Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 5509–5527
Research Article

ISSN : 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Some fixed point results for α-nonexpansive maps on partial metric spaces

Hassen Aydi[a,b], Abdelbasset Felhi[c,*]

[a]Department of Mathematics, College of Education of Jubail, Imam Abdulrahman Bin Faisal University, P. O.: 12020, Industrial Jubail, 31961, Saudi Arabia.

[b]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

[c]Department of Mathematics and Physics, Preparatory Engineering Institute, Bizerte, Carthage University, Tunisia.

Communicated by S. S. Chang

Abstract

In this paper, we prove some fixed point results for a class of α-nonexpansive single and multi-valued mappings in the setting of partial metric spaces. Our results generalize the analogous ones of Vetro [F. Vetro, Filomat, **29** (2015), 2011–2020]. Some examples are presented making our results effective.

Keywords: Fixed point, nonexpansive mapping, partial metric space.
2010 MSC: 47H10, 54H25, 46J10.

1. Introduction and preliminaries

It is well-known that the study of fixed point theorems for nonexpansive mappings has attracted the attention of many researchers, see for example [12–14, 17, 19]. On the other hand, the notion of partial metric spaces was introduced by Matthews [15] in 1994 as a part to study the denotational semantics of dataflow networks which play an important role in constructing models in the theory of computation. Many (common) fixed point results have been provided on partial metric spaces. For more details, see [1, 5–11, 18].

Definition 1.1 ([15]). A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$

(PM1) $p(x, x) = p(x, y) = p(y, y)$, then $x = y$;
(PM2) $p(x, x) \leqslant p(x, y)$;
(PM3) $p(x, y) = p(y, x)$;
(PM4) $p(x, z) + p(y, y) \leqslant p(x, y) + p(y, z)$.

The pair (X, p) is then called a partial metric space (PMS).

According to [15], each partial metric p on X generates a T_0 topology τ_p on X which has as a base of the family of open p-balls $\{B_p(x, \varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for

*Corresponding author
Email addresses: hmaydi@uod.edu.sa (Hassen Aydi), abdelbassetfelhi@gmail.com (Abdelbasset Felhi)

doi:10.22436/jnsa.010.10.32

Received 2016-12-26

all $x \in X$ and $\varepsilon > 0$. Following [15], several topological concepts can be defined as follows. A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$ and is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite. Moreover, a partial metric space (X, p) is said complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$. It is known [15] that if p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

for all $x, y \in X$, is a metric on X.

Note that if a sequence converges in a partial metric space (X, p) with respect to τ_{p^s}, then it converges with respect to τ_p.

Also, a sequence $\{x_n\}$ is Cauchy in a partial metric space (X, p) if and only if it is Cauchy in the metric space (X, p^s). Consequently, a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover, if $\{x_n\}$ is a sequence in a partial metric space (X, p) and $x \in X$, one has that

$$\lim_{n\to\infty} p^s(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Definition 1.2. Let (X, p) be a partial metric space. We say that $T : X \to X$ is (sequentially) continuous if $p(x_n, x) \to p(x, x)$, then $p(Tx_n, Tx) \to p(Tx, Tx)$ as $n \to \infty$.

Lemma 1.3. *Let (X, p) be a partial metric space. Then*

(1) *if $p(x, y) = 0$, we have $x = y$;*
(2) *if $x \neq y$, we have $p(x, y) > 0$.*

Let (X, p) be a partial metric space. We denote by $CB^p(X)$ the family of all nonempty, closed, and bounded subsets of X. For $A, B \in CB^p(X)$ and $x \in X$, we define

$$p(x, A) = \inf\{p(x, a) : a \in A\} \quad \text{and} \quad H_p(A, B) = \max\{\sup_{a\in A} p(a, B), \sup_{b\in B} p(b, A)\}.$$

Now, we introduce the following set.

$$F^p(X) = \{A \subseteq CB^p(X) : \forall x \in X, \exists y \in A : p(x, A) = p(x, y)\}.$$

The set F^p is not empty as illustrated by the following.

Example 1.4. Let $X = [0, 1]$ be equipped with the partial metric $p(x, y) = \max\{x, y\}$. Clearly, the subsets $A = \{0, \frac{1}{3}, \frac{1}{2}\}$ and $A = [0, \frac{1}{4}]$ are in $F^p(X)$.

Remark 1.5.

(1) If (X, p) is a metric space and $K(X)$ is the family of all nonempty compact subsets of X, then $K(X) \subseteq F^p(X)$. Indeed, for $A \subseteq X$, we know that the function $x \longmapsto p(x, A)$ is continuous. So its infimum is achieved on a compact. Then if A is a compact in X, so there exists an element $y \in A$ such that $p(x, A) = p(x, y)$. Hence $A \in F^p(x)$.
(2) Let X be a nonempty finite dimension vectorial space equipped with the norm N. Take the metric $p(x, y) = N(x - y)$. Then the set of closed subsets of X belongs to $F^p(X)$. Indeed, if A is closed in X (which is a finite dimension metric space), so A is bounded and closed. Then A is compact. Similarly, $A \in F^p(x)$.

Lemma 1.6 ([2]). *Let (X, p) be a partial metric space and A any nonempty set in (X, p), then $a \in \overline{A}$ if and only if $p(a, A) = p(a, a)$, where $\overline{A}$ denotes the closure of A with respect to the partial metric p.*

Proposition 1.7 ([3, 4]). *Let (X, p) be a partial metric space. For all $A, B, C \in CB^p(X)$, we have*

(h1) $H_p(A, A) \leqslant H_p(A, B)$;
(h2) $H_p(B, A) = H_p(A, B)$;
(h3) $H_p(A, B) \leqslant H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$;
(h4) $H_p(A, B) = 0 \Rightarrow A = B$.

Recently, Popescu [16] introduced the concept of α-orbital admissible maps.

Definition 1.8 ([16]). For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is α-orbital admissible if for all $x \in X$, we have

$$\alpha(x, Tx) \geqslant 1 \Longrightarrow \alpha(Tx, T^2x) \geqslant 1.$$

Definition 1.9. Let (X, p) be a partial metric space and $T : X \to X$ be a mapping. We say that T is a nonexpansive mapping if

$$p(Tx, Ty) \leqslant p(x, y) \quad \text{for all } x, y \in X.$$

We generalize Definition 1.9 by introduction of the concept of α-nonexpansive mappings.

Definition 1.10. Let (X, p) be a partial metric space. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two given mappings. We say that T is α-nonexpansive if for all $x, y \in X$, we have

$$\alpha(x, y) \geqslant 1 \Longrightarrow p(Tx, Ty) \leqslant p(x, y).$$

The following example illustrates the concept of α-nonexpansive mappings.

