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Existence of solutions to boundary value problems for a higher-dimensional difference system

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Abstract

By using critical point theory, some new criteria are obtained for the existence of a nontrivial homoclinic orbit to a higher order difference system containing both many advances and retardations. The proof is based on the Mountain Pass Lemma in combination with periodic approximations. Related results in the literature are generalized and improved.

Keywords: Boundary value problems, higher-dimensional, mountain pass lemma, critical point theory.
2010 MSC: 39A10, 47J30, 58E05.

1. Introduction

Difference equations, the discrete analogs of differential equations, have been applied as models in vast areas such as finance insurance, biological populations, disease control, genetic study, physical field, and computer application technology, etc. Because of their importance, many literature and monographs deal with its existence and uniqueness problems. For example, see [1–7, 12–20, 22, 24–27].

We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of all natural numbers, integers and real numbers respectively. Throughout this paper, for all a, $b \in \mathbb{Z}$, we define $\mathbb{Z}(a) = \{a, a+1, \cdots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a \leqslant b$. Let the symbol $*$ denote the transpose of a vector. In the following and in the sequel, for any $n \in \mathbb{N}$, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$, defined by

$$|X| = \left(\sum_{i=1}^{n} X_i^2\right)^{\frac{1}{2}}, \quad \forall X = (X_1, X_2, \cdots, X_n) \in \mathbb{R}^n.$$

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Consider the nonlinear higher order difference system

$$\sum_{i=0}^{n} r_i(X_{k-i}+X_{k+i}) = f(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}), \quad n \in \mathbb{N}, \ k \in \mathbb{Z}(1, T), \tag{1.1}$$

with boundary value conditions

$$X_{1-\tau} = X_{2-\tau} = \cdots = X_0 = 0, \quad X_{T+1} = X_{T+2} = \cdots = X_{T+\tau} = 0, \tag{1.2}$$

where r_i is a real number, Γ is a given nonnegative integer, T and m are given positive integers,

$$f = (f_1, f_2, \cdots, f_m)^* \in C(\mathbb{R}^{2\Gamma+2} \times \mathbb{R}^m, \mathbb{R}),$$

$\tau = \max\{n, \Gamma\}$, $f \in C(\mathbb{R}^{2\Gamma+2}, \mathbb{R})$.

We may regard (1.1) as being a discrete analogue of the following $2n$-th order functional differential equation

$$\left[r(t)X^{(n)}\right]^{(n)} = f(t, X(t+\Gamma), \cdots, X(t), \cdots, X(t-\Gamma)), \quad t \in \mathbb{R}.$$

When $m = 1, n = 1, \Gamma = 1$, Smets and Willem [28] proved the existence of solitary waves with prescribed speed on infinite lattices of particles with nearest neighbor interaction for the following forward and backward differential difference equation

$$c^2u''(t) = V'\left(u(t+1) - u(t)\right) - V'\left(u(t) - u(t-1)\right), \quad t \in \mathbb{R}. \tag{1.3}$$

Equations similar in structure to (1.3) arise in the study of the existence of periodic solutions and homoclinic orbits for functional differential equations, see [8–11].

In 2007, Cai and Yu [2] established some criteria for the existence of periodic solutions of a $2n$-th order difference equation

$$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right) + f(k, u_k) = 0, \quad n \in \mathbb{Z}(3), \ k \in \mathbb{Z},$$

by using the Linking Theorem.

If $\Gamma = 0$, Hu and Huang in 2008 [12] applied the critical point theorem to prove the existence of periodic solution of a higher order difference equation as the following type

$$\sum_{i=0}^{n} r_i(X_{k-i}+X_{k+i}) + f(k, X_k) = 0, \quad n \in \mathbb{N}, \ k \in \mathbb{Z}.$$

Chen and Tang [3] in 2011 obtained some new existence criteria to guarantee the $2n$-th order nonlinear difference equation

$$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right) + q_k u_k = f(k, u_{k+n}, \cdots, u_k, \cdots, u_{k-n}), \quad n \in \mathbb{Z}(3), \ k \in \mathbb{Z},$$

has at least one or infinitely many homoclinic orbits by establishing a proper variational framework and using the critical point theory.

