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Modelling the movement of groundwater pollution with variable order derivative

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Abstract

In this paper, a new concept of variable differentiation is used to revisit the model of groundwater pollution. The new variable order derivation has a non-singular kernel and can be used for analytical and numerical purposes. The novel model is solved via Fourier transform method. We solve numerically the new equation using the implicit finite difference scheme and study the stability and convergence of that scheme.

Keywords: New variable order derivative, advection-dispersion equation with variable order, stability, convergence, finite difference scheme.
2010 MSC: 34A08, 34A12, 44AXX.

1. Introduction

The first observation of the dispersion was done in 1905 by Slichter who studied the movement of groundwater using an electrolyte as a tracer [17]. In 1980, Bachmat et al. realised a numerical model related to the groundwater management. In 1987, Naymik and Freeze [13] presented a review of 44 technical advanced articles on mathematical modelling of solute transport in the surface system. Abriola [1] presented also a review paper on modelling of solute transport in groundwater. Many other models have been developed to evaluate the impact of non-point source pollution on water. We have the Hydrological simulation Program Fortran, Water Loading Function model to evaluate the sources of organic carbon to the tidal, freshwater [7]. But these models will not really touch the area impacted by the groundwater. Hazardous waste management research has advanced the capability of the modelling to predict hazardous waste plumes. These models usually insist on the transport in the saturated zone and deal with localised sources. Many models have been developed for the problem of pollution of the non-point sources which include unsaturated zone modelling with varying degrees of complexity. In 1990, Leonard

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et al. [11] studied the groundwater loading effects of agricultural management systems. The method of underground solute evaluation was done by Pacenka and Steenhuis in 1984. In 1987, Angelakis et al. [2] described the transport and simultaneous transformation of two solutes with two "one-dimensional" partial differential equations. They used the linear equilibrium absorbtion-desorption relationship for both solutes and irreversible microbial first-order kinetics. They also used the Laplace transform for zero initial conditions to obtain the analytical solution. Lassey [10] derived an analytical solution to the advection-dispersion equation for one-dimensional solute or tracer transport including sorption and first-order loss. Miller and Weber [18] described laboratory investigations and mathematical modelling of the sorption of hydrophobic solutes by aquifer materials. Rao and Hathaway [15] have developed the three dimensional mixing cell solute transport model using the principles of conservation of mass for water and solute. Shoemaker et al. [16] have described the modelling of the movement of pesticide in the groundwater.

Given that the advection-dispersion equation is the equation governing the standard model for contaminant model, the classical one-dimensional advection-dispersion equation is given by

$$\frac{\partial C(x,t)}{\partial t} = -\nu\frac{\partial C(x,t)}{\partial x} + D\frac{\partial^2 C(x,t)}{\partial x^2}, \qquad C(x,0)=\delta(x),$$

which describes the tracer plume at location $x = 0$ at time $t = 0$, ν represents the advective velocity, D the effect of molecular diffusion and advective dispersion. The problem of the classical models is that the tracer plumes generally behave differently from a Brownian motion with drift. Many tracer plumes are biased instead of symmetric and propagate faster than the square root of time. In 1999, Meerschaert et al. came out with the one-dimensional fractional advection-dispersion equation given as follows:

$$\frac{\partial C(x,t)}{\partial t} = -\nu\frac{\partial C(x,t)}{\partial x} + Dp\frac{\partial^\alpha C(x,t)}{\partial x^\alpha} + D(p-1)\frac{\partial^\alpha C(x,t)}{\partial (-x)^\alpha}, \qquad C(x,0)=\delta(x), \tag{1.1}$$

C is the expected concentration, t still the time, ν a constant mean velocity, x the distance in the direction of mean velocity, $0 \leqslant p \leqslant 1$ describes the skewness of the transport process and α the order of fractional differentiation. In 2000, Benson et al. [6] studied if the fractional-order Advection Dispersion Equation (ADE) is also a useful model for transport in relatively homogeneous material using (1.1). In 2009, Jaiswal et al. [9] solved analytically the advection-diffusion equation in one-dimensional semi-infinite medium using Laplace transform. In 2011, Jaiswal et al. [8] studied the transport of solute mass transport, originating from a uniform pulse-type stationary point source through a heterogeneous semi-infinite horizontal medium. In 2012 also, Yadav et al. [20] presented the mathematical model for dispersion problem in finite porous media in which the flow is two-dimensional where the seepage flow velocity is periodic, and dispersion parameter is proportional to the flow velocity.