Example 1.11. Let $X = [0, \infty)$ be endowed with the partial metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Consider the mapping $T : X \to X$ defined by

$$Tx = \frac{x^2 + x}{2} \quad \text{for all } x \in X.$$

Define $\alpha : X \times X \to [0, \infty)$ as follows

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{if not.} \end{cases}$$

Let $x, y \in X$ be such that $\alpha(x, y) \geqslant 1$. Then $x, y \in [0, 1]$. In this case, we have

$$p(Tx, Ty) = \max\{\frac{x^2 + x}{2}, \frac{y^2 + y}{2}\} \leqslant \max\{x, y\} = p(x, y).$$

Then T is an α-nonexpansive mapping. Note that T is not a nonexpansive mapping. In fact, we have

$$p(T0, T2) = p(0, 3) = 3 > 2 = p(0, 2).$$

We extend Definitions 1.9 and 1.10 to multi-valued mappings.

Definition 1.12. Let (X, p) be a partial metric space. Given $T : X \to CB^p(X)$ and $\alpha : X \times X \to [0, \infty)$. We say that T is an α-nonexpansive multi-valued mapping if for all $x, y \in X$, we have

$$\alpha(x, y) \geqslant 1 \Longrightarrow H_p(Tx, Ty) \leqslant p(x, y).$$

Definition 1.13. Let (X, p) be a partial metric space. Let $T : X \to CB^p(X)$ be a mapping. We say that T is a nonexpansive multi-valued mapping if

$$H_p(Tx, Ty) \leqslant p(x, y) \quad \text{for all } x, y \in X.$$

Definition 1.14. For a nonempty set X, let $T : X \to CB^p(X)$ and $\alpha : X \times X \to [0, \infty)$ be two given mappings. We say that T is α-admissible if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geqslant 1$, we have $\alpha(y, z) \geqslant 1$ for all $z \in Ty$.

We have the following auxiliary lemma.

Lemma 1.15 ([20]). *If $\{a_n\}$ is a nonincreasing sequence of nonnegative real numbers, then $\{\frac{a_n+a_{n+1}}{a_n+a_{n+1}+1}\}$ is nonincreasing too.*

In this paper, we establish some fixed points results in the setting of partial metric spaces by using the concept of α-nonexpansive mappings for single and multi-valued mappings. We present some examples making effective the given results.

2. Main results

2.1. Fixed point theorems for single-valued mappings

The first main result is as following.

Theorem 2.1. *Let (X, p) be a complete partial metric space. Given $\alpha, \beta : X \times X \to [0, \infty)$, let $T : X \to X$ be an α-nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant \left(\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k\right) M(x, y) \tag{2.1}$$

for all $x, y \in X$ satisfying $\beta(x, y) \geqslant 1$, where $k \in [0, 1)$ and

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.$$

Assume that

(i) T *is α-orbital admissible;*
(ii) T *is β-orbital admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, $\beta(x_0, Tx_0) \geqslant 1$, and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1; \tag{2.2}$$

(iv) T *is continuous.*

Then, there exists $z \in X$ such that $p(z, z) = 0$. Assume in addition that

(v) $\alpha(x, x) \geqslant 1$ *for each $x \in X$ such that $p(x, x) = 0$.*

Then such z is a fixed point of T, that is, $Tz = z$. Moreover, if $z, w \in X$ are two distinct fixed points of T satisfying $\beta(z, w) \geqslant 1$, then $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

Proof. By assumption (iii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, $\beta(x_0, Tx_0) \geqslant 1$ and (2.2) holds. Define the sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \geqslant 1$. The mapping T is α-orbital admissible and is β-orbital admissible, so $\alpha(Tx_0, T^2x_0) \geqslant 1$, $\beta(Tx_0, T^2x_0) \geqslant 1$. By induction, we have

$$\alpha(x_n, x_{n+1}) = \alpha(T^n x_0, T^{n+1} x_0) \geqslant 1, \quad \beta(x_n, x_{n+1}) = \beta(T^n x_0, T^{n+1} x_0) \geqslant 1, \quad \forall n \geqslant 0. \tag{2.3}$$

Since $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and the mapping T is α-nonexpansive, we get

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leqslant p(x_{n-1}, x_n), \quad \forall n \geqslant 1. \tag{2.4}$$

Using (2.1) and the fact that $\beta(x_n, x_{n+1}) \geqslant 1$ for all n,

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leqslant (\frac{p(x_{n-1}, Tx_n) + p(Tx_{n-1}, x_n)}{1 + p(x_{n-1}, Tx_{n-1}) + p(x_n, Tx_n)} + k)M(x_{n-1}, x_n),$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n), \frac{1}{2}[p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})]\} \\ &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\}. \end{aligned}$$

Note that

$$\frac{1}{2}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \leqslant \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})].$$

Therefore, by (2.4)

$$M(x_{n-1}, x_n) = \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n).$$

Take

$$\theta := \frac{p(x_0, x_1) + p(x_1, x_2)}{1 + p(x_0, x_1) + p(x_1, x_2)} + k. \tag{2.5}$$

From assumption (2.2), we have $\theta \in [0, 1)$. Since $\{p(x_n, x_{n+1})\}$ is a nonincreasing sequence, by Lemma 1.15, we deduce that

$$\begin{aligned} p(x_n, x_{n+1}) &\leqslant (\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{1 + p(x_{n-1}, x_n) + p(x_n, x_{n+1})} + k)p(x_{n-1}, x_n) \\ &\leqslant (\frac{p(x_0, x_1) + p(x_1, x_2)}{1 + p(x_0, x_1) + p(x_1, x_2)} + k)p(x_{n-1}, x_n) = \theta p(x_{n-1}, x_n) \end{aligned}$$

for all $n \geqslant 1$. By induction, we have

$$p(x_n, x_{n+1}) \leqslant \theta^n p(x_0, x_1), \quad \forall n \geqslant 0.$$

Now, for $m > n \geqslant 0$, we have

$$p(x_n, x_m) \leqslant \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \leqslant \sum_{i=n}^{m-1} \theta^i p(x_{n_0-1}, x_{n_0}) \leqslant p(x_0, x_1) \sum_{i=n}^{\infty} \theta^i \to 0 \text{ as } n \to \infty.$$

Thus,

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

So $\{x_n\}$ is a Cauchy sequence in the complete partial metric space (X, p). Then there exists $z \in X$ such that

$$\lim_{n\to\infty} p(x_n, z) = p(z, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

We obtain $p(z, z) = 0$. Thus, by condition (v), we have $\alpha(z, z) \geqslant 1$. Consequently, since T is an α-nonexpansive mapping,

$$p(Tz, Tz) \leqslant p(z, z) = 0,$$

which implies that $p(Tz, Tz) = 0$. The mapping T is continuous at z, so

$$p(z, Tz) = \lim_{n\to\infty} p(x_{n+1}, Tz) = \lim_{n\to\infty} p(Tx_n, Tz) = p(Tz, Tz) = 0,$$

that is, $p(z, Tz) = 0$, i.e., $z = Tz$ and so z is a fixed point of T. Now, suppose that $z, w \in X$ are two distinct fixed points of T satisfying $\beta(z, w) \geqslant 1$, then by (2.1), we have

$$p(z, w) = p(Tz, Tw) \leqslant (\frac{p(z, Tw) + p(Tz, w)}{1 + p(z, Tz) + p(w, Tw)} + k)M(z, w),$$

where

$$\begin{aligned} M(z, w) &= \max\{p(z, w), p(z, Tz), p(w, Tw), \frac{1}{2}[p(w, Tz) + p(z, Tw)]\} \\ &= \max\{p(z, w), p(z, z), p(w, w)\} = p(z, w). \end{aligned}$$

Therefore

$$0 < p(z, w) \leqslant (\frac{2p(z, w)}{1 + p(z, z) + p(w, w)} + k)p(z, w),$$

which implies that

$$\frac{2p(z, w)}{1 + p(z, z) + p(w, w)} + k > 1,$$

that is, $\frac{p(z,w)}{1+p(z,z)+p(w,w)} > \frac{1-k}{2}$. This ends the proof of Theorem 2.1. □

In the next result, we replace the continuity hypothesis by the following condition:

(H) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in (X, p)$ as $n \to \infty$ with $p(x, x) = 0$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k.