By using the critical point theory, Deng and Shi [6] in 2010 obtained some sufficient conditions for the existence and multiplicity of the boundary value problems to a class of second order functional difference equations

$$Lu_k = f(k, u_{k+1}, u_k, u_{k-1}),$$

with boundary value conditions

$$\Delta u_0 = A, \quad u_{T+1} = B.$$

Recently, Liu et al. [20] studied the following $2n$-th order nonlinear difference equation

$$\Delta^n\left(\gamma_{i-n+1}\Delta^n u_{i-n}\right) = (-1)^n f(i, u_{i+1}, u_i, u_{i-1}), \quad n \in \mathbb{Z}(1), \ i \in \mathbb{Z}(1, T),$$

with boundary value conditions

$$\Delta u_{1-n} = \Delta u_{2-n} = \cdots = \Delta u_0 = 0, \quad u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0,$$

and gave some new results of solutions for the mixed boundary value problem by using the Mountain Pass Lemma.

Motivated by the above papers [6, 20], the intention of this paper is to consider the boundary value problem (1.1) with (1.2) in a more general sense. More exactly our results represent the extensions to higher-dimensional difference systems containing both many advances and retardations. We establish some new sufficient conditions ensuring the existence of solutions to boundary value problems for such a system. One of our results generalizes an existing result. In fact, one can see the following Remark 1.4 for details.

Throughout the paper, for a function F, we let $F_i'(Y_1, \cdots, Y_i \cdots, Y_n)$ denote the partial derivative of F on the i variable.

For basic knowledge of variational methods, the reader is referred to [21, 23].

Our main results are the following theorems.

Theorem 1.1. *Assume that F satisfies the following assumptions:*

(F_1) *there exists a function $F(t, Y_\Gamma, \cdots, Y_0) \in C^1(\mathbb{R}^{\Gamma+2} \times \mathbb{R}^m, \mathbb{R})$ such that*

$$\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t+i, Y_{\Gamma+i}, \cdots, Y_i) = f(t, Y_\Gamma, \cdots, Y_0, \cdots, Y_{-\Gamma});$$

(F_2) *there exist constants $c_1 > 0$, $c_2 > 0$ and $\alpha > 2$ such that*

$$F(t, Y_\Gamma, \cdots, Y_0) \geqslant c_1 \left(\sqrt{\sum_{j=0}^{\Gamma} Y_j^2} \right)^{\alpha} - c_2, \quad \forall (t, T_\Gamma, \cdots, Y_0) \in \mathbb{R}^{\Gamma+2}.$$

Then the boundary value problem (1.1) with (1.2) possesses at least one solution.

Remark 1.2. The results of Theorem 1.1 ensure that the boundary value problem (1.1) with (1.2) possesses at least one solution. However, in some cases, we are interested in the existence of nontrivial solutions.

Theorem 1.3. *Assume that r and F satisfy (F_1), (F_2) and the following assumptions:*

(r_1) $r_0 + \sum_{s=1}^{n} |r_s| < 0$;

(F_3) *for all* $(t, T_\Gamma, \cdots, Y_0) \in \mathbb{R}^{\Gamma+2}$,

$$\lim_{\rho \to 0} \frac{F(t, Y_\Gamma, \cdots, Y_0)}{\rho^2} = 0, \quad \rho = \sqrt{\sum_{j=0}^{\Gamma} Y_j^2}.$$

Then the boundary value problem (1.1) with (1.2) possesses at least two nontrivial solutions.

Remark 1.4. Theorem 1.3 generalizes [27, Theorem 1.2] which is the special case of our Theorem 1.3 by letting $m = 1$ and $\Gamma = 1$.

Theorem 1.5. *Assume that r and F satisfy (F_1) and the following assumptions are satisfied:*

(r_2) $-r_0 + \sum_{s=1}^{n} |r_s| \leqslant 0$;

(F_4) $Y_0 f(t, Y_\Gamma, \cdots, Y_0, \cdots, Y_{-\Gamma}) > 0$, *for* $Y_0 \neq 0$, *for all* $t \in \mathbb{R}$.

