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On nonlinear implicit fractional differential equations without compactness

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Abstract

The main purpose of this research paper is to develop some sufficient conditions for the existence of solution of a nonlinear problem of implicit fractional differential equations (IFDEs) with boundary conditions, using prior estimate method. The distinction of the method applied here is, it does not require compactness of the operator. This idea is the result of motivation from the book of O'Regan and other [R. P. Agarwal, D. O'Regan, Y. J. Cho, Y.-Q. Chen, Taylor & Francis Group, New York, (2006)]. Devising the respective conditions, we also developed some conditions for Hyers-Ulam type stability to the solution of the said problem. To justify the relevant results a suitable example is provided.

Keywords: Implicit fractional differential equations, Brouwer degree, topological degree theory, boundary conditions, Banach contraction theorem, Lebesgue dominated convergence theorem.
2010 MSC: 34A08, 35R11, 26A33.

1. Introduction

Fractional calculus generalizes integral and derivative to fractional orders. In the past few decades fractional calculus was appeared to play a vital tool to model a large number of phenomena, specially, the modeling of memory dependent phenomena and complex media such as porous media, etc. It has gathered importance for being useful in the study of those areas, where classical methods suffer from various limitations. Not only the mathematicians benefits from fractional calculus, the engineers and the physicists do as well. The fractional order differential equations exercise a great scope of applications in rheology, porous media, electrochemistry, electromagnetism, optics, geology, bioscience, bioengineering, medicine, probability and statistics. The fractional order differential equations are also applicable in ecology, control theory, splines, tomography, control of power electron, converter, polymer science, polymer physics and neural networks. Furthermore, It has many applications in other phenomena, for instance, nonlinear oscillations due to earthquake, seepage flow in porous media, absorption of drug in blood stream, image processing, mathematical biology, genetic properties and traffic model of fluid dynamic

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traffic model. These equations are also used for calculations of genetically and chemically acquired properties of different material and phenomena. On the other hand, real word physical phenomena described by mathematical models of fractional differential equations (FDEs) are more useful and realistic in memory as compared to the models of integer order differential equations. Particularly fractional differential equations (FDEs) have a large number of applications in the fields of fractals theory, electromagnetic theory, metallurgy, plasma physics, signal and image processing process, control theory in ecology, economics, and many more. For further detail see [1, 8, 14, 18, 19, 21, 23, 29, 31].

For the applications mentioned above, researchers have paid much attention to study (FDEs) from different aspects. One of the important aspect which has been greatly investigated is devoted to the existence theory of (FDEs). A considerable number of valuable research articles can be cited for this topic, (see for detail[3, 9, 12, 16, 22, 26, 37]). Researchers applied various technique of nonlinear analysis including fixed point theory, hybrid fixed point theory, iterative techniques for establishing existence theory, etc, (see for details[5, 17, 20]). The existence theory by using classical fixed point theory has been much explored. For instance in [24], authors applied a fixed point theorem in partially ordered sets and a contraction mapping principle to prove the existence of at least one positive solution for some fractional boundary value problems. In the same line authors in [2], investigated fractional finite difference equations which are very rarely investigated. In [7], authors developed some appropriate conditions by using fixed point results on cones, to study the existence and uniqueness of positive solutions to some nonlinear fractional differential equations via given boundary conditions. The study of [25] is devoted to the interesting results of multi-point boundary value problem of a fractional order differential inclusion on an infinite interval. In [6], authors established sufficient conditions for existence of solutions to higher order FDEs. Classical fixed point theory needs strong conditions of compactness for the operator which restrict the applicability of the said theory to limited classes of FDEs. That is in the mentioned theory compactness for the corresponding operator to the fractional integral equations from (FDEs) is necessary which needs some strong conditions. To relax the criteria and to replace strong conditions of compactness by some weaker conditions, Mahawin introduced degree theory of topological type for classical integral equations, see [22]. Further in many articles degree approach has been used to replace the stronger conditions with weaker one, see[10, 14, 22, 36]. Some of the degree theories, for example Schauder degree theory, Brouwer's degree theory and topological degree theory are wellknown. The topological degree methods has appeared as a strongest tool in the study of great numbers of problems which occurs in non linear analysis. For example, Isia applied the aforesaid degree theory to establish necessary conditions for existence of solutions to some nonlinear integral equations, see [14]. The topological degree theory has named some author the priori estimate method which has been used by Wang and his coauthors to find out the conditions for existence of solution to nonlinear differential equation of fractional order, see [10]. They studied the following problem of nonlinear FDEs

$$\begin{cases} {}^cD^\gamma v(t) = \mathcal{F}(t, v(t)); t \in J = [0, T], \\ v(t)|_{t=0} + b(v) = v_0. \end{cases}$$