In 2013, Atangana and Kilicman [5] took a one-dimensional model of advection-dispersion with a particular chemical at the time t_0 to make the concentration changes as an exponential function. Given by the following equation from [3]:

$$R\frac{\partial C(x,t)}{\partial t} = -\nu\frac{\partial C(x,t)}{\partial x} - D\frac{\partial^2 C(x,t)}{\partial x^2} - \lambda R C(x,t),$$

where D is the dispersion coefficient and R the retardation factor, with the following initial condition:

$$\begin{aligned}
&C(x,t)=0, \quad t=0,\\
&C(0,t)=C_0 e^{-\gamma t}, \quad 0<t\leqslant t_0, \quad C_0,\gamma\in\mathbb{R},\\
&\frac{\partial C(x,t)}{\partial x}=0, \quad x\to\infty, \quad \text{as a boundary condition,}
\end{aligned}$$

and generalised it using fractional derivative. They then obtained the following equation:

$$R\frac{\partial C(x,t)}{\partial t} = -\nu\frac{\partial^\beta C(x,t)}{\partial^\beta x} - D\frac{\partial^\alpha C(x,t)}{\partial x^\alpha} - \lambda R C(x,t),$$

where they had replaced the first derivative with respect to x i.e., $\partial_x C(x,t)$ by $\partial_x^{\beta} C(x,t)$, $0 < \beta \leqslant 1$ and the second derivative with respect to x by $\partial_x^{\alpha} C(x,t)$, $1 < \beta \leqslant l$, $l \in \mathbb{R}$, with the same initial condition as previously and the following boundary condition:

$$\frac{\partial^{\beta} C(x,t)}{\partial x^{\beta}} = 0, \quad x \to \infty,$$

while taking these fractional derivatives over space like the Caputo or Riemann-Liouville derivative of order $0 < \beta \leqslant 1$ and the second order space derivative with Caputo or Riemann-Liouville fractional derivative of order $1 < \alpha \leqslant 2$ and used the Laplace transform on the equation, they found the solution in terms of Mittag-Leffler functions. Given that the solution obtained here does not depend only on the function of times and space, but also on the function of the order of derivative, they concluded by saying that the fractional advection-dispersion equation has better prediction than the advection-dispersion equation.

2. New variable order derivative

In order to solve the problem posed by the existing derivatives with variable orders, Atangana and Koca suggested a new definition of variable order derivative and presented their properties. In this section, in order to accommodate readers that are not familiar to this new differentiation, we present also some interesting properties of the fractional derivative as shown in Atangana and Koca [4].

Atangana and Koca first of all defined the following variable order derivative:

$${}_0^{AK}D_t^{\psi(x)} f(t) = \int_0^t \frac{df(y)}{dy} e^{-\psi(x)(t-y)} dy,$$

where the function f needs to be differentiable. To solve this problem, they defined the following new variable order derivative with no condition on our function f.

Definition 2.1 ([4])**.** Let $f(t)$ be a continuous function which does not need to be differentiable on an open interval I, $\psi(x)$ a continuous function in a closed interval $[a, b]$. The derivative of $f(t)$ with variable order $\psi(x)$ is given by:

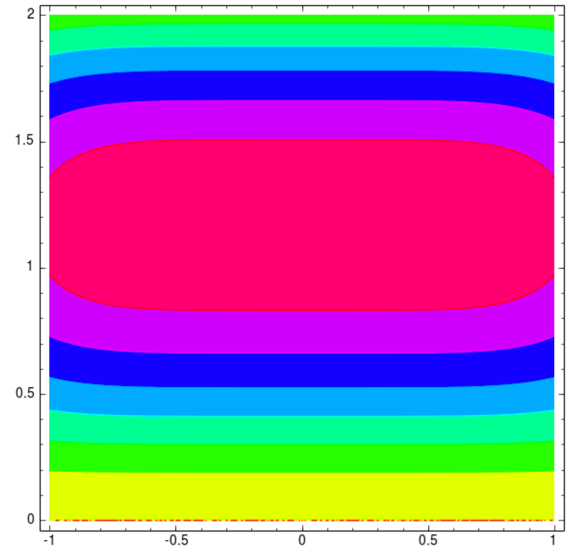
$${}_0^{A}D_t^{\psi(x)} f(t) = \frac{d}{dt} \int_0^t f(y) e^{-\psi(x)(t-y)} dy,$$

and is called the new variable order derivative.

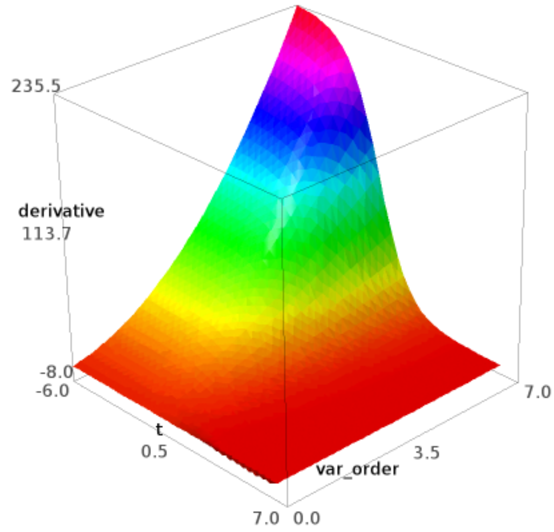
3. Plotting of the derivative of some functions

In this section, we give the contour plot and the plot in three dimension using SAGE of the new variable order derivative of three different functions to show how the new derivative can be used to solve problem analytically.

