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Robustness analysis of global exponential stability in neural networks evoked by deviating argument and stochastic disturbance

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Abstract

This paper studies the robustness of global exponential stability of neural networks evoked by deviating argument and stochastic disturbance. Given the original neural network is globally exponentially stable, we discuss the problem that the neural network is still globally exponentially stable when the deviating argument or both the deviating argument and stochastic disturbance is/are generated. By virtue of solving the derived transcendental equation(s), the upper bound(s) about the intensity of the deviating argument or both of the deviating argument and stochastic disturbance is/are received. The obtained theoretical results are the supplements to the existing literatures on global exponential stability of neural networks. Two numerical examples are offered to demonstrate the effectiveness of theoretical results.

Keywords: Global exponential stability, robustness, neural networks, deviating argument, stochastic disturbance.
2010 MSC: 34D23, 93D09.

1. Introduction

In recent decades, neural networks have been widely applied in many fields such as associative memory, speech recognition, neural computing, and so on. In order to tackle different kinds of tasks expediently, a variety of types of neural networks have been presented, for instance, cellular neural networks, Cohen-Grossberg neural networks, fuzzy neural networks, etc.. And all these neural networks have attracted much attention from various theoretical and engineering fields ([3, 14, 22, 32, 35]).

In most of the practical applications, for example, robot control, speech synthesis, and associative memory, it is essential that the neural networks are of stability. And there are many publications about different types of stability of all sorts of types of neural networks (see, for instance, [4, 6, 8, 9, 13, 15, 16, 19, 20, 22–30, 32–36, 38, 40]). In [4], the global asymptotic stability of recurrent neural networks was reported. It was analyzed the global Mittag-Leffler stability of memristor-based fractional-order neural networks in [8]. A new method was proposed for complete stability of the time-delayed cellular neural networks in [9]. Multistability analysis for time-varying delayed recurrent neural networks with non-monotonous activation functions was addressed in [22]. Exponential stability for time-delayed memristor-based neural networks was discussed in [27]. And Lagrange stability for memristor-based neural networks with

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discrete and distributed delays was investigated in [32].

It should be noted that stochastic disturbance is inevitable in practical applications of neural networks, which may aggravate the performance and even derail the stable neural networks. And therefore, neural networks evoked by stochastic disturbance have been investigated extensively ([5, 7, 10–12, 17, 18, 25, 31, 37, 39]). In [5], it was discussed the mean square exponential stability of neural networks with uncertain random time-delays. The multistability for Hopfield neural networks with time-varying delay and stochastic disturbance was analyzed in [7]. It was showed that the n-neuron network can have 2^n positive invariant sets with probability 1, in which every invariant set contained an asymptotically stable equilibrium point. It was reported in [37] that complex networks can be of exponential synchronization in mean square with the delayed impulsive controller.

As a particular type of neural networks, generalized type neural networks with piecewise constant argument, which can be seemed as neural networks evoked by deviating argument, have been explored widely in recent years ([1, 2, 21]). Different from the traditional neural networks, this class of neural networks can change the deviation types (alternately advanced and retarded) as the time t goes on. Based on the fact that the generalized type neural networks with piecewise constant argument play an important role in the electromagnetic field, and meanwhile, the disparate types of deviation argument may rely on the occurrence of traveling waves possessing potential applications, the generalized type neural networks with piecewise constant argument were addressed in [1]. In [21], the global mean square exponential stability for this type of neural networks with stochastic disturbances was studied via constructing the Lyapunov function.

From the analysis in [1, 25], it can be seen that the deviating argument and stochastic disturbance are some key factors destablizing the neural networks, when the intensity of the deviating argument or stochastic disturbance surpasses a certain limit. It should be stressed that there are many references which analyze the stability properties of neural networks, whereas there are rare about the robustness stability of neural networks evoked by deviating argument and stochastic disturbance. The deviating argument contains the information about the past and future, meanwhile the stochastic disturbance is ubiquitous in practical applications, hence it is meaningful to investigate: (1) Given a globally exponentially stable neural network, how much intensity of the deviating argument can the neural network sustain to maintain globally exponentially stable? (2) Given a globally exponentially stable neural network, how much intensity of the deviating argument and stochastic disturbance can the neural network sustain simultaneously to maintain globally exponentially stable?

Motivated by the above discussion, in this paper, we investigate the following two issues:

(1) for an originally globally exponentially stable neural network, the upper bound of the intensity of the deviating argument maintaining the neural network to be globally exponentially stable;
(2) for an originally globally exponentially stable neural network, the upper bounds of the intensity of the deviating argument and stochastic disturbance that maintain the neural network to be globally exponentially stable.

By establishing the transcendental equations about the intensity of deviating argument and stochastic disturbance, the upper bounds of the intensity of deviating argument and stochastic disturbance are derived. The validity of theoretical results is demonstrated well via some numerical examples.

The rest of the paper is outlined as follows. The preliminaries and model descriptions are introduced in Section 2. In Section 3, it formulates the impact of deviating argument and the impact of both deviating argument and stochastic disturbance on the global exponential stability of neural networks. Two numerical examples are provided to verify the effectiveness of theoretical results in Section 4. Concluding remarks are made in Section 5.

2. Preliminaries and model descriptions

Throughout this paper, denote R^n and $R^{n\times m}$ as the n-dimensional Euclidean space and the set of $n \times m$ matrices, respectively. For a vector $x \in R^n$ and a matrix $A \in R^{n\times n}$, the operator norm of A

is indicated as $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$, where $\|\cdot\|$ is the Euclidean norm with $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$. $E(\cdot)$ signifies the mathematical expectation. N indicates the natural number set. Select two real-valued sequences ξ_k, σ_k, satisfying $\xi_k < \xi_{k+1}$, $\xi_k \leqslant \sigma_k \leqslant \xi_{k+1}$, for any $k \in N$. And $\xi_k \to +\infty$, $\sigma_k \to +\infty$ as $k \to +\infty$. Then we consider the following neural network model

$$\dot{w}(t) = -Aw(t) + Bf(w(t)) + Cf(w(\vartheta(t))), \quad w(t_0) = w_0 \in R^n, \tag{2.1}$$

where $\vartheta(t)$ is a deviating argument satisfying $\vartheta(t) = \sigma_k \in [\xi_k, \xi_{k+1}]$, if $t \in [\xi_k, \xi_{k+1})$. And $w(t) = (w_1(t), \dots, w_n(t))^T \in R^n$ is the state vector, $A = \mathrm{diag}\{a_1, \dots, a_n\} \in R^{n\times n}$ is the self-feedback connection weight matrix. $B = (b_{ij})_{n\times n} \in R^{n\times n}$ and $C = (c_{ij})_{n\times n} \in R^{n\times n}$ denote the connection weight matrices about the state $w(t)$ and deviating argument state $w(\vartheta(t))$, respectively. $f(w(t)) = (f_1(w_1(t)), \dots, f_n(w_n(t)))^T \in R^n$, $f(w(\vartheta(t))) = (f_1(w_1(\vartheta(t))), \dots, f_n(w_n(\vartheta(t))))^T \in R^n$ indicate the vector-valued activation functions at time t and $\vartheta(t)$, respectively.

We say the type of neural network (2.1) is mixed. For the deviating argument $\vartheta(t) = \sigma_k$, $t \in [\xi_k, \xi_{k+1})$, when $\xi_k \leqslant t < \vartheta(t) = \sigma_k$, $w(\vartheta(t))$ is an advanced argument. When $\vartheta(t) = \sigma_k < t < \xi_{k+1}$, $w(\vartheta(t))$ is a retarded argument. Therefore, neural network (2.1) changes the type of deviating argument state $w(\vartheta(t))$ with the increase of time t.

A solution $w(t) = (w_1(t), w_2(t), \dots, w_n(t))$ of neural network (2.1) is continuous such that:

(1) the derivative $\dot{w}(t)$ exists at each point when $t \geqslant 0$ with the possible exception of the points ξ_k, $k \in N$, where a one-sided derivative exists;
(2) $w(t)$ satisfies neural network (2.1) on each interval (ξ_k, ξ_{k+1}), $k \in N$.

In what follows we will introduce some assumptions that will be needed in this paper.

(A1) For the activation functions $f_i(\cdot) \in C(R, R)$ satisfying $f_i(0) = 0$, there exist Lipschitz constants $L_i > 0$ such that

$$|f_i(u_i) - f_i(v_i)| \leqslant L_i |u_i - v_i|,$$

for any $u_i, v_i \in R$ and $i = 1, 2, \dots n$. It can be written in the vector format as

$$\|f(u) - f(v)\| \leqslant L\|u - v\|,$$

for any $u, v \in R^n$, where $L = \max_{1\leqslant i\leqslant n}(L_i)$.

(A2) There exists a positive constant ξ such that $\xi_{k+1} - \xi_k \leqslant \xi$ for any $k \in N$.

(A3) There exist positive constants ξ, ρ, and υ such that

$$\xi(\rho + 2\upsilon)\exp(\rho\xi) < 1,$$

where $\rho = \max_{1\leqslant i\leqslant n}\left(a_i + L_i \sum_{j=1}^{n} |b_{ji}|\right)$, $\upsilon = \max_{1\leqslant i\leqslant n}\left(L_i \sum_{j=1}^{n} |c_{ji}|\right)$.

Remark 2.1. It can be seen from Theorem 2.2 of [1] that the existence and uniqueness of the solution of neural network (2.1) are guaranteed by (A1), (A2), and (A3) jointly.