Where ${}^cD^\gamma$ is the Caputo fractional derivative order $\gamma \in (0,1), v_0$ is a real number,$\mathcal{F} \in C([0,T] \times \mathcal{R}, \mathcal{R})$ is continuous. The nonlocal term $b : C(J, \mathcal{R}) \to \mathcal{R}$ is a given function. Being inspired from, Wang and his co-authors work, the authors in [17] investigated the following (BVP) of (FDEs) using coincidence degree theory

$$\begin{cases} {}^cD^\gamma v(t) = \mathcal{F}(t, v(t)); t \in J = [0, T], \gamma \in (1,2], \\ v(t)|_{t=0} = b(v), \quad v(1) - \Sigma_{k=1}^{m-2} \lambda_k v(t)|_{t=\eta_k} = g(v), \end{cases}$$

where $\mathcal{F} \in C([0,T] \times \mathcal{R}, \mathcal{R})$, $\lambda_k, \eta_k \in (0,1)$ with $\Sigma_{k=1}^{m-2} \lambda_k \eta_k < 1, b, g \in C(J, R)$. In most of the situations it is utterly very hard to prove the exact solutions of nonlinear problems, in such situations, it is sufficient to come up with the approximate solutions for the nonlinear problems of (FDEs). Stability analysis is usually considered in such situation. From optimization and numerical point of view, stability analysis plays significant roles in studding of FDEs and FPDEs. Therefore in last few decades the researchers gave attentions to study various form of stability including exponential stability, Mittag-Leffler stability and Lyapunov stability etc, see [13, 30, 32] for the nonlinear problems. Another form of stability which has recently given much attentions by researchers is known as Hyers-Ulam stability. This form of stability was for the first time pointed out by Ulam [33], in 1940 and was introduced by Hyers [13] in 1941. Later on, this topic was much explored for functional, integral and ordinary differential equations, (see[28, 31, 34]). But it was very rarely investigated for (FDEs). However, some literature on the Hyers-Ulam stability of (FDEs) can be cited recently with initial conditions, (see [15, 27, 35]). The implicit fractional fractional differential equations represent a very important class of fractional differential equations. The implicit fractional differential equations (IFDEs) which have been studied so far on different standard fixed point theorems.But here, we investigate the (IFDEs) with boundary condition by using topological degree theory as given by

$$\begin{cases} {}^cD^\gamma v(t) = \mathcal{F}(t, v(t), {}^cD^\gamma v(t)), \ t \in I = [0,1], \ \gamma \in (0,1], \\ v(t)|_{t=0} = \lambda b(v) + \lambda v(t)|_{t=\xi}, \ \xi, \lambda \in (0,1), \end{cases} \tag{1.1}$$

where $\mathcal{F} \in C([0,1] \times \mathcal{R}, \mathcal{R})$ and the nonlocal function $b : C(J, \mathcal{R}) \to \mathcal{R}$ is also continuous. Existence and uniqueness results are developed through topological degree theory which is also called prior estimate method. Also, we establish some adequate conditions about Hyers-Ulam stability. The main result is also illustrated by providing an example. Here, we remark that such type of problems have many applications in managerial sciences and economics. Investigating, IFDEs with boundary conditions by topological degree theory is quiet new approach in the existence theory of FDEs.

2. Preliminaries

Here we remind few basic theorems, lemmas and results from fractional calculus, functional analysis, topological degree theory which can be found in [1, 3, 8, 9, 11, 13, 14, 18, 22].

Definition 2.1. Let $z : \mathcal{R}^+ \to \mathcal{R}$ be the given function, then the fractional integral of order $\gamma > 0$ is defined as below

$$I^\gamma z(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\sigma)^{\gamma-1} z(\sigma) d\sigma,$$

subject to the condition the function z pointwise defined on set $\mathcal{R}^+$.

Definition 2.2. The fractional order Caputo derivative of a function $z : \mathcal{R}^+ \to \mathcal{R}$ is given as below

$${}^cD^\gamma z(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\sigma)^{n-\gamma-1} \left(\frac{d}{d\sigma}\right)^n z(\sigma) d\sigma,$$

subject to the condition the function z pointwise defined on set $\mathcal{R}^+$, such that $n = [\gamma] + 1$ where $[\gamma]$ represents the integer part of γ.

Lemma 2.3. *Fractional order differential equation*

$${}^cD^\gamma z(t) = 0,$$

satisfies the result given below

$$I^\gamma({}^cD^\gamma z(t)) = z(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}$$

for arbitrary $a_i \in \mathcal{R}, \ i = 0, 1, 2, \cdots, n-1.$

Definition 2.4. The norm on $\mathcal{Y} = C(J, \mathcal{R})$ is defined by

$$\|v\|_{\mathcal{Y}} = \sup(|v(t)| : t \in J) \text{ is a Banach space.}$$

We state here the results given below from [4].

Definition 2.5. The Kuratowski's measure of non-compactness $\epsilon : \mathcal{P} \to \mathcal{R}_+$ is given below as

$$\epsilon(P) = \inf\{\rho > 0 \text{ where } P \in \mathcal{P} \text{ has a finite cover by sets of diameter } \leqslant \rho\}.$$

Proposition 2.6. The Kuratowski's measure ϵ satisfies the following properties:

(i) $\epsilon(P) = 0$ if and only if P is relatively compact;

(ii) ϵ is a seminorm, that is $\epsilon(\lambda P) = |\lambda|\epsilon(P), \lambda \in \mathcal{R}$ and $\epsilon(P_1 + P_2) \leqslant \epsilon(P_1) + \epsilon(P_2)$;

(iii) $P_1 \subset P_2$ implies $\epsilon(P_1) \leqslant \epsilon(P_2)$; $\epsilon(P_1 \cup P_2) = \max\{\epsilon(P_1), \epsilon(P_2)\}$;

(iv) $\epsilon(conv(P)) = \epsilon(P)$;

(v) $\epsilon(\bar{P}) = \epsilon(P)$.