1. We have: $\begin{cases} \psi(x) = e^x, \\ f(t) = 4t^2 + 5t + 6; \end{cases}$
2. $\begin{cases} \psi(x) = x^6 + 10, \\ f(t) = t\sin(t); \end{cases}$
3. $\begin{cases} \psi(x) = \frac{e^x - \sin(x)}{10}, \\ f(t) = t^3 + 5t + 20. \end{cases}$

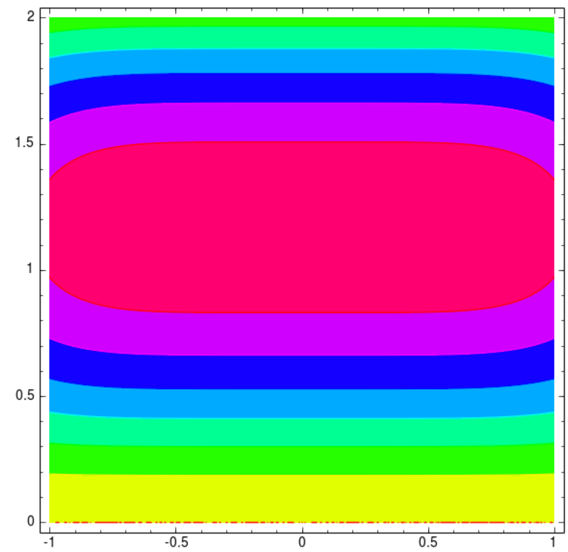


(a) ContourPlot for the derivative of $t\sin(t)$ with order x^6+10.

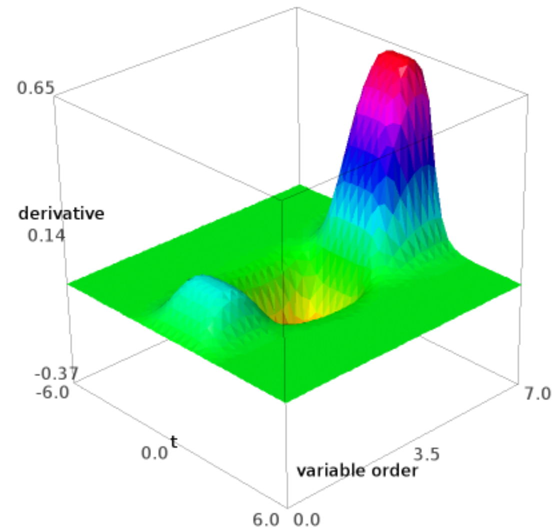


(b) Plot in 3D for the derivative of $4t^2+5t+6$ with order e^x.

Figure 1: Case 1.

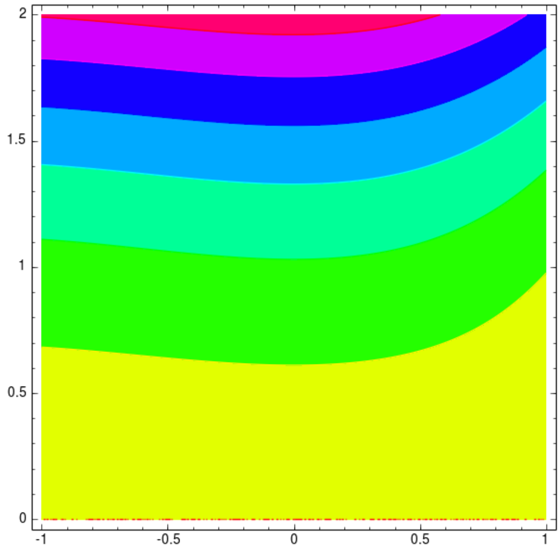


(a) ContourPlot for the derivative of $t\sin(t)$ with order x^6+10.

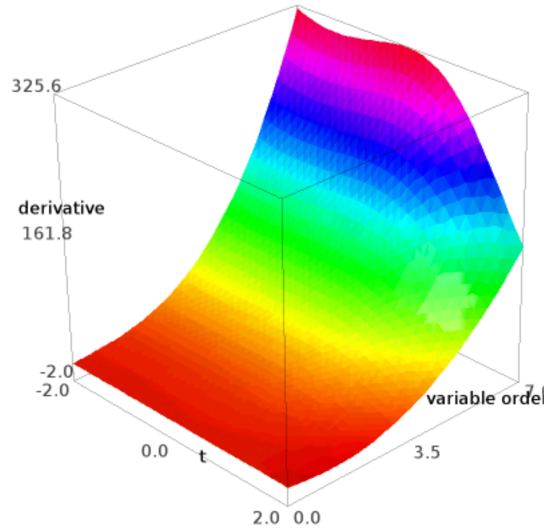


(b) Plot in 3D for the derivative of $t\sin(t)$ with order x^6+10.