In case of $\vartheta(t) = t$ (i.e., $\xi = 0$), for any $t \geqslant t_0$, neural network (2.1) degenerates into an ordinary neural network as follows

$$\dot{x}(t) = -Ax(t) + Bf(x(t)) + Cf(x(t)), \quad x(t_0) = x_0. \tag{2.2}$$

Definition 2.2 ([25]). The state of neural network (2.2) is globally exponentially stable if, for any $(t_0, x_0) \in ([0, +\infty), R^n)$, there exist $\alpha > 0$, $\beta > 0$ such that

$$\|x(t, t_0, x_0)\| \leqslant \alpha\|x(t_0)\| \exp(-\beta(t - t_0)),$$

where $x(t, t_0, x_0)$ is the state of neural network (2.2).

3. Main results

3.1. *The impact of deviating argument on stability*

Under (A1), (A2), and (A3), neural network (2.1) has a unique solution $w(t, t_0, w_0)$ for any initial state (t_0, w_0) when $t > t_0$. And it is clear that $w = 0$ is the equilibrium point. Then, one question arises: provided that neural network (2.2) is globally exponentially stable, how much intensity of the deviating argument can neural network (2.1) withstand and maintain globally exponentially stable as before? Based on this fact, in this subsection, we consider the robustness of global exponential stability of neural network (2.1) for the deviating argument when neural network (2.2) is globally exponentially stable.

Before giving the main theorem of this subsection, a useful lemma is presented. And the following assumptions are needed.

(A4) There exist positive constants ξ, μ and ν such that

$$6\xi^2[n\nu + 2\mu(1 + 3n\nu\xi^2)\exp(6\mu\xi^2)] < 1,$$

where $\mu = \max_{1\leqslant i\leqslant n}\left(a_i^2 + nL_i^2\sum_{j=1}^n b_{ji}^2\right)$, $\nu = \max_{1\leqslant i\leqslant n}\left(L_i^2\sum_{j=1}^n c_{ji}^2\right)$, and n corresponds to the number of units in neural network (2.1).

(A5) The parameters of neural network (2.2) satisfy the following inequality

$$2\alpha^2\exp(-2\beta\Delta) + 144\|C\|^2L^2\Delta\alpha^2\exp\left\{12\Delta^2\left(\|A\|^2 + \|B\|^2L^2 + 2\|C\|^2L^2\right) + 288\|C\|^2L^2\Delta^2\right\}\Big/\beta < 1,$$

where L is the Lipschitz constant, and $\Delta > \ln(2\alpha^2)/(2\beta)$.

Lemma 3.1. *Under* (A1), (A2), (A3), *and* (A4), *for* (2.1), *the following inequality holds,*

$$\|w(\vartheta(t))\| \leqslant \rho\|w(t)\|$$

for any $t \geqslant t_0$, *where* $\rho = 2\left(1 - 6\xi^2[n\nu + 2\mu(1 + 3n\nu\xi^2)\exp(6\mu\xi^2)]\right)^{-1}$, $\mu = \max_{1\leqslant i\leqslant n}\left(a_i^2 + nL_i^2\sum_{j=1}^n b_{ji}^2\right)$, $\nu = \max_{1\leqslant i\leqslant n}\left(L_i^2\sum_{j=1}^n c_{ji}^2\right)$, *and* n *corresponds to the number of units in neural network* (2.1).

Proof. For any $t \geqslant t_0$, from the property of $\vartheta(t)$ and the sequences $\{\xi_k\}$, $\{\sigma_k\}$, there exists a unique $k \in N$, such that

$$\vartheta(t) = \sigma_k \in [\xi_k, \xi_{k+1}), t \in [\xi_k, \xi_{k+1}),$$

and we have if $t \geqslant \sigma_k$,

$$w_i(t) = w_i(\sigma_k) + \int_{\sigma_k}^t \Big[-a_iw_i(s) + \sum_{j=1}^n b_{ij}f_j(w_j(s)) + \sum_{j=1}^n c_{ij}f_j(w_j(\sigma_k))\Big]ds, \tag{3.1}$$

for $i = 1, 2, \ldots, n$, then

$$\begin{aligned}
w_i^2(t) =& \left(w_i(\sigma_k) + \int_{\sigma_k}^t \Big[-a_iw_i(s) + \sum_{j=1}^n b_{ij}f_j(w_j(s)) + \sum_{j=1}^n c_{ij}f_j(w_j(\sigma_k))\Big]ds\right)^2\\
\leqslant& 2w_i^2(\sigma_k) + 2\left(\int_{\sigma_k}^t \Big[-a_iw_i(s) + \sum_{j=1}^n b_{ij}f_j(w_j(s)) + \sum_{j=1}^n c_{ij}f_j(w_j(\sigma_k))\Big]ds\right)^2\\
=& 2w_i^2(\sigma_k) + 2\left(\int_{\sigma_k}^t 1\times\Big[-a_iw_i(s) + \sum_{j=1}^n \Big(b_{ij}\times f_j(w_j(s))\Big) + \sum_{j=1}^n c_{ij}f_j(w_j(\sigma_k))\Big]ds\right)^2.
\end{aligned}$$

From the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
w_i^2(t) \leqslant & 2w_i^2(\sigma_k) + 2\int_{\sigma_k}^{t} 1^2 ds \times \int_{\sigma_k}^{t} \Big[- a_i w_i(s) + \sum_{j=1}^{n} b_{ij} f_j(w_j(s)) + \sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k)) \Big]^2 ds \\
\leqslant & 2w_i^2(\sigma_k) + 2\xi \int_{\sigma_k}^{t} \Big[- a_i w_i(s) + \sum_{j=1}^{n} b_{ij} f_j(w_j(s)) + \sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k)) \Big]^2 ds \\
\leqslant & 2w_i^2(\sigma_k) + 6\xi \int_{\sigma_k}^{t} \Big[a_i^2 w_i^2(s) + \Big(\sum_{j=1}^{n} b_{ij} f_j(w_j(s)) \Big)^2 + \Big(\sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k)) \Big)^2 \Big] ds \\
\leqslant & 2w_i^2(\sigma_k) + 6\xi \int_{\sigma_k}^{t} \Big[a_i^2 w_i^2(s) + n \sum_{j=1}^{n} b_{ij}^2 f_j^2(w_j(s)) + n \sum_{j=1}^{n} c_{ij}^2 f_j^2(w_j(\sigma_k)) \Big] ds,
\end{aligned}
$$

then

$$
\begin{aligned}
\sum_{i=1}^{n} w_i^2(t) \leqslant & 2\sum_{i=1}^{n} w_i^2(\sigma_k) + 6\xi \int_{\sigma_k}^{t} \Big[\sum_{i=1}^{n} a_i^2 w_i^2(s) + n\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2 f_j^2(w_j(s)) + n\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^2 f_j^2(w_j(\sigma_k)) \Big] ds \\
= & 2\sum_{i=1}^{n} w_i^2(\sigma_k) + 6\xi \int_{\sigma_k}^{t} \Big[\sum_{i=1}^{n} a_i^2 w_i^2(s) + n\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}^2 f_i^2(w_i(s)) + n\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ji}^2 f_i^2(w_i(\sigma_k)) \Big] ds \\
= & 2\sum_{i=1}^{n} w_i^2(\sigma_k) + 6\xi \int_{\sigma_k}^{t} \Big[\sum_{i=1}^{n} a_i^2 w_i^2(s) + n\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}^2 f_i^2(w_i(s)) + n\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^2 f_i^2(w_i(\sigma_k)) \Big] ds \\
\leqslant & 2\sum_{i=1}^{n} w_i^2(\sigma_k) + 6\xi \int_{\sigma_k}^{t} \Big[\sum_{i=1}^{n} a_i^2 w_i^2(s) + n\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}^2 L_i^2 w_i^2(s) + n\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^2 L_i^2 w_i^2(\sigma_k) \Big] ds \\
= & 2\sum_{i=1}^{n} w_i^2(\sigma_k) + 6\xi \int_{\sigma_k}^{t} \Big[\sum_{i=1}^{n} \Big(a_i^2 + nL_i^2 \sum_{j=1}^{n} b_{ji}^2 \Big) w_i^2(s) + n\sum_{i=1}^{n} \Big(L_i^2 \sum_{j=1}^{n} c_{ji}^2 \Big) w_i^2(\sigma_k) \Big] ds \\
\leqslant & 2\sum_{i=1}^{n} w_i^2(\sigma_k) + 6\xi \int_{\sigma_k}^{t} \Big[\sum_{i=1}^{n} \mu w_i^2(s) + n\sum_{i=1}^{n} \nu w_i^2(\sigma_k) \Big] ds,
\end{aligned}
$$

that is

$$
\begin{aligned}
\|w(t)\|^2 \leqslant & 2\|w(\sigma_k)\|^2 + 6\xi \int_{\sigma_k}^{t} \Big(\mu \|w(s)\|^2 + n\nu \|w(\sigma_k)\|^2 \Big) ds \\
= & 2\|w(\sigma_k)\|^2 + 6\xi\mu \int_{\sigma_k}^{t} \|w(s)\|^2 ds + 6n\xi\nu \int_{\sigma_k}^{t} \|w(\sigma_k)\|^2 ds \\
\leqslant & 2\|w(\sigma_k)\|^2 + 6\xi\mu \int_{\sigma_k}^{t} \|w(s)\|^2 ds + 6n\xi^2\nu \|w(\sigma_k)\|^2 \\
= & 2(1+3n\nu\xi^2)\|w(\sigma_k)\|^2 + 6\xi\mu \int_{\sigma_k}^{t} \|w(s)\|^2 ds.
\end{aligned}
$$

Based on the Gronwall-Bellman inequality, we obtain

$$
\|w(t)\|^2 \leqslant 2(1+3n\nu\xi^2)\|w(\sigma_k)\|^2 \exp\Big(\int_{\sigma_k}^{t} 6\xi\mu ds \Big) \leqslant 2(1+3n\nu\xi^2)\|w(\sigma_k)\|^2 \exp\big(6\mu\xi^2\big). \tag{3.2}
$$

