, \quad s\in[0,1].$$

4. Let $\lambda[u''] = 2u''(\frac{1}{2}) - u''(\frac{3}{4})$. Note that the coefficients $k_1 = 2$, $k_2 = -1$, i.e., not all of the coefficients must be positively, so some coefficients of k_i can be negative. In this case, we have

$$\lambda[\gamma] = \int_0^1 t^{\frac{1}{2}}dA(t) = \sqrt{2} - \frac{\sqrt{3}}{2} \approx 0.548188 < 1,$$

$$\mathscr{G}(s) = \begin{cases} 2G_2\left(\frac{1}{2},s\right) - G_2\left(\frac{3}{4},s\right), & 0 \leqslant s < \frac{1}{2},\\ 2G_1\left(\frac{1}{2},s\right) - G_2\left(\frac{3}{4},s\right), & \frac{1}{2} \leqslant s \leqslant \frac{3}{4},\\ 2G_1\left(\frac{1}{2},s\right) - G_1\left(\frac{3}{4},s\right), & \frac{3}{4} < s \leqslant 1.\end{cases}$$

Then $0 \leqslant \mathscr{G}(s) < 1, s\in[0,1]$.

Seen from above, the condition (H_0) holds in different cases 1-4. Therefore, all conditions of Theorem 3.3 are satisfied. Thus, by Theorem 3.3, BVP (4.1) has at least one positive solution provided μ is small enough.

5. Conclusions

In this manuscript, by making use of fixed point theorem on Banach spaces, explicit range for μ is derived such that for μ lying in this interval, the existence of at least one positive solution to the BVP (1.1) is guaranteed. The obtained results show clearly that the nonnegative of the nonlinearity is no longer required to get the positive solutions in this paper. To sustain our results, an example was analyzed in detail.

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