Theorem 2.2. *Let (X, p) be a complete partial metric space. Given $\alpha, \beta : X \times X \to [0, \infty)$, let $T : X \to X$ be an α-nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant (\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k)M(x, y)$$

for all $x, y \in X$ satisfying $\beta(x, y) \geqslant 1$, where $k \in [0, 1)$ and

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.$$

Assume that

(i) *T is α-orbital admissible;*
(ii) *T is β-orbital admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, $\beta(x_0, Tx_0) \geqslant 1$, and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1;$$

(iv) *(H) holds.*

Then T has a fixed point. Moreover, if $z, w \in X$ are two distinct fixed points of T satisfying $\beta(z, w) \geqslant 1$, then

$$\frac{p(z, w)}{1 + p(z, z) + p(w, w)} \geqslant \frac{1 - k}{2}.$$

Proof. Following the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ in X such that (2.3) holds. Also, $\{x_n\}$ is Cauchy sequence. Since (X, p) is a complete partial metric space, then there exists $z \in X$ such that

$$\lim_{n\to\infty} p(x_n, z) = p(z, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

By hypothesis (H), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geqslant 1$ for all k. Now, since

T is an α-nonexpansive mapping, then

$$p(x_{n(k)+1}, Tz) = p(Tx_{n(k)}, Tz) \leqslant p(x_{n(k)}, z), \quad \forall k \geqslant 1.$$

Passing to limit as $k \to \infty$ in the above inequality, we get $p(z, Tz) \leqslant p(z, z) = 0$. This implies that $p(z, Tz) = 0$, that is z is a fixed point of T. The rest of proof is similar to Theorem 2.1. □

We provide the following example.

Example 2.3. Let $X = [0, \infty)$ be endowed with the partial metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Clearly, (X, p) is complete. Consider the mapping $T : X \to X$ defined by

$$Tx = x^2 \quad \text{for all } x \in X.$$

Take $k = \frac{1}{2}$. Define $\alpha, \beta : X \times X \to [0, \infty)$ as follows

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{if not.} \end{cases}$$

Let $x \in X$ be such that $\alpha(x, Tx) \geqslant 1$ and $\beta(x, Tx) \geqslant 1$. Then $x \in [0, 1]$. Hence $(Tx, T(Tx)) = (x^2, x^4) \in [0, 1]^2$ and so $\alpha(Tx, T^2x) \geqslant 1$ and $\beta(Tx, T^2x) \geqslant 1$. This implies that T is α-orbital and β-orbital admissible. Let $x, y \in X$ such that $\alpha(x, y) \geqslant 1$. Then $x, y \in [0, 1]$. In this case, we have

$$p(Tx, Ty) = \max\{x^2, y^2\} \leqslant \max\{x, y\} = p(x, y).$$

Then T is an α-nonexpansive mapping. Now, let $x, y \in X$ such that $\beta(x, y) \geqslant 1$. Then $x, y \in [0, 1]$. Without loss of generality, take $x \geqslant y > 0$. We have the following cases:

Case 1: If $x^2 \geqslant y$, we have

$$p(Tx, Ty) \leqslant \Big(\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k\Big)M(x, y) \Leftrightarrow x^2 \leqslant \Big(\frac{x + x^2}{1 + x + y} + k\Big)x \Leftrightarrow 2xy \leqslant 1 + x + y.$$

Clearly, this holds because $2xy \leqslant x^2 + y^2 \leqslant x + y$.

Case 2: If $x^2 < y$, we have

$$p(Tx, Ty) \leqslant \Big(\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k\Big)M(x, y) \Leftrightarrow x^2 \leqslant \Big(\frac{x + y}{1 + x + y} + k\Big)x \Leftrightarrow 2x^2 + 2xy \leqslant 1 + x + 3y.$$

Again, the above holds since $2xy \leqslant x^2 + y^2 \leqslant x + 1$ and $2x^2 \leqslant 2y \leqslant 3y$.

Thus, (2.1) is verified for all $x, y \in X$ satisfying $\beta(x, y) \geqslant 1$. Moreover, the condition (H) holds. For $x_0 = \frac{1}{2}$, we have $\alpha(x_0, Tx_0) = \alpha(\frac{1}{2}, \frac{1}{4}) = 1$ and $\beta(x_0, Tx_0) = \beta(\frac{1}{2}, \frac{1}{4}) = 1$. On the other hand,

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k = \frac{p(\frac{1}{2}, \frac{1}{4}) + p(\frac{1}{4}, \frac{1}{16})}{1 + p(\frac{1}{2}, \frac{1}{4}) + p(\frac{1}{4}, \frac{1}{16})} + k = \frac{\frac{1}{2} + \frac{1}{4}}{1 + \frac{1}{2} + \frac{1}{4}} + \frac{1}{2} = \frac{3}{7} + \frac{1}{2} = \frac{13}{14} < 1.$$

Hence all hypotheses of Theorem 2.1 are verified. Here T has two fixed points which are $z = 0$ and $z = 1$. Moreover, since $\beta(0, 1) = 1$, we have

$$\frac{p(0, 1)}{1 + p(0, 0) + p(1, 1)} = \frac{1}{2} > \frac{1}{4} = \frac{1 - k}{2}.$$

Note that T is not a Banach contraction in (X, p) because $p(T0, T1) = p(0, 1) = 1$. Moreover, T is not a nonexpansive mapping for the usual metric $d(x, y) = |x - y|$. In fact,

$$d(T\frac{1}{2}, T1) = \frac{3}{4} > \frac{1}{2} = d(\frac{1}{2}, 1).$$

Then Theorem 3.1 in [20] is not applicable.

We also state the following results.

Theorem 2.4. *Let* (X, p) *be a complete partial metric space. Given* $\alpha, \beta : X \times X \to [0, \infty)$, *let* $T : X \to X$ *be an* α*-nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant \Big(\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k\Big)E(x, y) \tag{2.6}$$

for all $x, y \in X$ *satisfying* $\beta(x, y) \geqslant 1$, *where* $k \in [0, 1)$ *and*

$$E(x, y) = p(x, y) + |p(x, Tx) - p(y, Ty)|.$$

Assume that

(i) T *is* α*-orbital admissible;*

(ii) T *is* β*-orbital admissible;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$, $\beta(x_0, Tx_0) \geqslant 1$, *and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1; \tag{2.7}$$

(iv) T *is a continuous mapping.*

Then there exists $z \in X$ *such that* $p(z, z) = 0$. *Assume in addition that*

(v) $\alpha(x, x) \geqslant 1$ *for each* $x \in X$ *such that* $p(x, x) = 0$.