Then the boundary value problem (1.1) *with* (1.2) *has no nontrivial solution.*

This paper is divided into five parts. First of all, Section 2 presents variational structure. The second, in Section 3, we shall recall some related fundamental results and present some lemmas. Then, Section 4 is dedicated to the proof of main results. Finally, in Section 5, we shall give two examples to illustrate the applicability of the main results.

2. Variational structure

Our main tool is the critical point theory. We shall establish the corresponding variational framework for the boundary value problem (1.1) with (1.2). We start by some basic notations for the reader's convenience.

Let $\mathbb{R}^{mT}$ be the real Euclidean space with dimension mT. $\mathbb{R}^{mT}$ can be equipped with the inner product $\langle X, Y\rangle$ and norm $\|X\|$ as follows

$$\langle X, Y\rangle := \sum_{j=1}^{T} X_j \cdot Y_j, \quad \forall X, Y \in \mathbb{R}^{mT},$$

and

$$\|X\| := \left(\sum_{j=1}^{T} \left|X_j\right|^2\right)^{\frac{1}{2}}, \quad \forall X \in \mathbb{R}^{mT},$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$, and $X_j \cdot Y_j$ denotes the usual scalar product in $\mathbb{R}^m$.

On the other hand, we define the norm $\|\cdot\|_s$ on $\mathbb{R}^{mT}$ as follows

$$\|X\|_s = \left(\sum_{j=1}^{T} |X_j|^s\right)^{\frac{1}{s}}$$

for all $X \in \mathbb{R}^{mT}$ and $s > 1$.

Since $\|X\|_s$ and $\|X\|_2$ are equivalent, there exist constants K_1, K_2 such that $K_2 \geqslant K_1 > 0$, and

$$K_1\|X\|_2 \leqslant \|X\|_s \leqslant K_2\|X\|_2, \quad \forall X \in \mathbb{R}^{mT}.$$

For all $X \in \mathbb{R}^{mT}$, define the functional J on $\mathbb{R}^{mT}$ as follows:

$$J(X) := \frac{1}{2}\sum_{k=1}^{T}\sum_{i=0}^{n} r_i\left(X_{k-i} + X_{k+i}\right) X_k - \sum_{k=1}^{T} F(k, X_{k+\Gamma}, \cdots, X_k).$$

Since $X_{1-\tau} = X_{2-\tau} = \cdots = X_0 = 0, X_{T+1} = X_{T+2} = \cdots = X_{T+\tau} = 0$, then

$$\frac{\partial J(X)}{\partial X_{k,l}} = \sum_{i=0}^{n} r_i(X_{k-i,l} + X_{k+i,l}) - f_l(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}), \quad l \in \mathbb{Z}(1, m), \quad k \in \mathbb{Z}(1, T).$$

Therefore, $X \in \mathbb{R}^{mT}$ is a critical point of J, i.e., $J'(X) = 0$ if and only if

$$\sum_{i=0}^{n} r_i(X_{k-i,l} + X_{k+i,l}) - f_l(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}) = 0, \quad l \in \mathbb{Z}(1, m), \quad k \in \mathbb{Z}(1, T).$$

That is,

$$\sum_{i=0}^{n} r_i(X_{k-i}+X_{k+i}) - f(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}) = 0, \quad k \in \mathbb{Z}(1,T).$$

Thus, we reduce the problem of finding boundary value problem (1.1) with (1.2) to that of seeking critical points of the functional J in $\mathbb{R}^{mT}$.

For all $X \in \mathbb{R}^{mT}$, J can be rewritten as

$$J(X) = -\frac{1}{2}\langle DMX, MX\rangle - \sum_{k=1}^{T} F(k, X_{k+\Gamma}, \cdots, X_k), \tag{2.1}$$

where $X = \{X_k\} \in E_T,\ X_k = (X_{k,1}, X_{k,2}, \cdots, X_{k,m})^*, k \in \mathbb{Z}(1,T)$, and

$$D = \begin{pmatrix} P & & & 0 \\ & P & & \\ & & \ddots & \\ 0 & & & P \end{pmatrix}_{mT\times mT},$$