Exchanging the location of $w_i(t)$ and $w_i(\sigma_k)$ in (3.1),

$$
\|w(\sigma_k)\|^2 \leqslant 2\|w(t)\|^2 + 6n\xi^2\nu\|w(\sigma_k)\|^2 + 6\xi\mu \int_{\sigma_k}^{t} \|w(s)\|^2 ds, \tag{3.3}
$$

substituting (3.2) into (3.3),

$$
\begin{aligned}
\|w(\sigma_k)\|^2 \leqslant & 2\|w(t)\|^2+6n\xi^2\nu\|w(\sigma_k)\|^2+6\xi\mu\int_{\sigma_k}^{t}\Big(2(1+3n\nu\xi^2)\|w(\sigma_k)\|^2\exp\left(6\mu\xi^2\right)\Big)ds\\
\leqslant & 2\|w(t)\|^2+6n\xi^2\nu\|w(\sigma_k)\|^2+12\xi^2\mu(1+3n\nu\xi^2)\|w(\sigma_k)\|^2\exp\left(6\mu\xi^2\right)\\
= & 2\|w(t)\|^2+6\xi^2\Big[n\nu+2\mu(1+3n\nu\xi^2)\exp\left(6\mu\xi^2\right)\Big]\|w(\sigma_k)\|^2,
\end{aligned}
$$

it follows that

$$\|w(\vartheta(t))\| \leqslant \rho\|w(t)\|.$$

For $t<\sigma_k$, we can get the same result with the method used above. And the proof is completed. □

Remark 3.2. The existence of deviating argument unifies the advance and retard, hence it brings a lot of difficulties to analyze the neural network evoked by deviating argument. Through the estimation of the norm of the deviating argument state vector $w(\vartheta(t))$ by the norm of the corresponding state vector $w(t)$, Lemma 3.1 provides an effective approach to study neural network (2.1).

Theorem 3.3. *Let* (A1), (A2), (A3), (A4), *and* (A5) *hold and neural network* (2.2) *be globally exponentially stable. Neural network* (2.1) *is globally exponentially stable if $\xi<\bar{\xi}$, where $\bar{\xi}$ is a unique positive solution of the transcendental equation*

$$2\alpha^2\exp\left\{-2\beta\Delta\right\}+2\hat{c}_2\exp\left\{2\hat{c}_1\Delta\right\}=1, \tag{3.4}$$

where $\hat{c}_1=6\Delta(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2)+48\|C\|^2L^2\Delta(1+\rho)$, $\hat{c}_2=24\|C\|^2L^2\Delta\alpha^2(1+\rho)/\beta$, $\rho=2\Big(1-6\xi^2\big[n\nu+2\mu(1+3n\nu\xi^2)\exp(6\mu\xi^2)\big]\Big)^{-1}$, $\Delta>\ln(2\alpha^2)/(2\beta)$.

Proof. For the sake of simplicity, $x(t,t_0,x_0)$ and $w(t,t_0,w_0)$ are denoted as $x(t)$ and $w(t)$, respectively. From (2.1), (2.2), and the initial condition $x_0=w_0$, we obtain

$$x(t)-w(t)=\int_{t_0}^{t}[-A(x(s)-w(s))+B(f(x(s))-f(w(s)))+C(f(x(s))-f(w(\vartheta(t))))]ds,$$

that is

$$\|x(t)-w(t)\|^2=\Big\|\int_{t_0}^{t}\Big[-A\big(x(s)-w(s)\big)+B\big(f(x(s))-f(w(s))\big)+C\big(f(x(s))-f(w(\vartheta(t)))\big)\Big]ds\Big\|^2.$$

By the Cauchy-Schwarz inequality

$$
\begin{aligned}
\|x(t)-w(t)\|^2 &\leqslant \Big(\int_{t_0}^{t}\big\|[-A(x(s)-w(s))+B(f(x(s))-f(w(s)))+C(f(x(s))-f(w(\vartheta(t))))]\big\|ds\Big)^2\\
&=\Big(\int_{t_0}^{t}\big\|1\times[-A(x(s)-w(s))+B(f(x(s))-f(w(s)))+C(f(x(s))-f(w(\vartheta(t))))]\big\|ds\Big)^2\\
&\leqslant \int_{t_0}^{t}1^2ds\times\int_{t_0}^{t}\big\|[-A(x(s)-w(s))+B(f(x(s))-f(w(s)))+C(f(x(s))-f(w(\vartheta(t))))]\big\|^2ds,
\end{aligned}
$$

and when $t_0\leqslant t\leqslant t_0+2\Delta$,

$$\|x(t)-w(t)\|^2 \leqslant 2\Delta\Big(3\int_{t_0}^{t}\|-A(x(s)-w(s))\|^2ds+3\int_{t_0}^{t}\|B(f(x(s))-f(w(s)))\|^2ds$$

$$
\begin{aligned}
&+3\int_{t_0}^{t}\|C(f(x(s))-f(w(\vartheta(s))))\|^2ds\Big)\\
\leqslant&2\Delta\Big[(3\|A\|^2+3\|B\|^2L^2)\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+3\|C\|^2\times\int_{t_0}^{t}\|f(x(s))-f(w(\vartheta(s)))\|^2ds\Big]\\
=&2\Delta(3\|A\|^2+3\|B\|^2L^2)\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+6\|C\|^2\Delta\\
&\times\int_{t_0}^{t}\|(f(x(s))-f(w(s)))+(f(w(s))-f(w(\vartheta(s))))\|^2ds\\
\leqslant&6\Delta(\|A\|^2+\|B\|^2L^2)\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+12\|C\|^2L^2\Delta\\
&\times\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+12\|C\|^2L^2\Delta\int_{t_0}^{t}\|w(s)-w(\vartheta(s))\|^2ds\\
\leqslant&6\Delta(\|A\|^2+\|B\|^2L^2)\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+12\|C\|^2L^2\Delta\\
&\times\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+24\|C\|^2L^2\Delta\int_{t_0}^{t}(\|w(s)\|^2+\|w(\vartheta(s))\|^2)ds.
\end{aligned}
$$

Applying Lemma 3.1,

$$
\begin{aligned}
\|x(t)-w(t)\|^2\leqslant&6\Delta(\|A\|^2+\|B\|^2L^2)\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+12\|C\|^2L^2\Delta\int_{t_0}^{t}\|x(s)-w(s)\|^2ds\\
&+24\|C\|^2L^2(1+\rho)\Delta\int_{t_0}^{t}\|w(s)\|^2ds\\
\leqslant&6\Delta(\|A\|^2+\|B\|^2L^2)\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+12\|C\|^2L^2\Delta\int_{t_0}^{t}\|x(s)-w(s)\|^2ds\\
&+24\|C\|^2L^2(1+\rho)\Delta\int_{t_0}^{t}\|(x(s)-w(s))+x(s)\|^2ds\\
\leqslant&6\Delta(\|A\|^2+\|B\|^2L^2)\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+12\|C\|^2L^2\Delta\int_{t_0}^{t}\|x(s)-w(s)\|^2ds\\
&+48\|C\|^2L^2(1+\rho)\Delta\int_{t_0}^{t}\|x(s)-w(s)\|^2ds+48\|C\|^2L^2(1+\rho)\Delta\int_{t_0}^{t}\|x(s)\|^2ds\\
=&[6\Delta(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2)+48\|C\|^2L^2(1+\rho)\Delta]\int_{t_0}^{t}\|x(s)-w(s)\|^2ds\\
&+48\|C\|^2L^2(1+\rho)\Delta\int_{t_0}^{t}\|x(s)\|^2ds.
\end{aligned}
$$

From the condition that neural network (2.2) is globally exponentially stable, we have

$$\|x(t)\|^2\leqslant\alpha^2\|x(t_0)\|^2\exp\{-2\beta(t-t_0)\},$$

and then

$$\int_{t_0}^{t}\|x(s)\|^2ds\leqslant\alpha^2\|x(t_0)\|^2/(2\beta).$$

Hence

$$
\begin{aligned}
\|x(t)-w(t)\|^2 \leqslant & [6\Delta(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2)+48\|C\|^2L^2(1+\rho)\Delta] \\
& \times \int_{t_0}^{t} \|x(s)-w(s)\|^2 ds + (24\|C\|^2L^2(1+\rho)\Delta\alpha^2/\beta)\|x(t_0)\|^2 \\
= & \hat{c}_2\|x(t_0)\|^2 + \hat{c}_1 \int_{t_0}^{t} \|x(s)-w(s)\|^2 ds.
\end{aligned}
$$

Based on the Gronwall-Bellman inequality and when $t_0 \leqslant t \leqslant t_0 + 2\Delta$,

$$
\|x(t)-w(t)\|^2 \leqslant \hat{c}_2\|x(t_0)\|^2 \exp(2\hat{c}_1\Delta). \tag{3.5}
$$

So when $t_0 + \Delta \leqslant t \leqslant t_0 + 2\Delta$, from (3.5) and the global exponential stability of (2.2), we can get

$$
\begin{aligned}
\|w(t)\|^2 = & \|(w(t)-x(t))+x(t)\|^2 \\
\leqslant & 2\|w(t)-x(t)\|^2 + 2\|x(t)\|^2 \\
\leqslant & 2\hat{c}_2\|x(t_0)\|^2 \exp(2\hat{c}_1\Delta) + 2\alpha^2\|x(t_0)\|^2 \exp\{-2\beta(t-t_0)\} \\
\leqslant & 2\hat{c}_2\|x(t_0)\|^2 \exp(2\hat{c}_1\Delta) + 2\alpha^2\|x(t_0)\|^2 \exp(-2\beta\Delta) \\
= & 2\{\hat{c}_2 \exp(2\hat{c}_1\Delta) + \alpha^2 \exp(-2\beta\Delta)\}\|x(t_0)\|^2 \\
\leqslant & \hat{c}(\sup_{t_0 \leqslant t \leqslant t_0+\Delta} \|w(t)\|^2),
\end{aligned} \tag{3.6}
$$

where $\hat{c} = 2\{\hat{c}_2 \exp(2\hat{c}_1\Delta) + \alpha^2 \exp(-2\beta\Delta)\}$. Denote

$$
H(\rho) = 2\left\{\hat{c}_2 \exp(2\hat{c}_1\Delta) + \alpha^2 \exp(-2\beta\Delta)\right\},
$$

then

$$
H(2) = 2\alpha^2 \exp(-2\beta\Delta) + 144\|C\|^2L^2\Delta\alpha^2 \exp\left\{12\Delta^2\left(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2\right) + 288\|C\|^2L^2\Delta^2\right\}\Big/\beta < 1.
$$