Figure 2: Case 2.



(a) ContourPlot for the derivative of $t^3+5t+20$ with order $\frac{e^x-\sin(x)}{10}$.



(b) Plot in 3D for the derivative of $t^{30}+5t+20$ with order $\frac{e^x-\sin(x)}{10}$.

Figure 3: Case 3.

4. Equation of groundwater pollution with new derivative

In this section, we present the advection-dispersion equation with variable order on the velocity and look for its analytical and numerical solution. The advection-dispersion equation with variable-order derivative for $y \in [0, Y]$, $(x, t) \in [0, X] \times [0, T]$ is given by

$$R\frac{\partial C(x,t)}{\partial t} = -v\frac{\partial^{\alpha(y)}C(x,t)}{\partial x^{\alpha(y)}} + D\frac{\partial^2 C(x,t)}{\partial x^2} - \lambda RC(x,t), \tag{4.1}$$

with $0 < \alpha(y) \leqslant 1$, v the linear average velocity of the water, D the dispersion coefficient, λ the radioactivity decay rate, R the retardation coefficient and the following initial condition and boundary condition:

$$C(x,0) = \phi(x), \quad 0 \leqslant x \leqslant X, \tag{4.2}$$
$$C(0,t) = 0, \quad 0 < t \leqslant T, \tag{4.3}$$
$$C(X,t) = \psi(t), \quad 0 < t \leqslant T. \tag{4.4}$$

The operator $\frac{\partial^{\alpha(y)}}{\partial x^{\alpha(y)}}$ denotes the new variable order fractional derivative.

4.1. Analytical solution

Let us use the Fourier transform to solve this equation. When applying the Fourier transform to equation (4.1), we have

$$\mathcal{F}\left(R\frac{\partial C(x,t)}{\partial t}\right)(\zeta) = \mathcal{F}\left(-v\frac{\partial^{\alpha(y)}C(x,t)}{\partial x^{\alpha(y)}}\right)(\zeta) + \mathcal{F}\left(D\frac{\partial^2 C(x,t)}{\partial x^2}\right)(\zeta) - \mathcal{F}\left(\lambda RC(x,t)\right)(\zeta)$$
$$\Rightarrow R\mathcal{F}\left(\frac{\partial C(x,t)}{\partial t}\right)(\zeta) = -v\mathcal{F}\left(\frac{\partial^{\alpha(y)}C(x,t)}{\partial x^{\alpha(y)}}\right)(\zeta) + D\mathcal{F}\left(\frac{\partial^2 C(x,t)}{\partial x^2}\right)(\zeta) - \lambda R\mathcal{F}\left(C(x,t)\right)(\zeta)$$

$$\Rightarrow R\frac{\partial \hat{C}(\zeta,t)}{\partial t} = -i\nu\hat{C}(\zeta,t)\frac{2\zeta\alpha(y)}{\alpha(y)^2+\zeta^2} - \zeta^2 D\hat{C}(\zeta,t) - \lambda R\hat{C}(\zeta,t)$$

$$\Rightarrow \frac{\partial \hat{C}(\zeta,t)}{\partial t} = \left(i\nu\frac{2\zeta\alpha(y)}{R(\alpha(y)^2+\zeta^2)} + \zeta^2\frac{D}{R} + \lambda\right)\hat{C}(\zeta,t).$$

Using some property of integration, we have

$$\hat{C}(\zeta,t) = K_0 e^{-f(\zeta)t},$$

where $f(\zeta) = i\nu\frac{2\zeta\alpha(y)}{R(\alpha(y)^2+\zeta^2)} + \zeta^2\frac{D}{R} + \lambda$ and K_0 a constant.