$$-P = \begin{pmatrix}
2r_0 & r_1 & r_2 & \cdots & r_{n-1} & r_n & 0 & 0 & \cdots & 0 & r_n & r_{n-1} & \cdots & r_2 & r_1 \\
r_1 & 2r_0 & r_1 & \cdots & r_{n-2} & r_{n-1} & r_n & 0 & \cdots & 0 & 0 & r_n & \cdots & r_3 & r_2 \\
r_2 & r_1 & 2r_0 & \cdots & r_{n-3} & r_{n-2} & r_{n-1} & r_n & \cdots & 0 & 0 & 0 & \cdots & r_4 & r_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
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\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
r_2 & r_3 & r_4 & \cdots & 0 & 0 & 0 & 0 & \cdots & r_n & r_{n-1} & r_{n-2} & \cdots & 2r_0 & r_1 \\
r_1 & r_2 & r_3 & \cdots & r_n & 0 & 0 & 0 & \cdots & r_{n-1} & r_{n-2} & r_{n-3} & \cdots & r_1 & 2r_0
\end{pmatrix},$$

is a $T \times T$ matrix. Assume that the eigenvalues of P are $\lambda_1, \lambda_2, \cdots, \lambda_T$ respectively. It is easy to see that P is a circulant matrix [29] denoted by

$$P := \mathrm{Circ}\{-2r_0, -r_1, -r_2, \cdots, -r_n, 0, \cdots, 0, -r_n, -r_{n-1}, \cdots, -r_2, -r_1\}.$$

By [29], the eigenvalues of P are

$$\lambda_j = -2r_0 - \sum_{s=1}^{n} r_s \left\{\exp i\frac{2j\pi}{T}\right\}^s - \sum_{s=1}^{n} r_s \left\{\exp i\frac{2j\pi}{T}\right\}^{T-s} = -2\sum_{s=0}^{n} r_s \cos\left(\frac{2js\pi}{T}\right), \tag{2.2}$$

where $j = 1, 2, \cdots, T$.

By (2.2), we have

$$-2r_0 - 2\sum_{s=1}^{n} |r_s| \leqslant \lambda_j \leqslant -2r_0 + 2\sum_{s=1}^{n} |r_s|, \quad j = 1, 2, \cdots, T.$$

3. Main lemmas

In order to apply critical point theory to study the existence of boundary value problem (1.1) with (1.2), we shall state some lemmas, which will be used in the proofs of our main results.

Let E be a real Banach space, $J \in C^1(E, \mathbb{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{X^{(n)}\}_{n\in\mathbb{N}} \subset E$ for which $\{J(X^{(n)})\}_{n\in\mathbb{N}}$ is bounded and $J'(X^{(n)}) \to 0$ $(n \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 3.1 (Mountain Pass Lemma [21, 23])**.** *Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*

- (J_1) *there exist constants ρ, $\alpha > 0$ such that $J|_{\partial B_\rho} \geqslant \alpha$; and*
- (J_2) *there exists $e \in E \setminus B_\rho$ such that $J(e) \leqslant 0$.*

Then J possesses a critical value $c \geqslant \alpha$ given by

$$c = \inf_{g\in\Upsilon} \max_{s\in[0,1]} J(g(s)),$$

where

$$\Upsilon = \{g \in C([0,1], E) | g(0) = 0, \ g(1) = e\}.$$

Lemma 3.2. *Assume that (r_1) and (F_1)-(F_3) are satisfied. Then J satisfies the P.S. condition.*

Proof. It follows from (r_1) that P is positive definite. Denote

$$\lambda_{\max} = \max\left\{\lambda_j | j = 1, 2, \cdots, T\right\}. \tag{3.1}$$