It is obvious that

$$
H(+\infty) > 1.
$$

And $H(\rho)$ is strictly increasing for ρ, so there exists a unique $\bar{\rho} \in (2, +\infty)$ such that

$$
H(\bar{\rho}) = 1.
$$

Denote

$$
S(\xi) = 6\xi^2\left[n\nu + 2\mu(1+3n\nu\xi^2)\exp(6\mu\xi^2)\right],
$$

and denote $\hat{\xi}$ as the unique positive solution for

$$
S(\xi) = 1,
$$

then

$$
\rho = 2\left(1 - 6\xi^2[n\nu + 2\mu(1+3n\nu\xi^2)\exp(6\mu\xi^2)]\right)^{-1} \in (2, +\infty)
$$

for $\xi \in (0, \hat{\xi})$. And ρ is strictly increasing for ξ, therefore, there exists a unique positive $\bar{\xi} \in (0, \hat{\xi})$ such that

$$
\rho = \bar{\rho},
$$

namely, $\bar{\xi}$ is the unique positive solution for (3.4).

Hence, when $\xi < \bar{\xi}$, we have

$$\hat{c} = 2\{\hat{c}_2 \exp(2\hat{c}_1\Delta) + \alpha^2 \exp(-2\beta\Delta)\} < 1.$$

Select $\gamma = -\ln \hat{c}/\Delta > 0$, and from (3.6), we can get

$$\sup_{t_0+\Delta \leqslant t \leqslant t_0+2\Delta} \|w(t)\|^2 \leqslant \exp(-\gamma\Delta)(\sup_{t_0 \leqslant t \leqslant t_0+\Delta} \|w(t)\|^2). \tag{3.7}$$

For any positive integer $l = 1, 2, \ldots$, according to the existence and uniqueness of the solution $w(t)$ of neural network (2.1), when $t \geqslant t_0 + (l-1)\Delta$, we can get

$$w(t, t_0, w_0) = w(t, t_0 + (l-1)\Delta, w(t_0 + (l-1)\Delta, t_0, w_0)). \tag{3.8}$$

From (3.7) and (3.8),

$$\begin{aligned}
&\sup_{t_0+l\Delta \leqslant t \leqslant t_0+(l+1)\Delta} \|w(t, t_0, w_0)\|^2 \\
&\quad = \Big(\sup_{t_0+(l-1)\Delta+\Delta \leqslant t \leqslant t_0+(l-1)\Delta+2\Delta} \|w(t, t_0 + (l-1)\Delta, w(t_0 + (l-1)\Delta, t_0, w_0))\|^2\Big) \\
&\quad \leqslant \exp(-\gamma\Delta)\Big(\sup_{t_0+(l-1)\Delta \leqslant t \leqslant t_0+l\Delta} \|w(t, t_0, w_0)\|^2\Big) \\
&\quad \leqslant \exp(-l\gamma\Delta)\Big(\sup_{t_0 \leqslant t \leqslant t_0+\Delta} \|w(t, t_0, w_0)\|^2\Big) \\
&\quad = M \exp(-l\gamma\Delta),
\end{aligned}$$

where $M = \sup_{t_0 \leqslant t \leqslant t_0+\Delta} \|w(t, t_0, w_0)\|^2$.

For any $t > t_0 + \Delta$, there exists a unique positive integer l such that $t_0 + l\Delta \leqslant t \leqslant t_0 + (l+1)\Delta$, and we get

$$\|w(t, t_0, w_0)\|^2 \leqslant M \exp\{-\gamma(t - t_0) + \gamma\Delta\} = [M \exp(\gamma\Delta)] \exp\{-\gamma(t - t_0)\},$$

that is

$$\|w(t, t_0, w_0)\| \leqslant [\bar{c} \exp(\bar{\gamma}\Delta)] \exp\{-\bar{\gamma}(t - t_0)\}, \tag{3.9}$$

where $\bar{c} = M^{\frac{1}{2}}$, $\bar{\gamma} = \gamma/2$. It is obvious that (3.9) holds for $t_0 \leqslant t \leqslant t_0 + \Delta$. Therefore, neural network (2.1) is globally exponentially stable, and the proof is completed. □

Theorem 3.3 shows that, when neural network (2.2) is globally exponentially stable, neural network (2.1), which is evoked by the deviating argument, is globally exponentially stable, provided that the intensity of the deviating argument satisfies (A3) and is smaller than the estimated upper bound.

Remark 3.4. Theorem 3.3 seems to involve three transcendental inequalities and one transcendental equation to be solved. In fact, (3.4) has a positive solution if and only if (A5) is satisfied. And (3.4) can be solved easily by MATLAB. Meanwhile, (A4) is satisfied in condition that (3.4) has a positive solution. Afterwards, it only needs to verify (A3). Hence, the conditions in Theorem 3.3 are easy to be validated.

3.2. The impact of deviating argument and stochastic disturbance on stability

In the preceding subsection, the conditions that guarantee the neural network evoked by deviating argument to be globally exponentially stable are received. In this subsection, we consider the impact of both the deviating argument and stochastic disturbance on the global exponential stability of neural network, when the original neural network is globally exponentially stable.

Consider the model of neural network evoked by the deviating argument and stochastic disturbance as follows

$$dw(t) = [-Aw(t) + Bf(w(t)) + Cf(w(\vartheta(t)))]dt + \delta w(t)dX(t), \tag{3.10}$$

with the initial state $w(t_0) = w_0$, where matrices A, B, C and $\vartheta(t)$ are the same as in (2.1), δ indicates the noise intensity, $X(t)$ is a one-dimensional Brownian motion, which is defined in the complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geqslant t_0}$ generated by $\{X(s) : t_0 \leqslant s \leqslant t\}$. Without the deviating argument and stochastic disturbance, neural network (3.10) degrades into (2.2).

For the sake of exposition, the following definition and lemma will be needed in the presence of stochastic disturbance.

Definition 3.5 ([25])**.** Neural network (3.10) is said to be almost surely globally exponentially stable if, for any $t_0 \geqslant 0$, $w_0 \in R^n$, the Lyapunov exponent $\limsup_{t\to+\infty}(\ln \|w(t, t_0, w_0)\|/t) < 0$ almost surely. Neural network (3.10) is said to be mean square globally exponentially stable if, for any $t > t_0$, $w_0 \in R^n$, the Lyapunov exponent $\limsup_{t\to+\infty}(\ln E\|w(t, t_0, w_0)\|^2/t) < 0$, where $w(t, t_0, w_0)$ is the state of neural network (3.10).

Lemma 3.6 ([25])**.** *Under* (A1), (A2), *and* (A3), *the mean square global exponential stability of* (3.10) *implies the almost sure global exponential stability of* (3.10).

Remark 3.7. Lemma 3.6 provides a convenient approach for proving the almost sure global exponential stability of (3.10).

In this subsection, the following assumptions are useful.

(A6) The parameters of neural network (2.2) satisfy the following inequality

$$\left(480\|C\|^2L^2\Delta\alpha^2/\beta\right)\exp\left\{24\Delta^2\left(\|A\|^2 + \|B\|^2L^2 + 42\|C\|^2L^2\right)\right\} + 2\alpha^2\exp(-2\beta\Delta) < 1,$$

where L is the Lipschitz constant decided by

$$\|f(u) - f(v)\| \leqslant L\|u - v\|,$$

and $\Delta > \ln(2\alpha^2)/(2\beta)$.

(A7) There exist positive constants ξ, μ, ν, and δ such that

$$12n\nu\xi^2 + (12\mu\xi + 2\delta^2)(4 + 12n\nu\xi^2)\xi\exp\left\{(12\mu\xi + 2\delta^2)\xi\right\} < 1,$$

where

$$\mu = \max_{1\leqslant i\leqslant n}\left(a_i^2 + nL_i^2\sum_{j=1}^n b_{ji}^2\right), \nu = \max_{1\leqslant i\leqslant n}\left(L_i^2\sum_{j=1}^n c_{ji}^2\right),$$

and δ and n indicate the intensity of stochastic disturbance and the number of units in neural network (3.10), respectively.