Using the initial condition, we have $\hat{C}(\zeta,0) = K_0$. From the fact that $C(x,0) = \phi(x) \Rightarrow \hat{C}(\zeta,0) = \hat{\phi}(\zeta)$, we obtain $\hat{\phi}(\zeta) = K_0$ and finally

$$\hat{C}(\zeta,t) = \hat{\phi}(\zeta)e^{-f(\zeta)t}. \tag{4.5}$$

To find the value of $C(x,t)$ in (4.5), we use the inverse Fourier transform:

$$\begin{aligned} C(x,t) &= \mathcal{F}^{-1}\left(\hat{\phi}(\zeta)e^{-f(\zeta)t}\right) \\ &= \mathcal{F}^{-1}\left(\hat{\phi}(\zeta)\right) * \mathcal{F}^{-1}\left(e^{-f(\zeta)t}\right) \quad \text{where } * \text{ is the convolution product} \\ &= \phi(x) * \mathcal{F}^{-1}\left(e^{-f(\zeta)t}\right). \end{aligned}$$

The problem we encounter here is to find the inverse Fourier transform $\mathcal{F}^{-1}\left(e^{-f(\zeta)t}\right)$ with

$$f(\zeta) = i\nu\frac{2\zeta\alpha(y)}{R(\alpha(y)^2+\zeta^2)} + \zeta^2\frac{D}{R} + \lambda.$$

We have tried to solve it with the software we have to no avail.

4.2. *Numerical analysis*

Here, we present the numerical way to approximate the problem (4.1), (4.2), (4.3), (4.4). To discretize the fractional derivative, we apply the standard Grünwald formula [14] for a given $0 < \alpha \leqslant 1$, given by

$$\frac{d^\alpha f}{dx^\alpha} = \frac{1}{h^\alpha}\sum_{l=0}^{[X/h]} \omega_l^\alpha f(x-(l)h) + A_1, \tag{4.6}$$

and the shifted Grünwald formula [12] for $\alpha = 2$

$$\frac{d^\alpha f}{dx^\alpha} = \frac{1}{h^\alpha}\sum_{l=0}^{1+[X/h]} \omega_l^\alpha f(x-(l-1)h) + A_2, \tag{4.7}$$

where

$$\omega_l^\alpha = (-1)^l\frac{\alpha(\alpha-1)\cdots(\alpha-l+1)}{l!}, \quad l \in \mathbb{N},$$

is called the Grünwald coefficient and the expression $[X/h]$ is the floor of $\frac{X}{h}$. We also have A_1 and A_2 as remainder terms. We have from [12] that when $f \in \mathcal{C}^3$, there exist two constant K_1 and K_2 such that

$$|A_1| \leqslant K_1 h, \qquad |A_2| \leqslant K_2 h,$$

uniformly on a given interval $[0,c]$. The Grünwald coefficient has the following properties:

- $\omega_0^\alpha = 1, \ \omega_1^\alpha = -\alpha.$

- We have $\omega_l^\alpha < 0$ for $0 < \alpha < 1$ and $l = 1, 2, 3, \cdots$.
- We have $\sum_{l=0}^{p} \omega_l^\alpha > 0$ for $0 < \alpha < 1$ and $p = 1, 2, 3, \cdots$.
- $\sum_{l=0}^{p} \omega_l^2 = \omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \cdots = 1 - 2 + \frac{2}{2} + 0 + \cdots = 0 \Rightarrow \sum_{l=0}^{p} \omega_l^2 < 0$ for $p = 1$ and $\sum_{l=0}^{p} \omega_l^2 = 0$, for $p = 2, 3, \cdots$.
- $\omega_l^\alpha = (1 - \frac{\alpha+1}{l})\omega_{l-1}^\alpha$.

We have the following lemma.

Lemma 4.1. *Let* $a, a_1, a_2, a_3, a_4, \cdots, a_n$ *be all real numbers. Then we have*

$$|a| - (|a_1| + |a_2| + \cdots + |a_n|) \leqslant |a + a_1 + a_2 + a_3 + a_4 + \cdots + a_n|.$$

Let us discretize the temporal and spatial variable as follows:

$$\begin{aligned} t_k &= k\tau, \quad k = 0, 1, 2, \cdots, n, \quad \tau = \frac{T}{n} \text{ is the temporal step,} \\ x_i &= ih, \quad i = 0, 1, 2, \cdots, N, \quad h = \frac{X}{N} \text{ is the spacial step,} \\ y_j &= pj, \quad j = 0, 1, 2, \cdots, M, \quad p = \frac{Y}{M}. \end{aligned}$$

We have

$$\frac{\partial C(x_i, t_k)}{\partial t} = \frac{C(x_i, t_k) - C(x_i, t_{k-1})}{\tau} + A_t, \tag{4.8}$$

where $|A_t| \leqslant K_0 \tau$ for K_0 be a constant. (4.1) becomes

$$R\frac{\partial C(x_i, t_k)}{\partial t} = -\nu \frac{\partial^{\alpha(y_j)} C(x_i, t_k)}{\partial x^{\alpha(y_j)}} + D\frac{\partial^2 C(x_i, t_k)}{\partial x^2} - \lambda R C(x_i, t_k)$$

for $j = 0, 1, 2, \cdots, M$. Using (4.6), (4.7) and (4.8), the previous equation becomes