Let $\{X^{(n)}\}_{n\in\mathbb{N}} \subset \mathbb{R}^{mT}$ be such that $\{J(X^{(n)})\}_{n\in\mathbb{N}}$ is bounded and $J'(X^{(n)}) \to 0$ as $n \to \infty$. By (F_2), (2.1), (2.2) and (3.1), we have

$$\begin{aligned}
J\left(X^{(n)}\right) &= -\frac{1}{2}\left\langle DMX^{(n)}, MX^{(n)}\right\rangle - \sum_{k=1}^{T} F\left(k, X^{(n)}_{k+\Gamma}, \cdots, X^{(n)}_{k}\right) \\
&\leqslant \frac{1}{2}\lambda_{\max}\left\|X^{(n)}\right\|^2 - c_1 \sum_{k=1}^{T}\left[\sqrt{\sum_{j=0}^{\Gamma}\left(X^{(n)}_{k+j}\right)^2}\right]^{\alpha} + c_2 T \\
&\leqslant \frac{1}{2}\lambda_{\max}\left\|X^{(n)}\right\|^2 - c_1 K_1^{\alpha}\left\|X^{(n)}\right\|^{\alpha} + c_2 T.
\end{aligned}$$

Since $J(X^{(n)})$ is anti-coercive and $\alpha > 2$, then the P.S. condition follows immediately. □

4. Proof of the main results

Now, we shall finish proof of our main results by using the variational method and critical point theory.

Proof of Theorem 1.1. For any $X = (X_1, X_2, \cdots, X_T)^* \in \mathbb{R}^{mT}$, we have

$$\begin{aligned}
J(X) &= -\frac{1}{2}\langle DMX, MX\rangle - \sum_{k=1}^{T} F(k, X_{k+\Gamma}, \cdots, X_k) \\
&\leqslant \frac{1}{2}\lambda_{\max}\|X\|^2 - c_1 \sum_{k=1}^{T}\left[\sqrt{\sum_{j=1}^{\Gamma} X_{k+j}^2}\right]^{\alpha} + c_2 T \\
&\leqslant \frac{1}{2}\lambda_{\max}\|X\|^2 - c_1 K_1^{\alpha}\|X\|^{\alpha} + c_2 T \to -\infty,
\end{aligned}$$

as $\|X\| \to +\infty$. Due to the continuity of $J(X)$, the above inequality implies that there exist upper bounds of values of functional J. Classical calculus shows that J attains its maximal value at some point which is just the critical point of J and the result follows. □

Proof of Theorem 1.3. By (F_3), for any $\varepsilon = \frac{1}{4(\Gamma+2)}\lambda_{\min}$ ($\lambda_{\min}$ can be referred to (3.1)), there exists $\delta > 0$, such that

$$|F(k, Y_\Gamma, \cdots, Y_0)| \leqslant \frac{1}{4(\Gamma+2)}\lambda_{\min}\sum_{j=0}^{\Gamma} Y_j^2, \quad \forall k \in \mathbb{Z}(1,T),$$

for $\sqrt{\sum\limits_{j=0}^{\Gamma} Y_j^2} \leqslant \sqrt{\Gamma+2}\delta$.

For any $X = (X_1, X_2, \cdots, X_T)^* \in \mathbb{R}^{mT}$ and $\|X\| \leqslant \delta$, we have $|X_k| \leqslant \delta,\ k \in \mathbb{Z}(1,T)$. Then,

$$\begin{aligned} J(X) &= -\frac{1}{2}\langle DMX, MX\rangle - \sum_{k=1}^{T} F(k, X_{k+\Gamma}, \cdots, X_k) \\ &\geqslant \frac{1}{2}\lambda_{\min}\|X\|^2 - \frac{1}{4(\Gamma+2)}\lambda_{\min}\sum_{k=1}^{T}\sum_{j=0}^{\Gamma} X_{k+j}^2 \\ &\geqslant \frac{1}{2}\lambda_{\min}\|X\|^2 - \frac{1}{4}\lambda_{\min}\|X\|^2 \\ &= \frac{1}{4}\lambda_{\min}\|X\|^2. \end{aligned}$$

Take $a \triangleq \frac{1}{4}\lambda_{\min}\delta^2 > 0$. Therefore, $J(X) \geqslant a > 0$, for all $X \in \partial B_\delta$. At the same time, we have also proved that there exist constants $a > 0$ and $\delta > 0$ such that $J|_{\partial B_\delta} \geqslant a$. That is to say J satisfies the condition (J_1) of the Mountain Pass Lemma.

The rest of the proof is similar to that of [27], but for the sake of completeness, we give the details.