Definition 2.7. Let $\mathcal{M} : \mathcal{A} \to \mathcal{Y}$ be a continuous bounded map and $\mathcal{A} \subset \mathcal{Y}$. The operator $\mathcal{M}$ is said to be ϵ-Lipschitz if we can find a constant $k \geqslant 0$ satisfying the following condition,

$$\epsilon(\mathcal{M}(P)) \leqslant k\epsilon(P), \text{for all} \quad P \subset \mathcal{A} \quad \text{is bounded.}$$

Moreover, $\mathcal{M}$ is called strict ϵ-contraction if $k < 1$.

Definition 2.8. The function $\mathcal{M}$ is called ϵ-condensing if

$$\epsilon(\mathcal{M}(P)) < \epsilon(P), \text{for all} \quad P \subset \mathcal{A} \quad \text{bounded with} \quad \epsilon(P) > 0.$$

In other words, $\epsilon(\mathcal{M}(P)) > \epsilon(P)$ implies $\epsilon(P) = 0$.

The collection of all strict ϵ-contractions $\mathcal{M} : \mathcal{A} \to \mathcal{Y}$ is represented by $kC_\epsilon(\Omega)$ and the collection of all ϵ-condensing maps $\mathcal{M} : \mathcal{A} \to \mathcal{Y}$ by $C_\epsilon(\Omega)$.

Remark 2.9. $kC_\epsilon(\mathcal{A}) \subset C_\epsilon(\mathcal{A})$ and every $\mathcal{M} \in C_\epsilon(\mathcal{A})$ is ϵ-Lipschitz with constant $k = 1$.

Moreover, recall that $\mathcal{M} : \mathcal{A} \to \mathcal{Y}$ is Lipschitz if we can find $k > 0$ such that

$$|\mathcal{M}(v) - \mathcal{M}(w)| \leqslant k|v - w|, \quad \text{for all} \quad v, w \in \mathcal{A},$$

if $k < 1$, $\mathcal{M}$ is said to be strict contraction.

Proposition 2.10. If $\mathcal{M}, \mathcal{N} : \mathcal{A} \to \mathcal{Y}$ are ϵ-Lipschitz mapping with constants k_1 and k_2 respectively, then $\mathcal{M} + \mathcal{N} : \mathcal{A} \to \mathcal{Y}$ is ϵ-Lipschitz with constant $k_1 + k_2$.

Proposition 2.11. If $\mathcal{M} : \mathcal{A} \to \mathcal{Y}$ is compact, then $\mathcal{M}$ is ϵ-Lipschitz with constant $k = 0$.

Proposition 2.12. If $\mathcal{M} : \mathcal{A} \to \mathcal{Y}$ is Lipschitz with constant k, then $\mathcal{M}$ is ϵ-Lipschitz with the same constant k.

The theorem given below due to Isaia [14] is of key importance for the proof of our main result.

Theorem 2.13. *Let $\mathcal{M} : \mathcal{Y} \to \mathcal{Y}$ be γ-condensing and*

$$\mathcal{V} = \{a \in \mathcal{Y} : \textit{there exist} \quad \vartheta \in [0, 1] \quad \textit{such that} \quad a = \vartheta\mathcal{M}a\}.$$

If $\mathcal{V}$ is a bounded set in $\mathcal{Y}$, so there exists $r > 0$ such that $\mathcal{V} \subset B_r(0)$, then the degree

$$\deg(I - \vartheta\mathcal{M}, B_r(0), 0) = 1, \quad \textit{for all} \quad \vartheta \in [0, 1].$$

Consequently, $\mathcal{M}$ has at least one fixed point and the set of fixed points of $\mathcal{M}$ lies in $B_r(0)$.

Lemma 2.14. *Let* $x \in (J, \mathcal{R})$*, then the following (BVP) of (IFDEs)*

$$\begin{cases} {}^{c}D^{\gamma}v(t) = x(t); \; t \in J, \; \gamma \in (0,1], \\ v(t)|_{t=0} = \lambda b(v) + \lambda v(t)|_{t=\xi}, \; \xi, \lambda \in (0,1), \end{cases} \tag{2.1}$$

has a solution given by

$$v(t) = \frac{\lambda}{1-\lambda} b(v) + \int_0^1 \mathcal{G}(t,\sigma)x(\sigma)d\sigma, \; , \frac{\lambda}{1-\lambda} \leqslant 1,$$

where $\mathcal{G}(t,\sigma)$ *is the Green's function given as*

$$\mathcal{G}(t,\sigma) = \frac{1}{\Gamma(\gamma)} \begin{cases} \frac{\lambda}{1-\lambda}(\xi-\sigma)^{(\gamma-1)} + (t-\sigma)^{(\gamma-1)}; \; t \leqslant \sigma, \\ \frac{\lambda}{1-\lambda}(\xi-\sigma)^{(\gamma-1)}; \; \sigma \leqslant t. \end{cases} \tag{2.2}$$

Proof. Applying I^{γ} to both sides of equation (2.1), we get

$$v(t) = c_0 + I^{\gamma}x(t). \tag{2.3}$$

In view of conditions $v(t)|_{t=0} = \lambda b(v) + \lambda v(\xi)$, we get

$$c_0 = \lambda b(v) + \lambda[c_0 + I^{\gamma}x(\xi)].$$

Hence

$$c_0 = \frac{\lambda}{1-\lambda} b(v) + \frac{\lambda}{1-\lambda} I^{\gamma}x(\xi).$$