Then such z *is a fixed point of* T, *that is,* $Tz = z$. *Moreover, if* $z, w \in X$ *are two distinct fixed points of* T *satisfying* $\beta(z, w) \geqslant 1$, $p(z, z) = p(w, w) = 0$, *then* $p(z, w) \geqslant \frac{1-k}{2}$.

Proof. By assumption (iii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, $\beta(x_0, Tx_0) \geqslant 1$ and (2.7) holds. Define the sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \geqslant 1$. Similar to Theorem 2.1, we have

$$\alpha(x_n, x_{n+1}) = \alpha(T^n x_0, T^{n+1}x_0) \geqslant 1,\ \beta(x_n, x_{n+1}) = \beta(T^n x_0, T^{n+1}x_0) \geqslant 1, \quad \forall n \geqslant 0,$$

and

$$p(x_n, x_{n+1}) \leqslant p(x_{n-1}, x_n), \quad \forall n \geqslant 1. \tag{2.8}$$

By (2.6) and the fact that $\beta(x_n, x_{n+1}) \geqslant 1$ for all n, we have

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leqslant \Big(\frac{p(x_{n-1}, Tx_n) + p(Tx_{n-1}, x_n)}{1 + p(x_{n-1}, Tx_{n-1}) + p(x_n, Tx_n)} + k\Big)E(x_{n-1}, x_n),$$

where

$$\begin{aligned} E(x_{n-1}, x_n) &= p(x_{n-1}, x_n) + |p(x_{n-1}, Tx_{n-1}) - p(x_n, Tx_n)| \\ &= p(x_{n-1}, x_n) + |p(x_{n-1}, x_n) - p(x_n, x_{n+1})|. \end{aligned}$$

Therefore, by (2.8)

$$E(x_{n-1}, x_n) = 2p(x_{n-1}, x_n) - p(x_n, x_{n+1}).$$

Since $\{p(x_n, x_{n+1})\}$ is a nonincreasing sequence, by Lemma 1.15, we deduce that

$$\begin{aligned} p(x_n, x_{n+1}) &\leqslant \Big(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{1 + p(x_{n-1}, x_n) + p(x_n, x_{n+1})} + k\Big)E(x_{n-1}, x_n) \\ &\leqslant \Big(\frac{p(x_0, x_1) + p(x_1, x_2)}{1 + p(x_0, x_1) + p(x_1, x_2)} + k\Big)E(x_{n-1}, x_n) \\ &= \theta[2p(x_{n-1}, x_n) - p(x_n, x_{n+1})], \end{aligned}$$

where θ is defined by (2.5). Again, from assumption (2.7), we have $\theta \in [0,1)$. Finally, we get

$$p(x_n, x_{n+1}) \leqslant \frac{2\theta}{1+\theta} p(x_{n-1}, x_n).$$

Put

$$\lambda = \frac{2\theta}{1+\theta}.$$

Note that

$$0 < \theta < 1 \Leftrightarrow 0 < \lambda < 1.$$

We have

$$p(x_n, x_{n+1}) \leqslant \lambda p(x_{n-1}, x_n)$$

for all $n \geqslant 1$. Proceeding as in the proof of Theorem 2.1, we conclude that $\{x_n\}$ is a Cauchy sequence in the complete partial metric space (X, p). Then there exists $z \in X$ such that

$$\lim_{n\to\infty} p(x_n, z) = p(z,z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Also, we show that z is a fixed point of T. Now, suppose that $z, w \in X$ are two distinct fixed points of T satisfying $\beta(z,w) \geqslant 1$, then by (2.6), we have

$$p(z,w) = p(Tz, Tw) \leqslant (\frac{p(z,Tw) + p(Tz,w)}{1 + p(z,Tz) + p(w,Tw)} + k)E(z,w),$$

where

$$E(z,w) = p(z,w) + |p(z,Tz) - p(w,Tw)| = p(z,w) + |p(z,z) - p(w,w)| = p(z,w).$$

Therefore

$$0 < p(z,w) \leqslant (2p(z,w) + k)p(z,w),$$

which implies that

$$p(z,w) \geqslant \frac{1-k}{2}.$$

□

Similar to Theorem 2.2, we state the following.

Theorem 2.5. *Let* (X,p) *be a complete partial metric space. Given* $\alpha, \beta : X \times X \to [0,\infty)$, *let* $T : X \to X$ *be an* α*-nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant (\frac{p(x,Ty) + p(Tx,y)}{1 + p(x,Tx) + p(y,Ty)} + k)E(x,y)$$

for all $x, y \in X$ *satisfying* $\beta(x,y) \geqslant 1$, *where* $k \in [0,1)$ *and*

$$E(x,y) = p(x,y) + |p(x,Tx) - p(y,Ty)|.$$

Assume that

(i) T *is* α*-orbital admissible;*
(ii) T *is* β*-orbital admissible;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$, $\beta(x_0, Tx_0) \geqslant 1$, *and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1;$$

(iv) (H) *holds.*

Then T *has a fixed point. Moreover, if* $z, w \in X$ *are two distinct fixed points of* T *satisfying* $\beta(z,w) \geqslant 1$ *and* $p(z,z) = p(w,w) = 0$, *then*

$$p(z,w) \geqslant \frac{1-k}{2}.$$

2.2. Consequences

In this section, we present some consequences of our obtained results.

2.2.1. Some classical fixed point results

We have the following results.

Corollary 2.6. *Let* (X, p) *be a complete partial metric space. Given* $\alpha, \beta : X \times X \to [0, \infty)$, *let* $T : X \to X$ *be an* α*-nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant (\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k)p(x, y)$$

for all $x, y \in X$ *satisfying* $\beta(x, y) \geqslant 1$, *where* $k \in [0, 1)$. *Assume that*

(i) T *is* α*-orbital admissible;*

(ii) T *is* β*-orbital admissible;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$, $\beta(x_0, Tx_0) \geqslant 1$, *and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1;$$

(iv) T *is continuous.*

Then, there exists $z \in X$ *such that* $p(z, z) = 0$. *Assume in addition that*

(v) $\alpha(x, x) \geqslant 1$ *for each* $x \in X$ *such that* $p(x, x) = 0$.