Lemma 3.8. *Under* (A1), (A2), (A3), *and* (A7), *for* (3.10), *the following inequality holds*

$$E\|w(\vartheta(t))\|^2 \leqslant \eta E\|w(t)\|^2,$$

where

$$\eta = 4\left\{1 - \left[12n\nu\xi^2 + \left(12\mu\xi + 2\delta^2\right)\left(4 + 12n\nu\xi^2\right)\xi\exp\left\{\left(12\mu\xi + 2\delta^2\right)\xi\right\}\right]\right\}^{-1},$$

$$\mu = \max_{1\leqslant i\leqslant n} \Big(a_i^2 + nL_i^2 \sum_{j=1}^{n} b_{ji}^2\Big), \nu = \max_{1\leqslant i\leqslant n} L_i^2\Big(\sum_{j=1}^{n} c_{ji}^2\Big),$$

and n *and* δ *correspond to the number of units and the intensity of stochastic disturbance in neural network* (3.10)*, respectively.*

Proof. For any $t \geqslant t_0$, by the property of $\vartheta(t)$ and the sequences $\{\xi_k\}$, $\{\sigma_k\}$, there exists a unique $k \in N$, such that

$$\vartheta(t) = \sigma_k \in [\xi_k, \xi_{k+1}), t \in [\xi_k, \xi_{k+1}),$$

and if $t \geqslant \sigma_k$,

$$w_i(t) = w_i(\sigma_k) + \int_{\sigma_k}^{t} \Big[-a_i w_i(s) + \sum_{j=1}^{n} b_{ij} f_j(w_j(s)) + \sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k))\Big] ds + \int_{\sigma_k}^{t} \delta w_i(s) dX(s) \tag{3.11}$$

for $i = 1, 2, \dots, n$, then

$$\begin{aligned} w_i^2(t) =& \Big\{w_i(\sigma_k) + \int_{\sigma_k}^{t} \Big[-a_i w_i(s) + \sum_{j=1}^{n} b_{ij} f_j(w_j(s)) + \sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k))\Big] ds + \int_{\sigma_k}^{t} \delta w_i(s) dX(s)\Big\}^2 \\ \leqslant& 2\Big\{w_i(\sigma_k) + \int_{\sigma_k}^{t} \Big[-a_i w_i(s) + \sum_{j=1}^{n} b_{ij} f_j(w_j(s)) + \sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k))\Big] ds\Big\}^2 + 2\delta^2\Big(\int_{\sigma_k}^{t} w_i(s) dX(s)\Big)^2 \\ \leqslant& 4w_i^2(\sigma_k) + 4\Big(\int_{\sigma_k}^{t} 1 \times \Big[-a_i w_i(s) + \sum_{j=1}^{n} b_{ij} f_j(w_j(s)) \\ &+ \sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k))\Big] ds\Big)^2 + 2\delta^2\Big(\int_{\sigma_k}^{t} w_i(s) dX(s)\Big)^2. \end{aligned}$$

From Cauchy-Schwarz inequality, we have

$$\begin{aligned} w_i^2(t) \leqslant& 4w_i^2(\sigma_k) + 4\int_{\sigma_k}^{t} 1^2 ds \times \int_{\sigma_k}^{t} \Big[-a_i w_i(s) + \sum_{j=1}^{n} b_{ij} f_j(w_j(s)) \\ &+ \sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k))\Big]^2 ds + 2\delta^2\Big(\int_{\sigma_k}^{t} w_i(s) dX(s)\Big)^2 \\ \leqslant& 4w_i^2(\sigma_k) + 12\xi \int_{\sigma_k}^{t} \Big[a_i^2 w_i^2(s) + \Big(\sum_{j=1}^{n} b_{ij} f_j(w_j(s))\Big)^2 \\ &+ \Big(\sum_{j=1}^{n} c_{ij} f_j(w_j(\sigma_k))\Big)^2\Big] ds + 2\delta^2\Big(\int_{\sigma_k}^{t} w_i(s) dX(s)\Big)^2 \\ \leqslant& 4w_i^2(\sigma_k) + 12\xi \int_{\sigma_k}^{t} \Big[a_i^2 w_i^2(s) + \Big(n\sum_{j=1}^{n} b_{ij}^2 f_j^2(w_j(s))\Big) \\ &+ \Big(n\sum_{j=1}^{n} c_{ij}^2 f_j^2(w_j(\sigma_k))\Big)^2\Big] ds + 2\delta^2\Big(\int_{\sigma_k}^{t} w_i(s) dX(s)\Big)^2. \end{aligned}$$

And hence we can get

$$\sum_{i=1}^{n} w_i^2(t) \leqslant 4\sum_{i=1}^{n} w_i^2(\sigma_k) + 12\xi \int_{\sigma_k}^{t} \Big[\sum_{i=1}^{n} a_i^2 w_i^2(s) + \Big(n\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2 f_j^2(w_j(s))\Big)$$

$$
\begin{aligned}
&+\Big(n\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}^2f_j^2(w_j(\sigma_k))\Big)\Big]ds+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2\\
\leqslant&4\sum_{i=1}^{n}w_i^2(\sigma_k)+12\xi\int_{\sigma_k}^{t}\Big[\sum_{i=1}^{n}a_i^2w_i^2(s)+\Big(n\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^2L_j^2w_j^2(s)\Big)\\
&+\Big(n\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}^2L_j^2w_j^2(\sigma_k)\Big)\Big]ds+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2\\
=&4\sum_{i=1}^{n}w_i^2(\sigma_k)+12\xi\int_{\sigma_k}^{t}\Big[\sum_{i=1}^{n}a_i^2w_i^2(s)+\Big(n\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ji}^2L_i^2w_i^2(s)\Big)\\
&+\Big(n\sum_{j=1}^{n}\sum_{i=1}^{n}c_{ji}^2L_i^2w_i^2(\sigma_k)\Big)\Big]ds+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2\\
=&4\sum_{i=1}^{n}w_i^2(\sigma_k)+12\xi\int_{\sigma_k}^{t}\Big[\sum_{i=1}^{n}a_i^2w_i^2(s)+\Big(n\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ji}^2L_i^2w_i^2(s)\Big)\\
&+\Big(n\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ji}^2L_i^2w_i^2(\sigma_k)\Big)\Big]ds+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2\\
=&4\sum_{i=1}^{n}w_i^2(\sigma_k)+12\xi\int_{\sigma_k}^{t}\Big[\sum_{i=1}^{n}\Big(a_i^2+nL_i^2\sum_{j=1}^{n}b_{ji}^2\Big)w_i^2(s)\\
&+n\sum_{i=1}^{n}\Big(L_i^2\sum_{j=1}^{n}c_{ji}^2\Big)w_i^2(\sigma_k)\Big]ds+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2\\
\leqslant&4\sum_{i=1}^{n}w_i^2(\sigma_k)+12\xi\int_{\sigma_k}^{t}\Big[\sum_{i=1}^{n}\mu w_i^2(s)+n\sum_{i=1}^{n}\nu w_i^2(\sigma_k)\Big]ds+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2\\
=&4\sum_{i=1}^{n}w_i^2(\sigma_k)+12\xi\mu\int_{\sigma_k}^{t}\sum_{i=1}^{n}w_i^2(s)ds+12\xi n\nu\int_{\sigma_k}^{t}\sum_{i=1}^{n}w_i^2(\sigma_k)ds+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2.
\end{aligned}
$$

That is

$$
\begin{aligned}
\|w(t)\|^2\leqslant&4\|w(\sigma_k)\|^2+12\xi\mu\int_{\sigma_k}^{t}\|w(s)\|^2ds+12\xi n\nu\int_{\sigma_k}^{t}\|w(\sigma_k)\|^2ds+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2\\
\leqslant&4\|w(\sigma_k)\|^2+12\xi\mu\int_{\sigma_k}^{t}\|w(s)\|^2ds+12\xi^2n\nu\|w(\sigma_k)\|^2+2\delta^2\sum_{i=1}^{n}\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2,
\end{aligned}
$$

and hence

$$
E\|w(t)\|^2\leqslant 4E\|w(\sigma_k)\|^2+12\xi\mu\int_{\sigma_k}^{t}E\|w(s)\|^2ds+12\xi^2n\nu E\|w(\sigma_k)\|^2+2\delta^2\sum_{i=1}^{n}E\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2.
$$

According to the Itô isometry

$$
E\Big(\int_{\sigma_k}^{t}w_i(s)dX(s)\Big)^2=E\Big(\int_{\sigma_k}^{t}w_i^2(s)ds\Big),
$$

we can obtain

$$
E\|w(t)\|^2\leqslant 4E\|w(\sigma_k)\|^2+12\xi\mu\int_{\sigma_k}^{t}E\|w(s)\|^2ds+12\xi^2n\nu E\|w(\sigma_k)\|^2+2\delta^2\sum_{i=1}^{n}E\Big(\int_{\sigma_k}^{t}w_i^2(s)ds\Big)
$$

$$
\begin{aligned}
&=4E\|w(\sigma_k)\|^2+12\xi\mu\int_{\sigma_k}^{t}E\|w(s)\|^2ds+12\xi^2n\nu E\|w(\sigma_k)\|^2+2\delta^2\Big(\int_{\sigma_k}^{t}E\|w(s)\|^2ds\Big)\\
&=(4+12\xi^2n\nu)E\|w(\sigma_k)\|^2+(12\xi\mu+2\delta^2)\int_{\sigma_k}^{t}E\|w(s)\|^2ds.
\end{aligned}
$$

Based on the Gronwall-Bellman inequality, then

$$
\begin{aligned}
E\|w(t)\|^2\leqslant&(4+12\xi^2n\nu)E\|w(\sigma_k)\|^2\exp\Big\{\int_{\sigma_k}^{t}(12\xi\mu+2\delta^2)ds\Big\}\\
\leqslant&(4+12\xi^2n\nu)\exp\Big\{(12\xi\mu+2\delta^2)\xi\Big\}E\|w(\sigma_k)\|^2.
\end{aligned}\tag{3.12}
$$