Thus (2.3) becomes

$$v(t) = \frac{\lambda}{1-\lambda} b(v) + \frac{\lambda}{(1-\lambda)\Gamma(\gamma)} \int_0^{\xi} (\xi-\sigma)^{\gamma-1}v(\sigma)d\sigma + \int_0^t (t-\sigma)^{\gamma-1}v(\sigma)d\sigma. \tag{2.4}$$

Thus (2.4) implies that

$$v(t) = \frac{\lambda}{1-\lambda} b(v) + \int_0^1 \mathcal{G}(t,\sigma)x(\sigma)d\sigma,$$

$\mathcal{G}(t,\sigma)$ is called the Green's function, defined as given in (2.2). □

In view of Lemma 2.14, our considered BVP (1.1) is given by the following second kind of Fredholm integral equation

$$v(t) = \frac{\lambda}{1-\lambda} b(v) + \int_0^1 \mathcal{G}(t,\sigma)\mathcal{F}(t, v(t), {}^{c}D^{\gamma}v(t))d\sigma, \; t \in J. \tag{2.5}$$

Lemma 2.15. *The Green's function* $\mathcal{G}(t,\sigma)$ *defined in Lemma 2.14, satisfies the properties given below:*

(i) $\mathcal{G}(t,\sigma)$ *is continuous over* $J \times J$*;*

(ii) $\max|\mathcal{G}(t,\sigma)| = \frac{1}{(1-\lambda)\Gamma(\gamma)}(1-\sigma)^{\gamma-1}$.

Proof. (i) and (ii) can be proved easily. □

Definition 2.16. The solution of the considered problem (1.1) is Hyers-Ulam stable if we can find a real number $c_{\mathcal{F}} > 0$ with the property that for each $\mu > 0$ and for each solution $v \in C(J, \mathcal{R})$ of the inequality

$$|{}^{c}D^{\gamma}v(t) - \mathcal{F}(t, v(t), {}^{c}D^{\gamma}v(t))| \leqslant \mu, \; t \in J, \tag{2.6}$$

there exists a unique solution $w \in C(J, \mathcal{R})$ of the considered equation (1.1) with a constant $c_{\mathcal{F}} > 0$ such that

$$|v(t) - w(t))| \leqslant c_{\mathcal{F}}\mu, \; t \in J.$$

Definition 2.17. The solution of the equation (1.1) is said to be generalized Hyers-Ulam stable, if we can find

$$\Upsilon_{\mathcal{F}} \in C(\mathcal{R}_+, \mathcal{R}_+),\ \Upsilon_{\mathcal{F}}(0) = 0,$$

such that for each solution $v \in C(J, \mathcal{R})$ of the inequality (2.6) we can find a unique solution $w \in C(J, \mathcal{R})$ of the equation (1.1) such that

$$|v(t) - w(t))| \leqslant \Upsilon_{\mathcal{F}}(\mu),\ t \in J.$$

Remark 2.18. A function $v \in C(J, \mathcal{R})$ is said to be the solution of inequality given in (2.6) if and only if, we can find a function $h \in C(J, \mathcal{R})$ depends on v only then

(i) $|h(t)| \leqslant \mu$, for all $t \in J$;

(ii) ${}^cD^{\gamma}v(t) = \mathcal{F}(t, v(t), {}^cD^{\gamma}v(t)) + h(t),$ for all $t \in J$.

3. Existence theory for at least one solution of BVP (1.1)

This section is devoted to the proof of some results required for existence of solution of the considered problem (1.1). Thank to Lemma 2.14, the considered problem (1.1) can be represented by the second kind of Fredholm integral equation given below,

$$v(t) = \frac{\lambda}{1-\lambda} b(v) + \int_0^1 \mathcal{G}(t, \sigma)\mathcal{F}(t, v(t), {}^cD^{\gamma}v(t)),\ t \in J. \tag{3.1}$$

Below we list some assumptions:

(P_1) The function $\mathcal{F} : J \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is continuous.

(P_2) There exist constants L_b, $N_b > 0$, such that for each $v \in (J, \mathcal{R})$,

$$|b(v)| \leqslant L_b\|v\|_{\mathcal{Y}} + N_b.$$

(P_3) There exist constants $L_{\mathcal{F}} > 0$, $0 < N_{\mathcal{F}} < 1$ and $v_1, w_1, v_2, w_2 \in \mathcal{R}$ such that,

$$|\mathcal{F}(t, v_1, w_1) - \mathcal{F}(t, v_2, w_2)| \leqslant L_{\mathcal{F}}\|v_1 - v_2\|_{\mathcal{Y}} + N_{\mathcal{F}}\|w_1 - w_2\|_{\mathcal{Y}}.$$

(P_4) There exist $p, q, r \in C(J, \mathfrak{R}_+)$ and, $v, w \in \mathcal{R}, t \in J$ such that

$$|\mathcal{F}(t, v, w)| \leqslant p(t) + q(t)|v| + r(t)|v|,$$

where

$$r^* = \sup\{|r(t)| : t \in J\},\ p^* = \sup\{|p(t)| : t \in J\},\ q^* = \sup\{|q(t)| : t \in J\}.$$