Then such z *is a fixed point of* T, *that is,* $Tz = z$. *Moreover, if* $z, w \in X$ *are two distinct fixed points of* T *satisfying* $\beta(z, w) \geqslant 1$, *then* $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

Corollary 2.7. *Let* (X, p) *be a complete partial metric space. Given* $\alpha, \beta : X \times X \to [0, \infty)$, *let* $T : X \to X$ *be an* α*-nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant (\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k)p(x, y)$$

for all $x, y \in X$ *satisfying* $\beta(x, y) \geqslant 1$, *where* $k \in [0, 1)$. *Assume that*

(i) T *is* α*-orbital admissible;*

(ii) T *is* β*-orbital admissible;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$, $\beta(x_0, Tx_0) \geqslant 1$, *and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1;$$

(iv) (H) *holds.*

Then T *has a fixed point. Moreover, if* $z, w \in X$ *are two distinct fixed points of* T *satisfying* $\beta(z, w) \geqslant 1$, *then*

$$\frac{p(z, w)}{1 + p(z, z) + p(w, w)} \geqslant \frac{1 - k}{2}.$$

Corollary 2.8. *Let* (X, p) *be a complete partial metric space. Given* $\beta : X \times X \to [0, \infty)$, *let* $T : X \to X$ *be a nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant (\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k)M(x, y)$$

for all $x, y \in X$ *satisfying* $\beta(x, y) \geqslant 1$, *where* $k \in [0, 1)$ *and*

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.$$

Assume that

(i) T *is* β*-orbital admissible;*

(ii) *there exists* $x_0 \in X$ *such that* $\beta(x_0, Tx_0) \geqslant 1$ *and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1.$$

Then T *has a fixed point. Moreover, if* $z, w \in X$ *are two distinct fixed points of* T *satisfying* $\beta(z, w) \geqslant 1$*, then* $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

Proof. It suffices to take $\alpha(x, y) = 1$ in Theorem 2.1. □

Corollary 2.9. *Let* (X, p) *be a complete partial metric space. Given* $\beta : X \times X \to [0, \infty)$*, let* $T : X \to X$ *be a nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant (\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k)p(x, y)$$

for all $x, y \in X$ *satisfying* $\beta(x, y) \geqslant 1$*, where* $k \in [0, 1)$*. Assume that*

(i) T *is* β*-orbital admissible;*

(ii) *there exists* $x_0 \in X$ *such that* $\beta(x_0, Tx_0) \geqslant 1$ *and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1.$$

Then T *has a fixed point. Moreover, if* $z, w \in X$ *are two distinct fixed points of* T *satisfying* $\beta(z, w) \geqslant 1$*, then* $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

Corollary 2.10. *Let* (X, p) *be a complete partial metric space. Let* $T : X \to X$ *be a nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant (\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k)M(x, y)$$

for all $x, y \in X$*, where* $k \in [0, 1)$ *and*

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.$$

If there exists $x_0 \in X$ *such that*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1,$$

then T *has a fixed point. Moreover, if* $z, w \in X$ *are two distinct fixed points of* T*, then* $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

Proof. It suffices to take $\beta(x, y) = 1$ in Corollary 2.8. □

Corollary 2.11. *Let* (X, p) *be a complete partial metric space. Let* $T : X \to X$ *be a nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant (\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k)E(x, y)$$

for all $x, y \in X$*, where* $k \in [0, 1)$ *and*

$$E(x, y) = p(x, y) + |p(x, Tx) - p(y, Ty)|.$$

If there exists $x_0 \in X$ *such that*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1,$$

then T *has a fixed point. Moreover, if* $z, w \in X$ *are two distinct fixed points of* T *such that* $p(z, z) = p(w, w) = 0$*, then* $p(z, w) \geqslant \frac{1-k}{2}$.

2.2.2. Some fixed point results on a partial metric endowed with an arbitrary binary relation

Definition 2.12. Let (X, p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. We say that $T : X \to X$ is a preserving mapping if for each $x \in X$ such that $x \,\mathcal{R}\, Tx$, we have $Tx \,\mathcal{R}\, T^2x$.

We state the following.

Corollary 2.13. *Let (X, p) be a complete partial metric space endowed with a binary relation $\mathcal{R}$ over X. Let $T : X \to X$ be a nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant \Big(\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k\Big)M(x, y)$$

for all $x, y \in X$ satisfying $x \,\mathcal{R}\, y$, where $k \in [0, 1)$ and

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.$$

Assume that

(i) T *is a preserving mapping;*

(ii) *there exists $x_0 \in X$ such that $x_0 \mathcal{R} Tx_0$ and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1.$$

Then T has a fixed point of T. Moreover, if $z, w \in X$ are two distinct fixed points of T satisfying $z\mathcal{R}w$ or $w\mathcal{R}z$, then $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

Proof. It suffices to consider $\beta : X \times X \to [0, \infty)$ such that

$$\beta(x, y) = \begin{cases} 1, & \text{if } x\mathcal{R}y, \\ 0, & \text{if not.} \end{cases}$$

From condition (ii), we get $\beta(x_0, Tx_0) \geqslant 1$. T is a preserving mapping, so T is β-orbital admissible. Thus all hypotheses of Corollary 2.8 are satisfied. This completes the proof. □

Corollary 2.14. *Let (X, p) be a complete partial metric space endowed with a binary relation $\mathcal{R}$ over X. Let $T : X \to X$ be a nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant \Big(\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k\Big)p(x, y)$$

for all $x, y \in X$ satisfying $x \,\mathcal{R}\, y$, where $k \in [0, 1)$. Assume that

(i) T *is a preserving mapping;*

(ii) *there exists $x_0 \in X$ such that $x_0 \,\mathcal{R}\, Tx_0$ and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1.$$

Then T has a fixed point of T. Moreover, if $z, w \in X$ are two distinct fixed points of T satisfying $z\mathcal{R}w$ or $w\mathcal{R}z$, then $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

2.2.3. Some fixed point results on a partial metric endowed with a partial order

Definition 2.15. Let X be a nonempty set. We say that $(X, p, \preceq)$ is a partially ordered partial metric space if (X, p) is a partial metric space and $(X, \preceq)$ is a partially ordered set.

Definition 2.16. Let X be a nonempty set endowed with the partial order $\preceq$. The mapping $T : X \to X$ is said non-decreasing if

$$(x, y) \in X \times X, \; x \preceq y \Rightarrow Tx \preceq Ty.$$

Corollary 2.17. *Let (X, p) be a complete partial metric space endowed with the partial order $\preceq$. Let $T : X \to X$ be a nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant \left(\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k\right)M(x, y)$$

for all $x, y \in X$ satisfying $x \preceq y$, where $k \in [0, 1)$ and

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.$$

Assume that

(i) T *is non-decreasing;*
(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1.$$

Then T has a fixed point of T. Moreover, if $z, w \in X$ are two distinct fixed points of T satisfying $z \preceq w$ or $w \preceq z$, then $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

Proof. It suffices to consider the binary relation $\mathcal{R}$ over X as

$$x\mathcal{R}y \Leftrightarrow x \preceq y.$$

All hypotheses of Corollary 2.13 are satisfied. □

Corollary 2.18. *Let (X, p) be a complete partial metric space endowed with the partial order $\preceq$. Let $T : X \to X$ be a nonexpansive mapping such that*

$$p(Tx, Ty) \leqslant \left(\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k\right)p(x, y)$$

for all $x, y \in X$ satisfying $x \preceq y$, where $k \in [0, 1)$. Assume that

(i) T *is non-decreasing;*
(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and*

$$\frac{p(x_0, Tx_0) + p(Tx_0, T^2x_0)}{1 + p(x_0, Tx_0) + p(Tx_0, T^2x_0)} + k < 1.$$

Then T has a fixed point of T. Moreover, if $z, w \in X$ are two distinct fixed points of T satisfying $z \preceq w$ or $w \preceq z$, then $\frac{p(z,w)}{1+p(z,z)+p(w,w)} \geqslant \frac{1-k}{2}$.