Exchanging the location of $w_i(t)$ and $w_i(\sigma_k)$ in (3.11), we can get

$$
E\|w(\sigma_k)\|\leqslant 4E\|w(t)\|^2+\big(12\xi\mu+2\delta^2\big)\int_{\sigma_k}^{t}E\|w(s)\|^2ds+12\xi^2n\nu E\|w(\sigma_k)\|^2,\tag{3.13}
$$

substituting (3.12) into (3.13),

$$
\begin{aligned}
E\|w(\sigma_k)\|^2\leqslant&4E\|w(t)\|^2+\big(12\xi\mu+2\delta^2\big)\int_{\sigma_k}^{t}(4+12\xi^2n\nu)\exp\Big\{(12\xi\mu+2\delta^2)\xi\Big\}\\
&\times E\|w(\sigma_k)\|^2ds+12\xi^2n\nu E\|w(\sigma_k)\|^2\\
\leqslant&4E\|w(t)\|^2+\Big\{12n\nu\xi^2+\Big(12\mu\xi+2\delta^2\Big)(4+12n\nu\xi^2)\xi\times\exp\Big(\big(12\mu\xi+2\delta^2\big)\xi\Big)\Big\}E\|w(\sigma_k)\|^2,
\end{aligned}
$$

it follows that

$$
E\|w(\vartheta(t))\|^2\leqslant\eta E\|w(t)\|^2.
$$

For the case when $t<\sigma_k$, the same conclusion can be gotten. And we complete the proof. □

Theorem 3.9. *Let* (A1), (A2), (A3), (A6), (A7) *hold, and let neural network* (2.2) *be globally exponentially stable. Neural network* (3.10) *is mean square globally exponentially stable, which implies that neural network* (3.10) *is almost surely globally exponentially stable, if* $|\delta|<\tilde{\delta}/\sqrt{2}$, $\xi<\tilde{\xi}$, *where* $\tilde{\delta}$ *is the unique positive solution of the transcendental equation*

$$
\begin{aligned}
&\big[480\Delta\|C\|^2L^2+4\delta^2\big]\alpha^2\exp\Big\{24\Delta^2\Big(\|A\|^2+\|B\|^2L^2\\
&+42\|C\|^2L^2\Big)+8\delta^2\Delta\Big\}\Big/\beta+2\alpha^2\exp(-2\beta\Delta)=1,
\end{aligned}\tag{3.14}
$$

and $\tilde{\xi}$ *is the unique positive solution of the following transcendental equation*

$$
2\tilde{c}_2\exp(2\Delta\tilde{c}_1)+2\alpha^2\exp(-2\beta\Delta)=1,\tag{3.15}
$$

where $\tilde{c}_1=12\Delta(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2(5+4\bar{\eta}))+2\tilde{\delta}^2$, $\tilde{c}_2=\Big[48\Delta\|C\|^2L^2(1+\bar{\eta})+\tilde{\delta}^2\Big]\alpha^2/\beta$, $\bar{\eta}=4\Big\{1-\Big[12n\nu\xi^2+\big(12\mu\xi+2\tilde{\delta}^2\big)(4+12n\nu\xi^2)\xi\exp\Big\{\big(12\mu\xi+2\tilde{\delta}^2\big)\xi\Big\}\Big]\Big\}^{-1}$, $\Delta>\ln(2\alpha^2)/(2\beta)$.

Proof. For simplicity, $x(t,t_0,x_0)$ and $w(t,t_0,w_0)$ are denoted as $x(t)$ and $w(t)$, respectively. When $t_0\leqslant t\leqslant t_0+2\Delta$, and from (2.2), (3.10), the initial condition $x_0=w_0$, Itô isometry, and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
E\|x(t)-w(t)\|^2\leqslant&\Big[12\Delta\big(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2\big)+4\delta^2\Big]\int_{t_0}^{t}E\|x(s)-w(s)\|^2ds\\
&+24\Delta\|C\|^2L^2\int_{t_0}^{t}E\|w(s)-w(\vartheta(s))\|^2ds+4\delta^2\int_{t_0}^{t}E\|x(s)\|^2ds.
\end{aligned}
$$

According to Lemma 3.8,

$$E\|w(s)-w(\vartheta(s))\|^2 \leqslant 2E\|w(s)\|^2+2E\|w(\vartheta(s))\|^2 \leqslant 2(1+\eta)E\|w(s)\|^2,$$

where

$$\eta = 4\left\{1-\left[12n\nu\xi^2+(12\mu\xi+2\delta^2)(4+12n\nu\xi^2)\xi\times\exp\left\{(12\mu\xi+2\delta^2)\xi\right\}\right]\right\}^{-1},$$

and

$$E\|w(s)\|^2 = E\|x(s)-w(s)-x(s)\|^2 \leqslant 2E\|x(s)-w(s)\|^2+2E\|x(s)\|^2,$$

then

$$\begin{aligned}
E\|x(t)-w(t)\|^2 \leqslant& \left[12\Delta\big(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2\big)+4\delta^2\right]\int_{t_0}^{t}E\|x(s)-w(s)\|^2 ds\\
&+96\Delta\|C\|^2L^2(1+\eta)\int_{t_0}^{t}E\|x(s)-w(s)\|^2 ds\\
&+96\Delta\|C\|^2L^2(1+\eta)\int_{t_0}^{t}E\|x(s)\|^2 ds+4\delta^2\int_{t_0}^{t}E\|x(s)\|^2 ds\\
=&\left\{12\Delta\Big(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2(5+4\eta)\Big)+4\delta^2\right\}\int_{t_0}^{t}E\|x(s)-w(s)\|^2 ds\\
&+\left[96\Delta\|C\|^2L^2(1+\eta)+4\delta^2\right]\int_{t_0}^{t}E\|x(s)\|^2 ds.
\end{aligned}$$

In view of the global exponential stability of (2.2), we get

$$\begin{aligned}
E\|x(t)-w(t)\|^2 \leqslant& \left\{12\Delta\Big(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2(5+4\eta)\Big)+4\delta^2\right\}\int_{t_0}^{t}E\|x(s)-w(s)\|^2 ds\\
&+\left[48\Delta\|C\|^2L^2(1+\eta)+2\delta^2\right]\alpha^2\|x(t_0)\|^2/\beta.
\end{aligned}$$

From the Gronwall-Bellman inequality and when $t_0 \leqslant t \leqslant t_0+2\Delta$,

$$\begin{aligned}
E\|x(t)-w(t)\|^2 \leqslant& \Big(\left[48\Delta\|C\|^2L^2(1+\eta)+2\delta^2\right]\alpha^2/\beta\Big)\|x(t_0)\|^2\exp\Big\{\Big[12\Delta\big(\|A\|^2\\
&+\|B\|^2L^2+2\|C\|^2L^2(5+4\eta)\big)+4\delta^2\Big](t-t_0)\Big\}\\
\leqslant& \Big(\left[48\Delta\|C\|^2L^2(1+\eta)+2\delta^2\right]\alpha^2/\beta\Big)\|x(t_0)\|^2\exp\Big\{\Big[12\Delta\big(\|A\|^2\\
&+\|B\|^2L^2+2\|C\|^2L^2(5+4\eta)\big)+4\delta^2\Big]2\Delta\Big\} = \tilde{c}_4\exp(2\Delta\tilde{c}_3)\|x(t_0)\|^2,
\end{aligned} \tag{3.16}$$

where $\tilde{c}_3 = 12\Delta\big(\|A\|^2+\|B\|^2L^2+2\|C\|^2L^2(5+4\eta)\big)+4\delta^2$, $\tilde{c}_4 = \left[48\Delta\|C\|^2L^2(1+\eta)+2\delta^2\right]\alpha^2/\beta$.

Hence, when $t_0+\Delta \leqslant t \leqslant t_0+2\Delta$, from (3.16) and the global exponential stability of (2.2), we can have

$$\begin{aligned}
E\|w(t)\|^2 =& E\|w(t)-x(t)+x(t)\|^2\\
\leqslant& 2E\|x(t)-w(t)\|^2+2E\|x(t)\|^2\\
\leqslant& 2\tilde{c}_4\exp(2\tilde{c}_3\Delta)\|x(t_0)\|^2+2\alpha^2\|x(t_0)\|^2\exp\{-2\beta(t-t_0)\}
\end{aligned}$$

$$\leqslant\Big(2\tilde{c}_4\exp(2\tilde{c}_3\Delta)+2\alpha^2\exp(-2\beta\Delta)\Big)\|x(t_0)\|^2.$$

Denote $J(\delta,\eta)=2\tilde{c}_4\exp(2\tilde{c}_3\Delta)+2\alpha^2\exp(-2\beta\Delta)$, then

$$J(0,4)=\Big(480\Delta\|C\|^2L^2\alpha^2/\beta\Big)\exp\Big\{24\Delta^2\Big(\|A\|^2+\|B\|^2L^2+42\|C\|^2L^2\Big)\Big\}+2\alpha^2\exp(-2\beta\Delta)<1.$$

It is clear that

$$J(+\infty,4)>1,$$

and $J(\delta,4)$ is strictly increasing for δ, so there exists a unique positive $\tilde{\delta}$ such that

$$J(\tilde{\delta},4)=1,$$

namely, $\tilde{\delta}$ is the unique positive solution for (3.14).

For a fixed $|\delta|<\tilde{\delta}$,

$$J(\delta,4)<1, J(\delta,+\infty)>1,$$

$J(\delta,\eta)$ is strictly increasing for η, so there exists a unique $\tilde{\eta}\in(4,+\infty)$ such that

$$J(\tilde{\delta}/\sqrt{2},\tilde{\eta})=1.$$

Denote

$$G(\delta,\xi)=12n\nu\xi^2+(12\mu\xi+2\delta^2)(4+12n\nu\xi^2)\xi\times\exp\{(12\mu\xi+2\delta^2)\xi\}.$$

Denote $\hat{\xi}$ as the unique positive solution of the following transcendental equation

$$G(\tilde{\delta},\xi)=1,$$

then

$$\bar{\eta}=4\Big\{1-\Big[12n\nu\xi^2+(12\mu\xi+2\tilde{\delta}^2)(4+12n\nu\xi^2)\xi\times\exp\{(12\mu\xi+2\tilde{\delta}^2)\xi\}\Big]\Big\}^{-1}\in(4,+\infty)$$

for $\xi\in(0,\hat{\xi})$. And $\bar{\eta}$ is strictly increasing for ξ, so there exists a unique positive $\tilde{\xi}\in(0,\hat{\xi})$ such that

$$\bar{\eta}=\tilde{\eta},$$

namely, $\tilde{\xi}$ is the unique positive solution of (3.15).