(P_5) For arbitrary $t \in J$ and $v, w \in C(J, \mathcal{R})$, constants $L_b > 0$ can be found, such that

$$|b(v) - b(w)| \leqslant L_b\|v - w\|_{\mathcal{Y}}.$$

Assume that (P_1) to (P_4) hold, here it will be shown that the fractional integral equation (3.1) has a unique solution $v \in C(J, \mathcal{R})$. We consider two operators $\mathcal{M}, \mathcal{N}$ on $C(J, \mathcal{R})$

$$\mathcal{M} :\ \mathcal{Y} \to \mathcal{Y} \text{ defined by } (\mathcal{M}v)t = \frac{\lambda}{1-\lambda} b(v),$$

and

$$\mathcal{N}: \mathcal{Y} \to \mathcal{Y} \text{ defined by } (\mathcal{N}v)t = \int_0^1 \mathcal{G}(t,\sigma)\mathcal{F}(t, v(t), {}^c D^{\gamma} v(t))d\sigma.$$

Let us consider another operator $\mathcal{U}$ on $C(J, \mathcal{R})$, such that

$$\mathcal{U}: \mathcal{Y} \to \mathcal{Y} \text{ defined by } \mathcal{U}(v) = \mathcal{M}v + \mathcal{N}v.$$

$\mathcal{U}$ is well-defined because $\mathcal{M}$ and $\mathcal{N}$ are well-defined. Moreover, $\mathcal{U}v = v$. Thus to find the solution of BVP (1.1) is equivalent to find fixed point for operator $\mathcal{U}$ in $\mathcal{Y}$.

Lemma 3.1. *$\mathcal{M}$ is Lipschitz with constant k_b . Moreover, $\mathcal{M}$ satisfies the growth condition given below*

$$\|\mathcal{M}(v\|_{\mathcal{Y}} \leqslant L_b\|v\| + N_b. \tag{3.2}$$

Proof. Let $v, w \in C(J, \mathcal{R})$, then consider

$$|\mathcal{M}(v) - \mathcal{M}(w)| = |\frac{\lambda}{1-\lambda}(b(v) - b(w))|,$$

putting $k_b = L_b\frac{\lambda}{1-\lambda} \leqslant 1$, which yields

$$|\mathcal{M}(v) - \mathcal{M}(w)| \leqslant k_b|v - w|$$
$$\|\mathcal{M}(v) - \mathcal{M}(w)\|_{\mathcal{Y}} \leqslant k_b\|v - w\|_{\mathcal{Y}}.$$

Consequently $\mathcal{M}$ is μ-Lipschitz with some constant k_b. The growth condition is a simple consequence of (P_2) as given by

$$\|\mathcal{M}v\|_{\mathcal{Y}} \leqslant L_b\|v\| + N_b.$$ □

Lemma 3.2. *The operator $\mathcal{N}$ is continuous and satisfies the growth condition given as below,*

$$\|\mathcal{N}(v)\|_{\mathcal{Y}} \leqslant \left(\frac{1}{\Gamma(\gamma+1)(1-\lambda)(1-r^*)}\right)(p^* + q^*\|v\|_{\mathcal{Y}}). \tag{3.3}$$

Proof. To prove that $\mathcal{N}$ is continuous, let $\{v_n\}$ be any sequence in bounded set $\mathfrak{B}_r$, such that $\mathfrak{B}_r = \{v_n : \|v_n\|_{\mathcal{Y}} \leqslant r\}$ and $v_n \to v$ as $n \to \infty$ in $\mathfrak{B}_r$. We are required to prove that $|\mathcal{N}v_n - \mathcal{N}v| \to 0$, $n \to \infty$. As $\mathcal{F}(t, v(t), {}^c D^{\gamma}v(t))$ is continuous, thus it follows that $\mathcal{F}(\sigma, v_n(\sigma), {}^c D^{\gamma}v_n(t)) \to \mathcal{F}(\sigma, v(\sigma), {}^c D^{\gamma}v(t))$, as $n \to \infty$. Using assumption (P_3), we get the relation given below

$$|\mathcal{N}v_n - \mathcal{N}v| = \int_0^1 |\mathcal{G}(t,\sigma)||\mathcal{F}(t, v_n(t), {}^c D^{\gamma}v_n(t)) - \mathcal{F}(t, v(t), {}^c D^{\gamma}v(t))d\sigma|. \tag{3.4}$$

Since $|\mathcal{F}(t, v_n(t), {}^c D^{\gamma}v_n(t)) - \mathcal{F}(t, v(t), {}^c D^{\gamma}v(t))| \leqslant \frac{L_{\mathcal{F}}}{1-N_{\mathcal{F}}}|v_n - v|$, thus (3.4) becomes

$$\left|\int_0^1 \mathcal{G}(t,\sigma)(\mathcal{F}(t, v_n(t), {}^c D^{\gamma}v_n(t)) - \mathcal{F}(t, v(t), {}^c D^{\gamma}v(t)))d\sigma\right| \leqslant \frac{L_{\mathcal{F}}}{(1-N_{\mathcal{F}})(1-\lambda)(\Gamma(\gamma+1)}|v_n - v| \tag{3.5}$$

the inequality (3.5) clearly implies that left side is integrable. Hence application of Lebesgue dominated convergence theorem yields the following relation,

$$\left|\int_0^1 \mathcal{G}(t,\sigma)(\mathcal{F}(t, v_n(t), {}^c D^{\gamma}v_n(t)) - \mathcal{F}(t, v(t), {}^c D^{\gamma}v(t)))d\sigma\right| \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Therefore $\mathcal{N}(v_n) \to \mathcal{N}(v)$, as $n \to \infty$. This means that the operator $\mathcal{N}$ is continuous. For the growth condition, we use (P_4) and obtain as

$$\|\mathcal{N}(v)\|_{\mathcal{Y}} \leqslant \left(\frac{1}{\Gamma(\gamma+1)(1-\lambda)(1-r^*)}\right)(p^* + q^*\|v\|_{\mathcal{Y}}).$$ □