2.3. *Fixed point theorems for multi-valued mappings*

In this section, we give some fixed point results for a class of multi-valued mappings in the setting of partial metric spaces, by using the concept of α-nonexpansive multi-valued mappings.

Theorem 2.19. *Let (X, p) be a complete partial metric space. Given $\alpha, \beta : X \times X \to [0, \infty)$, let $T : X \to F^p(X)$ be* *an α-nonexpansive multi-valued mapping. Assume that*

(i)
$$H_p(Tx,Ty) \leqslant \Big(\frac{p(x,Ty)+p(Tx,y)}{1+p(x,Tx)+p(y,Ty)}+k\Big)M(x,y) \tag{2.9}$$

for all $x,y \in X$ *satisfying* $\beta(x,y) \geqslant 1$, *where* $k \in [0,1)$ *and*

$$M(x,y) = \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(Tx,y)]\};$$

(ii) T *is α-admissible;*

(iii) T *is β-admissible;*

(iv) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0,x_1) \geqslant 1$, $\beta(x_0,x_1) \geqslant 1$, $p(x_0,x_1) = p(x_0,Tx_0)$, *and*

$$\frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{1+p(x_0,Tx_0)+p(x_1,Tx_1)}+k<1; \tag{2.10}$$

(v) T *is continuous.*

Then, there exists $z \in X$ *such that* $p(z,z)=0$. *Assume in addition that*

(vi) $\alpha(x,x) \geqslant 1$ *for each* $x \in X$ *such that* $p(x,x)=0$.

Then such z *is a fixed point of* T, *that is* $z \in Tz$.

Proof. By assumption (iv), there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geqslant 1$, $\beta(x_0,x_1) \geqslant 1$, $p(x_0,x_1) = p(x_0,Tx_0)$ and (2.10) holds. We have $Tx_1 \subseteq F^p(X)$, so there exists $x_2 \in Tx_1$ such that $p(x_1,x_2)=p(x_1,Tx_1)$. The mapping T is α-admissible and is β-admissible, then as $x_2 \in Tx_1$, so $\alpha(x_1,x_2) \geqslant 1$ and $\beta(x_1,x_2) \geqslant 1$. The mapping T is α-nonexpansive, hence

$$p(x_1,x_2) = p(x_1,Tx_1) \leqslant H_p(Tx_0,Tx_1) \leqslant p(x_0,x_1). \tag{2.11}$$

Again, from (2.9), we have

$$p(x_1,x_2) = p(x_1,Tx_1) \leqslant H_p(Tx_0,Tx_1) \leqslant \Big(\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{1+p(x_0,Tx_0)+p(x_1,Tx_1)}+k\Big)M(x_0,x_1),$$

where

$$\begin{aligned} M(x_0,x_1) &= \max\{p(x_0,x_1), p(x_1,Tx_1), p(x_0,Tx_0), \frac{1}{2}[p(x_0,Tx_1)+p(x_1,Tx_0)]\} \\ &= \max\{p(x_0,x_1), p(x_1,Tx_1), \frac{1}{2}[p(x_0,Tx_1)+p(x_1,Tx_0)]\}. \end{aligned}$$

Note that

$$\frac{1}{2}[p(x_0,Tx_1)+p(x_1,Tx_0)] \leqslant \frac{1}{2}[p(x_0,Tx_1)+p(x_1,x_1)] \leqslant \frac{1}{2}[p(x_0,x_1)+p(x_1,Tx_1)].$$

Therefore, by (2.11)

$$M(x_0,x_1) = \max\{p(x_0,x_1), p(x_1,Tx_1)\} = p(x_0,x_1).$$

Thus

$$p(x_1,x_2) \leqslant \Big(\frac{p(x_0,x_1)+p(x_1,Tx_1)}{1+p(x_0,Tx_0)+p(x_1,Tx_1)}+k\Big)p(x_0,x_1) = \Big(\frac{p(x_0,x_1)+p(x_1,x_2)}{1+p(x_0,x_1)+p(x_1,x_2)}+k\Big)p(x_0,x_1).$$

Having in mind $Tx_2 \subseteq F^p(X)$, there exists $x_3 \in Tx_2$ such that $p(x_2,x_3)=p(x_2,Tx_2)$. Then by assumptions

(ii) and (iii), we have $\alpha(x_2, x_3) \geqslant 1$, $\beta(x_2, x_3) \geqslant 1$. Again, T is α-nonexpansive and

$$p(x_2, x_3) = p(x_2, Tx_2) \leqslant H_p(Tx_1, Tx_2) \leqslant p(x_1, x_2).$$

From (2.9), we have

$$p(x_2, x_3) = p(x_2, Tx_2) \leqslant H_p(Tx_1, Tx_2) \leqslant (\frac{p(x_1, Tx_2) + p(x_2, Tx_1)}{1 + p(x_1, Tx_1) + p(x_2, Tx_2)} + k)M(x_1, x_2).$$

Similarly, we get

$$p(x_2, x_3) \leqslant (\frac{p(x_1, x_2) + p(x_2, x_3)}{1 + p(x_1, x_2) + p(x_2, x_3)} + k)p(x_1, x_2).$$

Continuing in this process, we construct a sequence $\{x_n\}$ in X such that

$$\alpha(x_n, x_{n+1}) \geqslant 1,\ \beta(x_n, x_{n+1}) \geqslant 1,\ p(x_n, x_{n+1}) = p(x_n, Tx_n) \quad \text{and } x_{n+1} \in Tx_n \quad \text{for all } n \geqslant 0. \tag{2.12}$$

Also

$$p(x_n, x_{n+1}) \leqslant p(x_{n-1}, x_n) \quad \text{for all } n \geqslant 1, \tag{2.13}$$

$$p(x_n, x_{n+1}) \leqslant (\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{1 + p(x_{n-1}, x_n) + p(x_n, x_{n+1})} + k)p(x_{n-1}, x_n) \quad \text{for all } n \geqslant 1. \tag{2.14}$$

Following the proof of Theorem 2.1, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, p). By completeness of (X, p), there exists $z \in X$ such

$$\lim_{n\to\infty} p(x_n, z) = p(z, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Since $p(z, z) = 0$, by condition (vi), we have $\alpha(z, z) \geqslant 1$. Consequently, in view of the fact that T is an α-nonexpansive mapping,

$$H_p(Tz, Tz) \leqslant p(z, z) = 0,$$

which implies that $H_p(Tz, Tz) = 0$. The mapping T is continuous at z, so

$$\lim_{n\to\infty} H_p(Tx_n, Tz) = H_p(Tz, Tz) = 0.$$

On the other hand

$$p(z, Tz) \leqslant p(z, x_{n+1}) + p(x_{n+1}, Tz) \leqslant p(z, x_{n+1}) + H_p(Tx_n, Tz) \quad \text{for all } n \geqslant 0.$$