When $0<\delta<\tilde{\delta}/\sqrt{2}$, $0<\xi<\tilde{\xi}$, we have

$$0<G(\delta,\xi)<G(\tilde{\delta}/\sqrt{2},\xi)<G(\tilde{\delta}/\sqrt{2},\tilde{\xi})<G(\tilde{\delta},\tilde{\xi})<1,$$

then

$$\eta=4\Big/\Big(1-G(\delta,\xi)\Big)<4\Big/\Big(1-G(\tilde{\delta}/\sqrt{2},\xi)\Big)<4\Big/\Big(1-G(\tilde{\delta}/\sqrt{2},\tilde{\xi})\Big)<\tilde{\eta}=4\Big/\Big(1-G(\tilde{\delta},\tilde{\xi})\Big),$$

hence

$$0<J(\delta,\eta)<J(\tilde{\delta}/\sqrt{2},\eta)<J(\tilde{\delta}/\sqrt{2},\tilde{\eta})=1.$$

Select $\gamma=-\ln\Big(J(\delta,\eta)\Big)/\Delta>0$, and we can have

$$\sup_{t_0+\Delta\leqslant t\leqslant t_0+2\Delta}E\|w(t,t_0,x_0)\|^2\leqslant\exp(-\gamma\Delta)\Big(\sup_{t_0\leqslant t\leqslant t_0+\Delta}E\|w(t,t_0,w_0)\|^2\Big).$$

For any positive integer $l = 1, 2, \ldots$, from the existence and uniqueness of the solution $w(t)$ of neural network (3.10), when $t \geqslant t_0 + (l-1)\Delta$, we can get

$$w(t, t_0, x_0) = w(t, t_0 + (l-1)\Delta, w(t_0 + (l-1)\Delta, t_0, w_0)).$$

And hence

$$\begin{aligned}\sup_{t_0+l\Delta\leqslant t\leqslant t_0+(l+1)\Delta} & E\|w(t,t_0,x_0)\|^2 \\ &= \left(\sup_{t_0+(l-1)\Delta+\Delta\leqslant t\leqslant t_0+(l-1)\Delta+2\Delta} E\|w(t, t_0+(l-1)\Delta, w(t_0+(l-1)\Delta, t_0, x_0))\|^2\right) \\ &\leqslant \exp(-\gamma\Delta)\left(\sup_{t_0+(l-1)\Delta\leqslant t\leqslant t_0+l\Delta} E\|w(t,t_0,x_0)\|^2\right) \\ &\leqslant \exp(-\gamma l\Delta)\left(\sup_{t_0\leqslant t\leqslant t_0+\Delta} E\|w(t,t_0,x_0)\|^2\right) \\ &= \tilde{c}\exp(-\gamma l\Delta),\end{aligned}$$

where $\tilde{c} = \sup_{t_0\leqslant t\leqslant t_0+\Delta} E\|w(t,t_0,x_0)\|^2$. For any $t > t_0 + \Delta$, there exists a unique positive integer l such that $t_0 + l\Delta \leqslant t \leqslant t_0 + (l+1)\Delta$, and then

$$E\|w(t,t_0,x_0)\|^2 \leqslant \tilde{c}\exp\left\{-\gamma(t-t_0)+\gamma\Delta\right\} = \left(\tilde{c}\exp(\gamma\Delta)\right)\exp\left\{-\gamma(t-t_0)\right\}. \tag{3.17}$$

It is obvious that (3.17) holds when $t_0 \leqslant t \leqslant t_0 + \Delta$. Hence, neural network (3.10) is mean square globally exponentially stable, which implies that neural network (3.10) is also almost surely globally exponentially stable according to Lemma 3.6. And the proof is completed. □

Remark 3.10. In [25], the robustness of global exponential stability of neural networks in the presence of time delays and random disturbances was discussed. In this paper, the robustness of neural network evoked by deviating argument and stochastic disturbance is investigated, in which the type of the neural network can vary in alternately advanced and retarded. Therefore, the result gained in this paper is an effective supplement to the existing references.

4. Numerical examples

In this section, two examples are introduced to illustrate the effectiveness of the proposed criteria.

Example 4.1. Consider the following neural network

$$\begin{cases}\dot{x}_1(t) = -x_1(t) - 2\sin(x_1(t)/20) + 2\sin(x_2(t)/20), \\ \dot{x}_2(t) = -x_2(t) + 2\sin(x_1(t)/20) - 2\sin(x_2(t)/20).\end{cases} \tag{4.1}$$

The state $x(t) = (x_1(t), x_2(t))^T$ of neural network (4.1) is globally exponentially stable with $\alpha = 1$, $\beta = 0.5$, which is depicted in Figure 1.

Adding the deviating argument into neural network (4.1), it becomes

$$\begin{cases}\dot{w}_1(t) = -w_1(t) - 1.9999\sin(w_1(t)/20) + 1.9998 \\ \qquad\qquad \times\sin(w_2(t)/20) - 0.0001\sin(w_1(\vartheta(t))) + 0.0002\sin(w_2(\vartheta(t))), \\ \dot{w}_2(t) = -w_2(t) + 2\sin(w_1(t)/20) - 1.9997\times\sin(w_2(t)/20) - 0.0003\sin(w_2(\vartheta(t))),\end{cases} \tag{4.2}$$

where $\vartheta(t)$ is the deviating argument. From (4.2), it can be seen that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1.9999 & 1.9998 \\ 2 & -1.9997 \end{pmatrix}, \quad C = \begin{pmatrix} -0.0001 & 0.0002 \\ 0 & -0.0003 \end{pmatrix}, \quad L = 1/20.$$

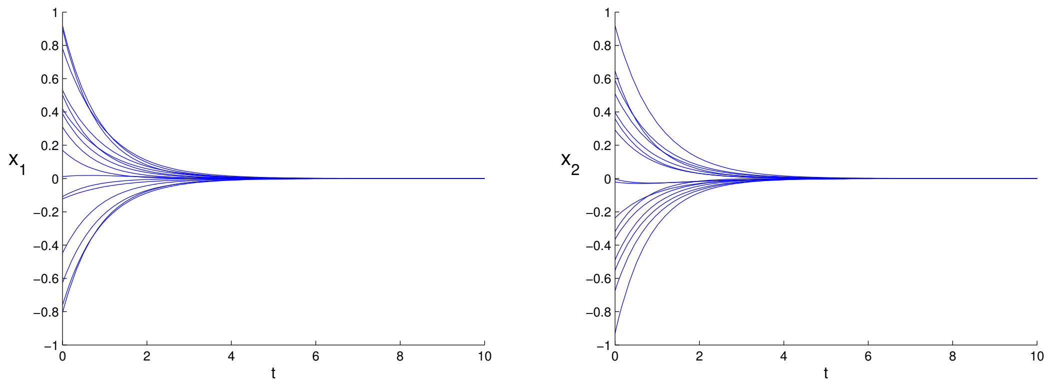

Figure 1: Transient behavior of $x_1(t)$ and $x_2(t)$ in (4.1).

Let $\Delta = 1 > \ln(2\alpha^2)/(2\beta) = 0.6931$. By computing, we get

$$\|A\|^2 = 1, \quad \|B\|^2 = 15.9976, \quad \|C\|^2 = 1.332 \times 10^{-7}, \quad \rho = 1.2, \upsilon = 2.5 \times 10^{-5}, \quad \mu = 1.04, \nu = 3.25 \times 10^{-10}.$$

From (A4), it can be calculated that

$$\xi < 0.2374.$$

From (3.4), we can get

$$\rho = 30.4206, \bar{\xi} = 0.2315.$$

Selecting $\vartheta(t) = \xi = 1/9 < \bar{\xi} = 0.2315$, it can be

$$1/9 \times (1.2 + 2 \times 2.5 \times 10^{-5}) \times \exp(1.2 \times 1/9) = 0.1524 < 1,$$

and (A3) is satisfied.

Therefore, the conditions in Theorem 3.3 are all satisfied. From Theorem 3.3, neural network (4.2) is globally exponentially stable, and the simulations in Figure 2 confirm well with the theoretical results.

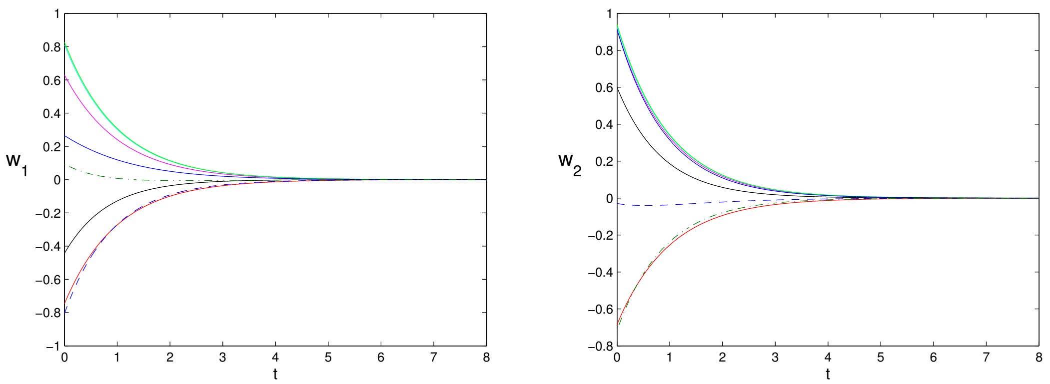

Figure 2: Transient behavior of $w_1(t)$ and $w_2(t)$ in (4.2) with $\xi = 1/9$.