$$\begin{aligned} R\frac{C(x_i, t_k) - C(x_i, t_{k-1})}{\tau} + RA_t = -\nu \frac{1}{h^{\alpha(y_j)}} \sum_{l=0}^{i} \omega_l^{\alpha(y_j)} C(x_{i-l}, t_k) - \nu A_1 \\ + D\frac{1}{h^2} \sum_{l=0}^{i+1} \omega_l^2 C(x_{i+1-l}, t_k) + DA_2 \\ - \lambda R C(x_i, t_k), \quad \text{for } j = 0, 1, 2, \cdots, M, \end{aligned}$$

$$\begin{aligned} C(x_i, t_k) - C(x_i, t_{k-1}) + \tau A_t = -\frac{\tau\nu}{Rh^{\alpha(y_j)}} \sum_{l=0}^{i} \omega_l^{\alpha(y_j)} C(x_{i-l}, t_k) - \frac{\tau\nu}{R} A_1 \\ + \frac{\tau D}{Rh^2} \sum_{l=0}^{i+1} \omega_l^2 C(x_{i+1-l}, t_k) + D\frac{\tau A_2}{R} \\ - \lambda\tau C(x_i, t_k), \quad \text{for } j = 0, 1, 2, \cdots, M, \end{aligned}$$

which implies that

$$(1 + \lambda\tau) C(x_i, t_k) - q_j^1 \sum_{l=0}^{i} \omega_l^{\alpha(y_j)} C(x_{i-l}, t_k) - q^2 \sum_{l=0}^{i+1} \omega_l^2 C(x_{i+1-l}, t_k) = C(x_i, t_{k-1}) + A,$$

where $q_j^1 = -\frac{\tau\nu}{Rh^{\alpha(y_j)}}$, $q^2 = \frac{\tau D}{Rh^2}$ and $A = (-\tau A_t - \frac{\tau\nu}{R}A_1 + D\frac{\tau A_2}{R})$, for $j = 0, 1, \cdots, M$. From this, we observe that $q^1 < 0$ and $q^2 > 0$. Taking C_i^k and α_j as the approximation of $C(x_i, t_k)$ and $\alpha(y_j)$, respectively, truncating the remainder, we obtain the implicit finite difference scheme to solve (4.1) as follows:

$$(1+\lambda\tau)C_i^k - q_j^1 \sum_{l=0}^{i} \omega_l^{\alpha_j} C_{i-l}^k - q^2 \sum_{l=0}^{i+1} \omega_l^2 C_{i+1-l}^k = C_i^{k-1} \tag{4.9}$$

for $j = 0, 1, \cdots, M$, $k = 1, 2, \cdots, n$, $i = 1, 2, \cdots, N-1$. With the following initial and boundary condition:

$$C_i^0 = \phi(x_i), \quad i = 0, 1, 2, \cdots, N, \tag{4.10}$$

$$C_0^k = 0, \quad k = 0, 1, \cdots, n, \tag{4.11}$$

$$C_N^k = \psi(t_k), \quad k = 0, 1, \cdots, n. \tag{4.12}$$

Therefore (4.9), (4.10), (4.11), (4.12) refer to the implicit finite difference method to solve (4.1)-(4.4).

4.2.1. Stability analysis

Let us assume that C_i^k and $\tilde{C}_i^k$ are two solution of (4.9)-(4.12) such that $\tilde{C}_i^0 = \tilde{\phi}(x_i)$.

Set $\psi_i^k = \tilde{C}_i^k - C_i^k$ for $i = 1, 2, \cdots, N-1$ and $k = 0, 1, \cdots, n$. Then from (4.9), we have

$$(1+\lambda\tau)\psi_i^k - q_j^1 \sum_{l=0}^{i} \omega_l^{\alpha_j} \psi_{i-l}^k - q^2 \sum_{l=0}^{i+1} \omega_l^2 \psi_{i+1-l}^k = \psi_i^{k-1}$$

for $j = 0, 1, 2 \cdots, M$.

Let us define the following vector $E^k = (\psi_1^k, \psi_2^k, \cdots, \psi_{N-1}^k)$ for $k = 0, 1, \cdots, n$. We then have

$$\begin{aligned}||E^0||_\infty = ||(\psi_1^0, \psi_2^0, \cdots, \psi_{N-1}^0)|| &= \max_{1\leqslant i\leqslant N-1} |\psi_i^0| \\ &= \max_{1\leqslant i\leqslant N-1} |\tilde{\phi}(x_i) - \phi(x_i)| \\ &\leqslant \max_{0\leqslant x\leqslant X} |\tilde{\phi}(x) - \phi(x)|.\end{aligned}$$

Then $||E^0||_\infty \leqslant \max_{0\leqslant x\leqslant X} |\tilde{\phi}(x) - \phi(x)|$.