Passing to limit as $n \to \infty$ in the above inequality, we get

$$p(z, Tz) \leqslant p(z, z) + H_p(Tz, Tz) = 0,$$

which implies that $p(z, Tz) = 0 = p(z, z)$ and so $z \in \overline{Tz}(= Tz)$, that is, z is a fixed point of T. □

Theorem 2.20. *Let* (X, p) *be a complete partial metric space. Given* $\alpha, \beta : X \times X \to [0, \infty)$, *let* $T : X \to F^p(X)$ *be an* α*-nonexpansive multi-valued mapping. Assume that*

(i)

$$H_p(Tx, Ty) \leqslant (\frac{p(x, Ty) + p(Tx, y)}{1 + p(x, Tx) + p(y, Ty)} + k)M(x, y)$$

for all $x, y \in X$ *satisfying* $\beta(x, y) \geqslant 1$, *where* $k \in [0, 1)$ *and*

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\};$$

(ii) T *is* α*-admissible;*
(iii) T *is* β*-admissible;*
(iv) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geqslant 1$, $\beta(x_0, x_1) \geqslant 1$, $p(x_0, x_1) = p(x_0, Tx_0)$, *and*

$$\frac{p(x_0, Tx_0) + p(x_1, Tx_1)}{1 + p(x_0, Tx_0) + p(x_1, Tx_1)} + k < 1;$$

(v) (H) *holds.*

Then T *has a fixed point.*

Proof. Proceeding as in the proof of Theorem 2.19, we construct a sequence $\{x_n\}$ in X such that (2.12), (2.13), and (2.14) hold. Also, $\{x_n\}$ is a Cauchy sequence in (X, p). By completeness of (X, p), there exists $z \in X$ such

$$\lim_{n\to\infty} p(x_n, z) = p(z, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

By hypothesis (H) and since T is an α-nonexpansive multi-valued mapping, we have

$$\begin{aligned} p(z, Tz) &\leqslant p(z, x_{n(k)+1}) + p(x_{n+1}, Tz) \\ &\leqslant p(z, x_{n(k)+1}) + H_p(Tx_{n(k)}, Tz) \\ &\leqslant p(z, x_{n(k)+1}) + p(x_{n(k)}, z) \quad \text{for all } k \geqslant 0. \end{aligned}$$

Passing to limit as $k \to \infty$ in the above inequality, we get

$$p(z, Tz) \leqslant 2p(z, z) = 0,$$

which implies that $p(z, Tz) = 0 = p(z, z)$ and so $z \in \overline{Tz}(= Tz)$. □

Using the same techniques, we have the following results.

Theorem 2.21. *Let* (X, p) *be a complete partial metric space. Given* $\alpha, \beta : X \times X \to [0, \infty)$, *let* $T : X \to F^p(X)$ *be an* α*-nonexpansive multi-valued mapping. Assume that*

(i)

$$H_p(Tx, Ty) \leqslant \Big(\frac{p(x, Ty) + p(y, Tx)}{1 + p(x, Tx) + p(y, Ty)} + k\Big)E(x, y)$$

for all $x, y \in X$ *satisfying* $\beta(x, y) \geqslant 1$, *where* $k \in [0, 1)$ *and*

$$E(x, y) = p(x, y) + |p(x, Tx) - p(y, Ty)|;$$

(ii) T *is* α*-admissible;*
(iii) T *is* β*-admissible;*
(iv) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geqslant 1$, $\beta(x_0, x_1) \geqslant 1$, $p(x_0, x_1) = p(x_0, Tx_0)$, *and*

$$\frac{p(x_0, Tx_0) + p(x_1, Tx_1)}{1 + p(x_0, Tx_0) + p(x_1, Tx_1)} + k < 1;$$

(v) T *is continuous.*

Then, there exists $z \in X$ *such that* $p(z, z) = 0$. *Assume in addition that*

(vi) $\alpha(x, x) \geqslant 1$ *for each* $x \in X$ *such that* $p(x, x) = 0$.

Then, such z *is a fixed point of* T, *that is,* $z \in Tz$.

Theorem 2.22. *Let* (X, p) *be a complete partial metric space. Given* $\alpha, \beta : X \times X \to [0, \infty)$, *let* $T : X \to F^p(X)$ *be* *an α-nonexpansive multi-valued mapping. Assume that*

(i)
$$H_p(Tx,Ty) \leqslant \left(\frac{p(x,Ty)+p(y,Tx)}{1+p(x,Tx)+p(y,Ty)}+k\right)E(x,y)$$
for all $x,y \in X$ *satisfying* $\beta(x,y) \geqslant 1$, *where* $k \in [0,1)$, *and*
$$E(x,y) = p(x,y) + |p(x,Tx) - p(y,Ty)|;$$

(ii) T *is* α*-admissible;*

(iii) T *is* β*-admissible;*

(iv) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0,x_1) \geqslant 1$, $\beta(x_0,x_1) \geqslant 1$, $p(x_0,x_1) = p(x_0,Tx_0)$ *and*
$$\frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{1+p(x_0,Tx_0)+p(x_1,Tx_1)}+k < 1;$$

(v) (H) *holds.*

Then T *has a fixed point.*

Example 2.23. Let $X = [0,1]$ be endowed with the partial metric $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. Consider the mapping $T : X \to F^p(X)$ defined by
$$Tx = \{0,x^2\} \quad \text{for all } x \in X.$$
Take $k = \frac{1}{2}$. Define $\alpha,\beta : X \times X \to [0,\infty)$ as follows
$$\alpha(x,y) = \beta(x,y) = 1 \quad \text{for all } x \in X.$$
Let $x,y \in X$. Without loss of generality, take $x \geqslant y > 0$. In this case, we have
$$H_p(Tx,Ty) = x^2 \leqslant x = \max\{x,y\} = p(x,y).$$
Then T is a nonexpansive mapping. We have the following cases:

Case 1: If $x^2 \geqslant y$, we have
$$H_p(Tx,Ty) \leqslant \left(\frac{p(x,Ty)+p(Tx,y)}{1+p(x,Tx)+p(y,Ty)}+k\right)M(x,y) \Leftrightarrow x^2 \leqslant \left(\frac{x+x^2}{1+x+y}+k\right)x \Leftrightarrow 2xy \leqslant 1+x+y.$$
Clearly, this holds due to the fact $2xy \leqslant x^2+y^2 \leqslant x+y$.