Lemma 3.3. *The operator* $\mathcal{N}:\mathcal{Y}\to\mathcal{Y}$ *is compact and* ϵ*-Lipschitz with constant* 0

Proof. In order to show that $\mathcal{N}$ is compact, let us take a bounded set $\mathfrak{B}\subset\mathfrak{B}_r$. We are required to show that $\mathcal{N}(\mathfrak{B})$ is relatively compact in $\mathcal{Y}$. For arbitrary $\nu_n\in\mathfrak{B}\subset\mathfrak{B}_r$, the growth condition is given as

$$\|\mathcal{N}(\nu)\|_{\mathcal{Y}}\leqslant \frac{1}{\Gamma(\gamma+1)(1-\lambda)(1-r(t))}\Big(p(t)+q(t)\|\nu\|_{\mathcal{Y}}\Big),$$

from this it is clear that $\mathcal{N}(\nu)$ is uniformly bounded. For equi-continuity of $\mathcal{N}$, we discuss the following possible cases.

Case I. When $t\leqslant\sigma$, then we have,

$$\begin{aligned}|\mathcal{N}(\nu_n)(t_1)-\mathcal{N}(\nu_n)(t_2)| &\leqslant \int_0^1|\mathcal{G}(t_1,\sigma)-\mathcal{G}(t_2,\sigma)||\mathcal{F}(t,\nu_n(t),{}^cD^\gamma\nu_n(t))|d\sigma\\ &\leqslant\left(\frac{p(t)+q(t)r}{(1-\lambda)(1-r^*)\Gamma(\gamma)}\right)\left(\int_0^1\Big|\mathcal{G}(t_1,\sigma)-\mathcal{G}(t_2,\sigma)\Big|d\sigma\right)\\ &\leqslant\left(\frac{p^*+q^*r}{(1-\lambda)(1-r^*)\Gamma(\gamma)}\right)\left(\int_0^{t_1}(t_1-\sigma)^{\gamma-1}-(t_2-\sigma)^{\gamma-1}d\sigma+\int_{t_1}^{t_2}(t_2-\sigma)^{\gamma-1}d\sigma\right)\\ &\leqslant\left(\frac{p^*+q^*r}{(1-\lambda)(1-r^*)\Gamma(\gamma+1)}\right)(t_1^\gamma-t_2^\gamma+2(t_2-t_1)^\gamma).\end{aligned}$$

From above relation , it follows clearly that $\|\mathcal{N}(\nu_n)(t_1)-\mathcal{N}(\nu_n)(t_2)\|\to 0,\ as\ t_1\to t_2$, which implies that $\mathcal{N}(\nu)$ is equi-continuous in this case.

Case II. When $\sigma\leqslant t$, then we have,

$$\begin{aligned}|\mathcal{N}(\nu_n)(t_1)-\mathcal{N}(\nu_n)(t_2)| &\leqslant \int_0^1|\mathcal{G}(t_1,\sigma)-\mathcal{G}(t_2,\sigma)||\mathcal{F}(t,\nu_n(t),{}^cD^\gamma\nu_n(t))|d\sigma\to 0ast_1\to t_2\\ &\leqslant 0.\end{aligned}$$

From above relation, it follows clearly that $\|\mathcal{N}(\nu_n)(t_1)-\mathcal{N}(\nu_n)(t_2)\|\to 0,\ as\ t_1\to t_2$, which implies that $\mathcal{N}(\nu)$ is equi-continuous.
Hence by Arzela-Ascoli theorem $\mathcal{N}(p)$ is compact and thus by Proposition 2.10 $\mathcal{N}$ is ϵ-Lipschitz with constant 0. □

Theorem 3.4. *Suppose that* (P_2)-(P_4) *are satisfied, then the (BVP)* (1.1) *has at least one solution* $\nu\in C(J,\mathcal{R})$ *and the set of the solutions is bounded in* $C(J,\mathcal{R})$.

Proof. Let $\mathcal{M},\mathcal{N},\mathcal{U}$ be the operators defined in the start of this section. These operators are continuous and bounded. Moreover, by Lemma 3.1, $\mathcal{M}$ is γ-Lipschitz with constant K_b and by Lemma 3.2, $\mathcal{N}$ is γ-Lipschitz with constant 0. Thus, $\mathcal{U}$ is γ-Lipschitz with constant K_b. Let us take the set given below

$$\mathcal{W}=\Big\{\nu:\text{there exists }\alpha\in[0,1],\text{such that }\nu=\alpha\mathcal{U}(\nu)\Big\}.$$

We will show that the set $\mathcal{W}$ is bounded. For $\nu\in\mathcal{W}$, we have $\nu=\alpha\mathcal{U}\nu=\alpha(\mathcal{M}(\nu)+\mathcal{N}(\nu))$, which implies that

$$\begin{aligned}|\nu| &\leqslant \alpha(|\mathcal{M}\nu|+|\mathcal{N}\nu|)\\ \|\nu\| &\leqslant L_b\|\nu\|_{\mathcal{Y}}+N_b+\frac{p^*+q^*\|\nu\|_{\mathcal{Y}}}{\Gamma(\gamma+1)(1-r^*)(1-\lambda)}.\end{aligned}$$