Theorem 4.2 ([19]). *The implicit finite difference scheme* (4.9)-(4.12) *is stable, if*

$$||E^k||_\infty \leqslant \max_{0\leqslant x\leqslant X} |\tilde{\phi}(x) - \phi(x)|, \quad \textit{for} \;\; k = 1, \cdots, n.$$

Proof. [19] According to the properties of ω_l^α, we have

$$q_j^1 \sum_{l=0}^{i} \omega_l^{\alpha_j} < 0 \text{ and } q^2 \sum_{l=0}^{i+1} \omega_l^2 < 0, \quad \text{for } j = 0, 1, 2, \cdots, M,$$

since $q^1 < 0$, $q^2 > 0$, $\sum_{l=0}^{i} \omega_l^{\alpha_j} > 0$ and $\sum_{l=0}^{i+1} \omega_l^2 < 0$, for $j = 0, 1, 2, \cdots, M$. We have

$$||E^k||_\infty = \max_{1\leqslant i\leqslant N-1} |(\psi_i^k)| \Rightarrow \exists i_0 \in [1, \cdots, N-1],$$

such that $||E^k||_\infty = |\psi_{i_0}^k|$ for $k \in \{1, 2, \cdots, n\}$. We have

$$|\psi_{i_0}^k| \leqslant (1+\lambda\tau)|\psi_{i_0}^k| \leqslant (1+\lambda\tau)|\psi_{i_0}^k| - \left(q_j^1 \sum_{l=0}^{i_0} \omega_l^{\alpha_j} + q^2 \sum_{l=0}^{i_0+1} \omega_l^2\right)(1+\lambda\tau)|\psi_{i_0}^k| \quad \text{for } \; j = 0, 1, 2, \cdots, M$$

$$
\begin{aligned}
&\leqslant \Big((1-q_j^1\omega_0^{\alpha_j}-q^2\omega_1^2)\,(1+\lambda\tau)|\psi_{i_0}^k| \\
&\quad - q_j^1\sum_{l=1}^{i_0}\omega_l^{\alpha_j}|\psi_{i_0-l}^k| - q^2\sum_{l=0,l\neq 1}^{i_0+1}\omega_l^2|\psi_{i_0+1-l}^k| \\
&\leqslant \left|(1+\lambda\tau)\psi_{i_0}^k - q_j^1\sum_{l=0}^{i_0}\omega_l^{\alpha_j}\psi_{i_0-l}^k - q^2\sum_{l=0}^{i_0+1}\omega_l^2\psi_{i_0+1-l}^k\right| \quad \text{for} \quad j=0,1,2,\cdots,M \\
&= |\psi_{i_0}^{k-1}|.
\end{aligned}
$$

Then $|\psi_{i_0}^k| \leqslant |\psi_{i_0}^{k-1}| \leqslant \|E^{k-1}\|_\infty$. Hence, $\|E^k\|_\infty \leqslant \|E^{k-1}\|_\infty \leqslant \cdots \leqslant \|E^0\|_\infty$, which ends the proof of the theorem. □

4.2.2. Study of the convergence

Let us assume that the problem (4.1)-(4.4) has $C(x,t)$ has a solution and C_i^k the numerical approximation of $C(x_i,t_k)$. Set $\Upsilon_i^k = C(x_i,t_k) - C_i^k$, $i=1,2,\cdots,N-1$ and $k=1,2,\cdots,n$. We then have

$$
(1+\lambda\tau)\Upsilon_i^k - q_j^1\sum_{l=0}^{i}\omega_l^{\alpha_j}\Upsilon_{i-l}^k - q^2\sum_{l=0}^{i+1}\omega_l^2\psi_{i+1-l}^k = \Upsilon_i^{k-1} + A
$$

for $j=0,1,\cdots,M$. We have

$$
\begin{aligned}
|A| &= |(-\tau A_t - \frac{\tau\nu}{R}A_1 + D\frac{\tau A_2}{R})| \\
&\leqslant |\tau A_t| + |\frac{\tau\nu}{R}A_1| + |D\frac{\tau A_2}{R}| \\
&\leqslant \tau\tau K_0 + \frac{\tau\nu}{R}K_1 h + D\frac{\tau A_2}{R}K_2 h \\
&\leqslant K^0\tau(2h+\tau).
\end{aligned}
$$

Let $Y^k = (\Upsilon_1^k, \Upsilon_2^k, \Upsilon_3^k, \cdots, \Upsilon_{N-1}^k)$. We have

$$
Y^0 = (\Upsilon_1^0, \Upsilon_2^0, \Upsilon_3^0, \cdots, \Upsilon_{N-1}^0),
$$

and

$$
\Upsilon_1^0 = C(x_i,t_0) - C_i^0 = C(x_i,0) - C_i^0 = \phi(x_i) - \phi(x_i) = 0,
$$

which implies that $Y^0 = (0,0,\cdots,0)$.