Case 2: If $x^2 < y$, we have
$$H_p(Tx,Ty) \leqslant \left(\frac{p(x,Ty)+p(Tx,y)}{1+p(x,Tx)+p(y,Ty)}+k\right)M(x,y) \Leftrightarrow x^2 \leqslant \left(\frac{x+y}{1+x+y}+k\right)x \Leftrightarrow 2x^2+2xy \leqslant 1+x+3y.$$
The above holds because $2xy \leqslant x^2+y^2 \leqslant x+1$ and $2x^2 \leqslant 2y \leqslant 3y$. Thus, (2.9) is satisfied for all $x,y \in X$. For $x_0 = \frac{1}{2}$ and $x_1 = \frac{1}{4} \in Tx_0 (= \{0,\frac{1}{4}\})$, we have
$$\frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{1+p(x_0,Tx_0)+p(x_1,Tx_1)}+k = \frac{\frac{1}{2}+\frac{1}{4}}{1+\frac{1}{2}+\frac{1}{4}}+\frac{1}{2} = \frac{13}{14} < 1.$$
Hence all hypotheses of Theorem 2.19 are verified. Here T has two fixed points which are $z=0$ and $z=1$. Note that T is not a contraction in (X,p) since $H_p(T0,T1) = H_p(\{0\},\{0,1\}) = 1 = p(0,1) = 1$. Moreover, T is not a nonexpansive mapping for the usual metric $d(x,y) = |x-y|$. In fact, we have
$$H(T\frac{1}{2},T1) = H(\{0,\frac{1}{4}\},\{0,1\}) = \frac{3}{4} > \frac{1}{2} = d(\frac{1}{2},1).$$
Then Theorem 5.1 in [20] is not applicable here.

Take $\alpha(x,y)=\beta(x,y)=1$ for all $x,y\in X$, we have the following results.

Corollary 2.24. *Let (X,p) be a complete partial metric space. Let $T:X\to F^p(X)$ be a nonexpansive multi-valued mapping such that*

$$H_p(Tx,Ty)\leqslant \Big(\frac{p(x,Ty)+p(Tx,y)}{1+p(x,Tx)+p(y,Ty)}+k\Big)M(x,y)$$

for all $x,y\in X$, where $k\in[0,1)$ and

$$M(x,y)=\max\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{2}[p(x,Ty)+p(Tx,y)]\}.$$

Assume that there exist $x_0\in X$ and $x_1\in Tx_0$ such that $p(x_0,x_1)=p(x_0,Tx_0)$ and

$$\frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{1+p(x_0,Tx_0)+p(x_1,Tx_1)}+k<1.$$

Then T has a fixed point.

Corollary 2.25. *Let (X,p) be a complete partial metric space. Let $T:X\to F^p(X)$ be a nonexpansive multi-valued mapping such that*

$$H_p(Tx,Ty)\leqslant \Big(\frac{p(x,Ty)+p(Tx,y)}{1+p(x,Tx)+p(y,Ty)}+k\Big)E(x,y)$$

for all $x,y\in X$, where $k\in[0,1)$ and

$$E(x,y)=p(x,y)+|p(x,Tx)-p(y,Ty)|.$$

Assume that there exist $x_0\in X$ and $x_1\in Tx_0$ such that $p(x_0,x_1)=p(x_0,Tx_0)$ and

$$\frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{1+p(x_0,Tx_0)+p(x_1,Tx_1)}+k<1.$$

Then T has a fixed point.

Corollary 2.26. *Let (X,p) be a complete partial metric space. Let $T:X\to F^p(X)$ be a nonexpansive multi-valued mapping such that*

$$H_p(Tx,Ty)\leqslant \Big(\frac{p(x,Ty)+p(Tx,y)}{1+p(x,Tx)+p(y,Ty)}+k\Big)p(x,y)$$

for all $x,y\in X$, where $k\in[0,1)$. Assume that there exist $x_0\in X$ and $x_1\in Tx_0$ such that $p(x_0,x_1)=p(x_0,Tx_0)$ and

$$\frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{1+p(x_0,Tx_0)+p(x_1,Tx_1)}+k<1.$$

Then T has a fixed point.

3. Conclusion

We have presented some fixed point results for both single and multi-valued mappings in the class of partial metric spaces via a new concept named as α-nonexpansive maps. This new concept generalizes the known concept of expansive mappings. We also presented some concrete examples illustrating the new concepts and results where some known theorems in literature are not applicable, such as the recent results of Vetro [20].

References

[1] T. Abdeljawad, H. Aydi, E. Karapınar, *Coupled fixed points for Meir-Keeler contractions in ordered partial metric spaces*, Math. Probl. Eng., **2012** (2012), 20 pages. 1

[2] I. Altun, H. Simsek, *Some fixed point theorems on dualistic partial metric spaces*, J. Adv. Math. Stud., **1** (2008), 1–8. 1.6

[3] H. Aydi, M. Abbas, C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology Appl., **159** (2012), 3234–3242. 1.7

[4] H. Aydi, M. Abbas, C. Vetro, *Common fixed points for multivalued generalized contractions on partial metric spaces*, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM, **108** (2014), 483–501. 1.7

[5] H. Aydi, M. Barakat, A. Felhi, H. Işik, *On ϕ-contraction type couplings in partial metric spaces*, J. Math. Anal., **8** (2017), 78–89. 1

[6] H. Aydi, M. Jellali, E. Karapınar, *Common fixed points for generalized α-implicit contractions in partial metric spaces: consequences and application*, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM, **109** (2015), 367–384.

[7] H. Aydi, E. Karapınar, *New Meir-Keeler type tripled fixed-point theorems on ordered partial metric spaces*, Math. Probl. Eng., **2012** (2012), 17 pages.

[8] H. Aydi, E. Karapınar, W. Shatanawi, *Coupled fixed point results for (ψ, ϕ)-weakly contractive condition in ordered partial metric spaces*, Comput. Math. Appl., **62** (2011), 4449–4460.

[9] H. Aydi, E. Karapınar, C. Vetro, *On Ekeland's variational principle in partial metric spaces*, Appl. Math. Inf. Sci., **9** (2015), 257–262.

[10] H. Aydi, C. Vetro, W. Sintunavarat, P. Kumam, *Coincidence and fixed points for contractions and cyclical contractions in partial metric spaces*, Fixed Point Theory Appl., **2012** (2012), 18 pages.

[11] L. Ćirić, B. Samet, H. Aydi, C. Vetro, *Common fixed points of generalized contractions on partial metric spaces and an application*, Appl. Math. Comput., **218** (2011), 2398–2406. 1

[12] M. Edelstein, *On nonexpansive mappings*, Proc. Amer. Math. Soc., **15** (1964), 689–695. 1

[13] K. Goebel, W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, (1990).

[14] M. A. Khamsi, S. Reich, *Nonexpansive mappings and semigroups in hyperconvex spaces*, Math. Japon., **35** (1990), 467–471. 1

[15] S. G. Matthews, *Partial metric topology*, Papers on general topology and applications, Flushing, NY, (1992), Ann. New York Acad. Sci., New York Acad. Sci., New York, **728** (1994), 183–197. 1, 1.1, 1

[16] O. Popescu, *Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl., **2014** (2014), 12 pages. 1, 1.8

[17] S. Reich, I. Shafrir, *The asymptotic behavior of firmly nonexpansive mappings*, Proc. Amer. Math. Soc., **101** (1987), 246–250. 1

[18] B. Samet, E. Karapınar, H. Aydi, V. Ćojbašić Rajić, *Discussion on some coupled fixed point theorems*, Fixed Point Theory Appl., **2013** (2013), 12 pages. 1

[19] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340** (2008), 1088–1095. 1

[20] F. Vetro, *Fixed point results for nonexpansive mappings on metric spaces*, Filomat, **29** (2015), 2011–2020. 1.15, 2.3, 2.23, 3