From the above inequalities, we conclude that $\mathcal{W}$ is bounded in $C(J,\mathcal{R})$. If it is not bounded, then dividing the above inequality by $a = \|v\|$ and letting $a \to \infty$, we arrive at

$$1 \leqslant \frac{(L_b\|v\|_{\mathcal{Y}}+N_b) + \left(\frac{(p^*+q^*\|v\|)}{\Gamma(\gamma+1)(1-r^*)(1-\lambda)}\right)}{a} \leqslant 0,$$

which is a contradiction. Thus the set $\mathcal{W}$ is bounded and the operator $\mathcal{U}$ has at least one fixed point which represents the solution of(BVP) (1.1). □

Theorem 3.5. *Under the assumptions* (P_1)-(P_4), *the (BVP)* (1.1) *has a unique solution if*

$$K_b + \frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)} < 1.$$

Proof. We will use Banach contraction theorem to show that equation (1.1) has a unique solution. The Banach contraction theorem states that if an operator $\mathcal{U}$ from Banach space $\mathcal{Y}$ into itself is a contraction mapping then the operator $\mathcal{U}$ has a unique fixed point in $\mathcal{Y}$. Consequently equation (1.1) has a unique solution in $\mathcal{Y}$. consider $v(.)$ and $w(.)$ are the solutions of (BVP) (1.1), then

$$\begin{aligned}
|\mathcal{U}(v)(t) - \mathcal{U}(w)(t)| &\leqslant |\frac{\lambda}{1-\lambda}(b(v) - b(w))| + \frac{1}{(1-\lambda)\Gamma(\gamma+1)}|\mathcal{F}(t, v(t), {}^c D^\gamma v(t)) - \mathcal{F}(t, w(t), {}^c D^\gamma w(t))| \\
&\leqslant \frac{\lambda}{1-\lambda}|b(v) - b(w)| + \frac{L_{\mathcal{F}}}{\Gamma(\gamma+1)(1-N_{\mathcal{F}})(1-\lambda)}|v - w| \\
&\leqslant K_b|v - w| + \frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)}|v - w| \\
\|\mathcal{U}(v) - \mathcal{U}(w)\| &\leqslant \left(K_b + \frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)}\right)\|v - w\|_{\mathcal{Y}}.
\end{aligned}$$

Since $K_b + \frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)} < 1$. Thus $\mathcal{U}$ is a contraction mapping and by Banach contraction principle $\mathcal{U}$ has a unique fixed point. □

4. Hyers-Ulam stability

Theorem 4.1. *Let* $v \in C(J, \mathcal{R})$ *be a solution of*

$$\begin{aligned}
{}^c D^\gamma v(t) &= \mathcal{F}(t, v(t), {}^c D^\gamma v(t)) + h(t), \ t \in J, \gamma \in (0, 1], \\
v(t)|_{t=0} &= \lambda b(v) + \lambda v(t)|_{t=\xi}, \ \xi, \lambda \in (0, 1).
\end{aligned}$$

Then the result given below holds,

$$\left| v(t) - \left(\frac{\lambda}{1-\lambda}b(v) + \int_0^1 \mathcal{G}(t, \sigma)\mathcal{F}(t, v(t), {}^c D^\gamma v(t))d\sigma\right)\right| \leqslant \frac{\mu}{(1-\lambda)\Gamma(\gamma+1)}.$$

Proof. The solution of the problem

$$\begin{aligned}
{}^c D^\gamma v(t) &= \phi(t, v(t)) + h(t), \ t \in J, \gamma \in (0, 1], \\
v(t)|_{t=0} &= \lambda b(v) + \lambda v(t)|_{t=\xi}, \ \xi, \lambda \in (0, 1),
\end{aligned}$$

is given by

$$v(t) = \frac{\lambda}{1-\lambda}b(v) + \int_0^1 \mathcal{G}(t, \sigma)\mathcal{F}(t, v(t), {}^c D^\gamma v(t))d\sigma + \int_0^1 \mathcal{G}(t, \sigma)h(\sigma)d\sigma.$$

From, which we have

$$\left|v(t)-\left(\frac{\lambda}{1-\lambda}b(v)+\int_0^1 \mathcal{G}(t,\sigma)\mathcal{F}(t,v(t),{}^cD^\gamma v(t))d\sigma\right)\right| \leqslant \left|\int_0^1 \mathcal{G}(t,\sigma)h(\sigma)d\sigma\right|,$$

which implies that,

$$\left|v(t)-\left(\frac{\lambda}{1-\lambda}b(v)+\int_0^1 \mathcal{G}(t,\sigma)\mathcal{F}(t,v(t),{}^cD^\gamma v(t))d\sigma\right)\right| \leqslant \frac{\mu}{(1-\lambda)\Gamma(\gamma+1)}.$$

□

Theorem 4.2. *Under the assumption* (P_3) *if* $(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)\neq L_{\mathcal{F}}$ *holds, then the solution of BVP* (1.1) *is Hyers-Ulam stable.*

Proof. Let $v\in C(J,\mathcal{R})$ be the solution of (2.6) and $w\in C(J,\mathcal{R})$ be the unique solution of

$$\begin{aligned}&{}^cD^\gamma w(t)=\mathcal{F}(t,w(t),{}^cD^\gamma w(t)),\ t\in J,\gamma\in(0,1],\\ &w(t)|_{t=0}=\lambda b(w)+\lambda v(t)|_{t=\xi}=\lambda b(v)+\lambda v(t)|_{t=\xi},\ \xi,\lambda\in(0,1).\end{aligned}$$