Theorem 4.3 ([19]). *Let us assume that the problem (4.1) has a solution $C(x,t)$ and let C_i^k be the numerical solution computed using (4.9)-(4.12), then there exist a positive constant K independent of i and K such that*

$$
\|Y^k\|_\infty \leqslant Kk\tau(\tau+2h) \tag{4.13}
$$

for $k=1,2,\cdots,n$. Given that $k\tau \leqslant T$, we have $\|Y^k\|_\infty \leqslant KT(\tau+2h)$ which implies that

$$
|C_i^k - C(x_i,t_k)| \leqslant K'(\tau+2h), \quad \textit{for } i=1,2,\cdots,N, \;\; j=1,2,\cdots,n, \;\; K'=KT.
$$

Proof. [19] The proof will be done by induction on k.

- For $k=1$, we have as above the existence of $1 \leqslant i_0 \leqslant N-1$ such that $\|Y^1\|_\infty = |\Upsilon_{i_0}^1|$. We then have

$$
\|Y^1\|_\infty = |\Upsilon_{i_0}^1| \leqslant (1+\lambda\tau)|\Upsilon_{i_0}^1| \leqslant (1+\lambda\tau)|\Upsilon_{i_0}^1| - q_j^1\sum_{l=0}^{i_0}\omega_l^{\alpha_j}|\Upsilon_{i_0-l}^1|
$$

$$
\begin{aligned}
&- q^2 \sum_{l=0}^{i_0+1} \omega_l^2 |\Upsilon^1_{i_0+1-l}| \quad \text{for } j = 0,1,2,\cdots,M \\
&\leqslant |(1+\lambda\tau)\Upsilon^1_{i_0} - q_j^1 \sum_{l=0}^{i_0} \omega_l^{\alpha_j} \Upsilon^1_{i_0-l} \\
&- q^2 \sum_{l=0}^{i_0+1} \omega_l^2 \Upsilon^1_{i_0+1-l}|, \quad \text{for } j = 0,1,2,\cdots,M \\
&= |\Upsilon^0_{i_0} + A| \leqslant \|\Upsilon^0_{i_0}\|_\infty + |A| \\
&\leqslant |A| \leqslant K^0\tau(2h+\tau) \quad \text{since } \|\Upsilon^0_{i_0}\|_\infty = 0,
\end{aligned}
$$

which gives the solution of (4.13) for $k = 1$.

- Let us assume that $\|Y^{k-1}\|_\infty \leqslant K(k-1)\tau(\tau+2h)$, we have

$$
\begin{aligned}
\|Y^k\|_\infty = |\Upsilon^k_{i_0}| \leqslant (1+\lambda\tau)|\Upsilon^k_{i_0}| &\leqslant (1+\lambda\tau)|\Upsilon^k_{i_0}| - q_j^1 \sum_{l=0}^{i_0} \omega_l^{\alpha_j} |\Upsilon^k_{i_0-l}| \\
&- q^2 \sum_{l=0}^{i_0+1} \omega_l^2 |\Upsilon^k_{i_0+1-l}| \quad \text{for } j = 0,1,2,\cdots,M \\
&\leqslant |(1+\lambda\tau)\Upsilon^k_{i_0} - q_j^1 \sum_{l=0}^{i_0} \omega_l^{\alpha_j} \Upsilon^k_{i_0-l} \\
&- q^2 \sum_{l=0}^{i_0+1} \omega_l^2 \Upsilon^k_{i_0+1-l}| \quad \text{for } j = 0,1,2,\cdots,M \\
&= |\Upsilon^{k-1}_{i_0} + A| \\
&\leqslant \|\Upsilon^{k-1}_{i_0}\|_\infty + |A| \\
&\leqslant K^0\tau(2h+\tau) + K(k-1)\tau(\tau+2h) \\
&= Kk\tau(\tau+2h),
\end{aligned}
$$

which proves also that (4.13) for k.

From the fact that $k\tau \leqslant T$ and using (4.13), we have

$$
|C_i^k - C(x_i, t_k)| \leqslant |\Upsilon^k_{i_0}| \leqslant \|Y^k\|_\infty \leqslant Kk\tau(\tau+2h) \leqslant KT(\tau+2h), = K'(\tau+2h),
$$

with $K' = KT$ which ends the proof of the theorem. □

5. Conclusion

We have modelled the movement of groundwater pollution with variable order derivative. Firstly, we presented the new variable order derivative. Secondly, we used the three dimensional plot and the contour plot to show how this new derivative can be applied to function and used to solve problem analytically and numerically. Finally, we presented the advection-dispersion equation with variable order derivative and solved it analytically using the Fourier transform. The numerical analysis of this equation was made using the implicit finite difference scheme from which we studied the stability and convergence of this scheme and we found that the scheme was stable and convergent.

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