For our setting, clearly $J(0) = 0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify other conditions of the Mountain Pass Lemma. By Lemma 3.2, J satisfies the P.S. condition. So it suffices to verify the condition (J_2).

From the proof of the P.S. condition, we know $J(X) \leqslant \frac{1}{2}\lambda_{\max}\|X\|^2 - c_1K_1^{\alpha}\|X\|^{\alpha} + c_2T$. Since $\alpha > 2$, we can choose $\bar{X}$ large enough to ensure that $J(\bar{X}) < 0$. By the Mountain Pass Lemma, J possesses a critical value $c \geqslant a > 0$, where $c = \inf\limits_{g\in\Upsilon}\sup\limits_{s\in[0,1]} J(g(\theta))$ and $\Upsilon = \{g \in C([0,1], \mathbb{R}^{mT}) \mid g(0) = 0,\ g(1) = \bar{X}\}$.

Let $\tilde{X} \in \mathbb{R}^{mT}$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{X}) = c$. Similar to the proof of the P.S. condition, we know that there exists $\hat{X} \in \mathbb{R}^{mT}$ such that $J(\hat{X}) = c_{\max} = \max\limits_{\theta\in[0,1]} J(g(\theta))$.

If $\tilde{X} \neq \hat{X}$, then the conclusion of Theorem 1.3 holds. Otherwise, $\tilde{X} = \hat{X}$. Then we get that all points on the critical level are solutions. Otherwise we have two of them. $\hat{X}$ must be nontrivial since the value is greater than the critical value which is positive. □

Proof of Theorem 1.5. By matrix theory, combining with (r_2), we have that the eigenvalues of P are non-positive, i.e., $\lambda_j \leqslant 0$ for all $j \in \mathbb{Z}(1,T)$. For the sake of contradiction, we assume that the boundary value problem (1.1) with (1.2) has a nontrivial solution. Then, J has a nonzero critical point $X^\star$. Since

$$\frac{\partial J}{\partial X_k^\star} = \sum_{i=0}^{n} r_i(X_{k-i}^\star + X_{k+i}^\star) - f(k, X_{k+\Gamma}^\star, \cdots, X_k^\star, \cdots, X_{k-\Gamma}^\star),$$

we get

$$\sum_{k=1}^{T} f(k, X_{k+\Gamma}^\star, \cdots, X_k^\star, \cdots, X_{k-\Gamma}^\star)X_k^\star = \sum_{k=1}^{T}\sum_{i=0}^{n} r_i(X_{k-i}^\star + X_{k+i}^\star)X_k^\star = -\langle DMX^\star, MX^\star\rangle \leqslant 0. \tag{4.1}$$

On the other hand, it follows from (F_4) that

$$\sum_{i=1}^{T} f(k, X^{\star}_{k+\Gamma}, \cdots, X^{\star}_{k}, \cdots, X^{\star}_{k-\Gamma}) X^{\star}_{k} > 0.$$

This contradicts (4.1) and hence the proof is finished. □

5. Examples

As an application of Theorems 1.3 and 1.5, we give two examples to illustrate our main results.

Example 5.1. For $n \in \mathbb{N}, k \in \mathbb{Z}(1, T)$, let

$$f(k, X_{k+\Gamma}, \cdots, X_{k}, \cdots, X_{k-\Gamma}) = \alpha X_k \sum_{j=0}^{\Gamma} (k-j) \left(\sum_{i=0}^{\Gamma} X^2_{k+i-j} \right)^{\frac{\alpha}{2}-1}.$$

It is easy to verify all the assumptions of Theorem 1.3 are satisfied. Consequently, two nontrivial solutions are obtained.

Example 5.2. For $n \in \mathbb{N}, k \in \mathbb{Z}(1, T)$, let

$$f(k, X_{k+\Gamma}, \cdots, X_{k}, \cdots, X_{k-\Gamma}) = 12 X_k \sum_{j=0}^{\Gamma} (k-j) \left(\sum_{i=0}^{\Gamma} X^6_{k+i-j} \right)^{5}.$$

It is easy to verify all the assumptions of Theorem 1.5 are satisfied. Consequently, there is no nontrivial solution.

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