Consider

$$\begin{aligned}|w(t)-v(t)| &= \Big|w(t)-(\frac{\lambda}{1-\lambda}b(v)+\int_0^1 \mathcal{G}(t,\sigma)\mathcal{F}(t,w(t),{}^cD^\gamma w(t))d\sigma\\ &\quad+\int_0^1 \mathcal{G}(t,\sigma)\mathcal{F}(t,w(t),{}^cD^\gamma w(t))d\sigma-\int_0^1 \mathcal{G}(t,\sigma)\mathcal{F}(t,v(t),{}^cD^\gamma v(t))d\sigma\Big|\\ &\leqslant \frac{\mu}{(1-\lambda)\Gamma(\gamma+1)}+\frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)}|w(t)-v(t)|\\ \|w-v\|_{\mathcal{Y}} &\leqslant \frac{\mu}{(1-\lambda)\Gamma(\gamma+1)}+\frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)}\|w-v\|_{\mathcal{Y}}\\ \left(1-\frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)}\right)\|w-v\|_{\mathcal{Y}} &\leqslant \frac{\mu}{(1-\lambda)\Gamma(\gamma+1)}\\ \|w-v\|_{\mathcal{Y}} &\leqslant \frac{\mu}{(1-\lambda)\Gamma(\gamma+1)\left(1-\frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)}\right)}\\ \|w-v\|_{\mathcal{Y}} &\leqslant \frac{\mu(1-N_{\mathcal{F}})}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)-L_{\mathcal{F}}},\end{aligned}$$

where $(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)\neq L_{\mathcal{F}}$. So the solution of BVP (1.1) is Hyer-Ulam stable. Further taking $\Upsilon(\mu)=\frac{\mu(1-N_{\mathcal{F}})}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)-L_{\mathcal{F}}}$, then clearly $\Upsilon(0)=0$. So the solution of (BVP) (1.1) is generalized Hyers-Ulam stable. □

5. Example

Example 5.1. Consider the following (BVP)

$$\begin{aligned}&{}^cD^{\frac{1}{2}}v(t)=\frac{|\sin(v(t))|+|\sin({}^cD^{\frac{1}{2}}v(t))|}{40+9e^t},\ t\in J,\ \gamma\in(0,1],\\ &v(t)|_{t=0}=\frac{1}{2}\Sigma_{k=0}^{10}\lambda_k v(\frac{1}{3})+\frac{1}{2}v(\frac{1}{3}),\ \lambda_k\in(0,1),\Sigma_{k=0}^{10}\lambda_k\leqslant\frac{1}{20}.\end{aligned}\tag{5.1}$$

We have $\gamma=\frac{1}{2},\lambda=\frac{1}{2},\ \xi=\frac{1}{3},\ K_b=\frac{1}{20}$ and the nonlinear function is given by

$$\mathcal{F}(t,v(t),{}^cD^\gamma v(t))=\frac{|\sin(v(t))|+|\sin({}^cD^{\frac{1}{2}}v(t))|}{40+9e^t}.$$

In view of Lemma 2.14, the Green function of(BVP) (5.1) is given by

$$\mathcal{G}(t,\sigma)=\frac{1}{\Gamma(\frac{1}{2})}\begin{cases}\frac{1}{4}\left(\frac{1}{3}-\sigma\right)^{\frac{-1}{2}}+(1-\sigma)^{\frac{-1}{2}}; & t\leqslant\sigma,\\ \frac{1}{4}\left(\frac{1}{3}-\sigma\right)^{\frac{-1}{2}}; & \sigma\leqslant t.\end{cases}$$

Now let $v, w \in \mathcal{R},\ t \in J$, we have

$$\begin{aligned}&|\mathcal{F}(t,v(t),{}^{c}D^{\gamma}v(t))-\mathcal{F}(t,w(t),{}^{c}D^{\gamma}w(t))|\\ &=\left|\frac{|\sin(v(t))|+|\sin({}^{c}D^{\frac{1}{2}}v(t))|}{40+9e^{t}}-\frac{|\sin(w(t))|+|\sin({}^{c}D^{\frac{1}{2}}w(t))|}{40+9e^{t}}\right|\\ &\leqslant\frac{1}{49}|v(t)-w(t)|+\frac{1}{49}|{}^{c}D^{\gamma}v(t)-{}^{c}D^{\gamma}w(t)|.\end{aligned}$$

So we have $L_{\mathcal{F}}=\frac{1}{49}$ and $N_{\mathcal{F}}=\frac{1}{49}$. Also $K_b+\frac{L_{\mathcal{F}}}{(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)}=\frac{10}{103}<1$. Hence by Theorem 3.5, the BVP (5.1) has a unique solution. Further $L_{\mathcal{F}}=\frac{1}{49}$ and $(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)=12\sqrt{\pi}$, which implies that $(1-\lambda)(1-N_{\mathcal{F}})\Gamma(\gamma+1)\neq L_{\mathcal{F}}$. So by Theorem 4.2 the solution of the considered BVP (5.1) is Hyer-Ulam stable and hence generalized Hyer-Ulam stable.

6. Conclusion

Some sufficient conditions for existence and uniqueness of solution to the considered BVP (1.1) of (IFDEs) are obtained as result of application of Arzela Ascoli's theorem and Banach contraction theorem coupled with topological degree theory. We have also developed results for Hyers-Ulam and generalized Hyers-Ulam types stability of the solutions for the considered BVP (1.1) under some specific conditions.

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