Example 4.2. The following single-state neural network is considered

$$\dot{x}(t) = -3.1x(t) + 0.1\tanh(x(t)). \tag{4.3}$$

Based on the comparison principle, the state $x(t)$ of (4.3) is globally exponentially stable with $\alpha = 1$, $\beta = 3$, which is illustrated in Figure 3.

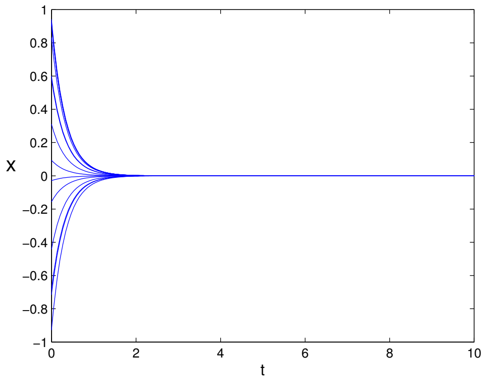

Figure 3: Transient behavior of $x(t)$ in (4.3).

When the deviating argument and stochastic disturbance to be generated, neural network (4.3) becomes

$$dw(t) = \big[-3.1w(t) + 0.099\tanh(w(t)) + 0.001 \times \tanh(w(\vartheta(t)))\big]dt + \delta w(t)dX(t), \tag{4.4}$$

where $\vartheta(t)$ is the deviating argument, δ is the noise intensity, and $X(t)$ is a one-dimensional Brownian motion, which is defined in the probability space.

From (4.4), we have

$$A = 3.1, B = 0.099, C = 0.001, L = 1.$$

And

$$\|A\|^2 = 9.61,\ \ \|B\|^2 = 9.801 \times 10^{-3},\ \ \|C\|^2 = 10^{-6},\ \ \rho = 3.1990,\ \ \upsilon = 0.001,\ \ \mu = 9.6198,\ \ \nu = 10^{-6}.$$

Let $\Delta = 0.12 > \ln\left(2\alpha^2\right)/(2\beta) = 0.1155$, and

$$\begin{aligned}\left(480\|C\|^2L^2\Delta\alpha^2/\beta\right)\exp\left\{24\Delta^2\left(\|A\|^2 + \|B\|^2L^2 + 42\|C\|^2L^2\right)\right\} + 2\alpha^2\exp(-2\beta\Delta)\\ = 5.3353 \times 10^{-4} + 0.9735 = 0.9740 < 1,\end{aligned}$$

then (A6) is satisfied. From (3.14),

$$(5.76 \times 10^{-5} + 4\delta^2) \times \exp(3.3246 + 0.96\delta^2)/3 + 2 \times \exp(-0.72) = 1,$$

we get its solution $\delta = \tilde{\delta} = 0.0265$. Combining with (A7), it derives $\xi < \hat{\xi} = 0.0420$. Substituting $\tilde{\delta} = 0.0265$ into (3.15),

$$(4.701 \times 10^{-4} + 3.84 \times 10^{-6}\bar{\eta}) \times \exp(3.3249 + 2.765 \times 10^{-6}\bar{\eta}) + 2 \times \exp(-0.72) = 1,$$

it gets $\bar{\eta} = 125.7645$, namely

$$\begin{aligned}\bar{\eta} =& 4\left\{1 - \left[12n\nu\xi^2 + \left(12\mu\xi + 2\tilde{\delta}^2\right) \times (4 + 12n\nu\xi^2) \times \xi \times \exp\left\{(12\mu\xi + 2\tilde{\delta}^2)\xi\right\}\right]\right\}^{-1}\\ =& 1.2 \times 10^{-5}\xi^2 + (115.4376\xi + 0.0014) \times (4 + 1.2\\ & \times 10^{-5}\xi^2) \times \xi \times \exp(115.4376\xi^2 + 0.0014\xi)\\ =& 125.7645,\end{aligned}$$

then $\xi = \tilde{\xi} = 0.0415$, namely, it is the solution of (3.15).

Choosing

$$\delta = 0.006 < \tilde{\delta}/\sqrt{2} = 0.0187, \quad \xi = 0.008 < \tilde{\xi} = 0.0415,$$

and by calculating, we can get

$$0.008 \times (3.1990 + 2 \times 0.001) \times \exp(3.1990 \times 0.008) = 0.0263 < 1,$$

namely, (A3) is satisfied. Hence, the conditions in Theorem 3.9 are all satisfied. And accordingly, neural network (4.4) is mean square globally exponentially stable and also almost surely globally exponentially stable with $\delta = 0.006$, $\xi = 0.008$. The simulations shown in Figure 4 agree well with the theoretical results.

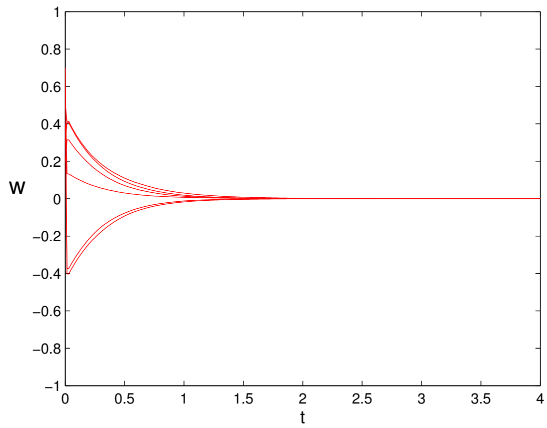

Figure 4: Transient behavior of $w(t)$ in (4.4) with $\delta = 0.006$ and $\xi = 0.008$.

It can be seen in Figure 5 that the state $w(t)$ of (4.4) is unstable with $\delta = 0.026 > \tilde{\delta}/\sqrt{2}$, $\xi = 0.0085$, when the conditions in Theorem 3.9 are not satisfied.

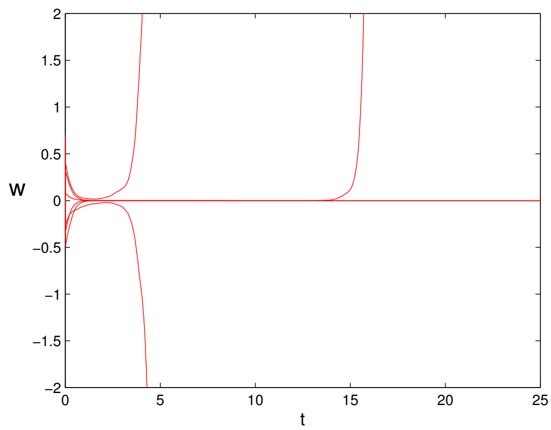

Figure 5: Transient behavior of $w(t)$ in (4.4) with $\delta = 0.026$ and $\xi = 0.0085$.

Figure 6 depicts the state $w(t)$ of (4.4) with $\delta = 0.016 < \tilde{\delta}/\sqrt{2}$, $\xi = 0.0416 > \tilde{\xi}$. Obviously, the conditions in Theorem 3.9 are not satisfied and the state $w(t)$ is unstable.

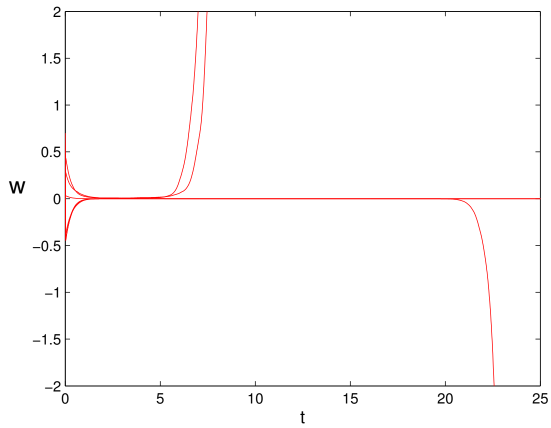

Figure 6: Transient behavior of $w(t)$ in (4.4) with $\delta = 0.016$ and $\xi = 0.0416$.

When $\delta = 0.021 > \tilde{\delta}/\sqrt{2}$, $\xi = 0.0418 > \tilde{\xi}$, it is clear that the conditions in Theorem 3.9 are not satisfied, and the state $w(t)$ of (4.4) is divergent, which can be easily seen in Figure 7.

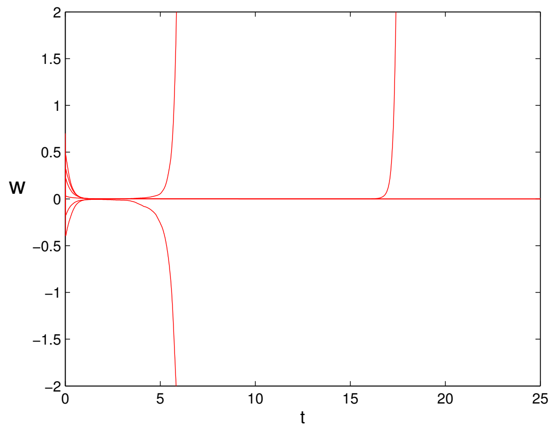

Figure 7: Transient behavior of $w(t)$ in (4.4) with $\delta = 0.021$ and $\xi = 0.0418$.

5. Concluding remarks

Neural networks evoked by deviating argument and stochastic disturbance have become a hotspot. In this paper, the robustness of this type of neural network is studied. For a given originally globally exponentially stable neural network, the upper bounds of the intensity of deviating argument and stochastic disturbance are derived to guarantee the disturbed neural network to be globally exponentially stable by means of the Gronwall-Bellman inequality and some mathematical analysis techniques. The results obtained provide some new approaches for the analysis and design of neural networks evoked by deviating argument and stochastic disturbance. The simulation results also confirm the validity of theoretical results.

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