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Some fixed point theorems for φ-contractive mappings in fuzzy normed linear spaces

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Abstract

In this paper a new concept of comparison function is introduced and discussed and some fixed point theorems are established for φ-contractive mappings in fuzzy normed linear spaces. In this way we obtain fuzzy versions of some classical fixed point theorems such as Nemytzki-Edelstein's theorem and Maia's theorem.

Keywords: Fuzzy normed linear spaces, φ-contractive mappings, fixed point theorems.
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1. Introduction

After Zadeh introduced in his famous paper [31] the brilliant concept of fuzzy set, many mathematicians became aware of the multitude of possibilities of extending the classical results in the new fuzzy framework and of their numerous applications. The fuzzification of classical structures started in 1968, when Chang published his seminal paper entitled "Fuzzy topological spaces" [9]. The fuzzification of algebraic structures has been initiated by Rosenfeld [24] in 1971. Other examples of fuzzification of classical structures are: fuzzy relations, fuzzy metric spaces, fuzzy topological vector spaces, fuzzy measure theory and fuzzy integrals, etc.. For an excellent overview on the evolution of mathematics of fuzziness we refer to the surveys of Kerre [21] and Dzitac [12].

The concept of fuzzy norm was introduced for the first time by Katsaras [20]. Since then, many mathematicians have introduced several notions of fuzzy norm from different points of view. Thus, Felbin [14] advanced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of a linear space. Following Cheng and Mordeson [10], in 2003, Bag and Samanta [3] introduced a more adequate notion of fuzzy norm. Another approaches for fuzzy norm was considered in papers [1, 16, 22, 27].

After the definition of fuzzy metric space, the fixed point theory on fuzzy metric spaces constitutes an attraction for many authors which have generalized and extended fixed point, common fixed point, and coincidence point theorems on fuzzy context (see [2, 11, 13, 17–19, 23]). Recently, fuzzy version of various fixed point theorems was discussed in the context of fuzzy normed linear spaces (see [5, 6, 26, 30, 32]).

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2. Preliminaries

In this section we recall some known notions and results.

Theorem 2.1 (Nemytzki-Edelstein's theorem)**.** *Let* (X, d) *be a compact metric space and* $f : X \to X$ *be a contractive mapping. Then* f *has a unique fixed point* x^* *and* $(f^n(x_0))$ *converges to* x^* *for all* $x_0 \in X$.

Theorem 2.2 (Maia's theorem)**.** *Let* X *be a nonempty set,* d *and* ρ *be two metrics on* X *and* $f : X \to X$ *such that:*

1. $d(x, y) \leqslant \rho(x, y),\ \forall x, y \in X$;
2. $f : (X, d) \to (X, d)$ *is continuous;*
3. $f : (X, \rho) \to (X, \rho)$ *is a contraction;*
4. (X, d) *is a complete metric space.*

Then f *has a unique fixed point* x^* *and* $(f^n(x_0))$ *converges in* (X, d) *to* x^* *for all* $x_0 \in X$.

Definition 2.3 ([29])**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called triangular norm (t-norm) if it satisfies the following conditions:

1. $a * b = b * a,\ \forall a, b \in [0, 1]$;
2. $a * 1 = a,\ \forall a \in [0, 1]$;
3. $(a * b) * c = a * (b * c),\ \forall a, b, c \in [0, 1]$;
4. If $a \leqslant c$ and $b \leqslant d$ with $a, b, c, d \in [0, 1]$, then $a * b \leqslant c * d$.

Example 2.4. Three basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0, 1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Definition 2.5 ([22])**.** Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0,\ \forall x \in X$;
(N2) $[N(x, t) = 1,\ \forall t > 0]$ if and only if $x = 0$;
(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right),\ \forall x \in X,\ \forall t \geqslant 0,\ \forall \lambda \in \mathbb{K}^*$;
(N4) $N(x + y, t + s) \geqslant N(x, t) * N(y, s),\ \forall x, y \in X,\ \forall t, s \geqslant 0$;
(N5) $\forall x \in X$, $N(x, \cdot)$ is left continuous and $\lim\limits_{t\to\infty} N(x, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNLS).

Remark 2.6.

a) Bag and Samanta [3, 4] gave a similar definition for $* = \wedge$, but in order to obtain some important results they assume that the fuzzy norm satisfies also the following conditions:
  (N6) $N(x, t) > 0,\ \forall t > 0 \Rightarrow x = 0$;
  (N7) $\forall x \neq 0, N(x, \cdot)$ is a continuous function and strictly increasing on the subset $\{t : 0 < N(x, t) < 1\}$ of $\mathbb{R}$.
  In this paper we do not need to assume these conditions.
b) Goleţ [16], and Alegre and Romaguera [1] gave also this definition in the context of real vector spaces.

Theorem 2.7 ([22])**.** *Let* $(X, N, *)$ *be a fuzzy normed linear space. For* $x \in X$, $r \in (0, 1), t > 0$ *we define the open ball* $B(x, r, t) := \{y \in X\ :\ N(x - y, t) > 1 - r\}$. *Then*

$$\mathcal{T}_N := \{T \subset X\ :\ x \in T \textit{ iff } (\exists)t > 0, r \in (0, 1)\ :\ B(x, r, t) \subseteq T\}$$

is a topology on X.

Definition 2.8 ([3]). Let $(X, N, *)$ be an FNLS and (x_n) be a sequence in X.

1. The sequence (x_n) is said to be convergent if there exists $x \in X$ such that
$$\lim_{n\to\infty} N(x_n - x, t) = 1 \ , \ \forall t > 0 \, .$$
In this case, x is called the limit of the sequence (x_n) and we denote $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
2. The sequence (x_n) is called Cauchy sequence if
$$\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1 \ , \ \forall t > 0, \forall p \in \mathbb{N}^* \, .$$
3. $(X, N, *)$ is said to be complete if any Cauchy sequence in X is convergent to a point in X. A complete FNLS will be called a fuzzy Banach space.

Definition 2.9 ([4]). Let $(X, N_1, *_1), (Y, N_2, *_2)$ be two fuzzy normed linear spaces. A mapping $T : X \to Y$ is said to be fuzzy continuous at $x_0 \in X$, if for all $\varepsilon > 0$, for all $\alpha \in (0, 1)$, there exist $\delta = \delta(\varepsilon, \alpha) > 0$, and $\beta = \beta(\varepsilon, \alpha) \in (0, 1)$ such that for all $x \in X$, $N_1(x - x_0, \delta) > \beta$ implies $N_2(T(x) - T(x_0), \varepsilon) > \alpha$. If T is fuzzy continuous at each point of X, then T is called fuzzy continuous on X.

Theorem 2.10 ([4]). *Let* $(X, N_1, *_1), (Y, N_2, *_2)$ *be two fuzzy normed linear spaces. A mapping* $T : X \to Y$ *is fuzzy continuous at* $x_0 \in X$*, if and only if for any sequence* $(x_n) \subseteq X$*, with* $x_n \to x_0$*, implies* $T(x_n) \to T(x_0)$*.*

Definition 2.11 ([28]). Let $(X, N, *)$ be a fuzzy normed linear space and $T \subseteq X$. An element $x_0 \in T$ is called an interior point of T if there exist $\alpha_0 \in (0, 1)$ and $t_0 > 0$ such that $B(x_0, \alpha_0, t_0) \subseteq T$. We will denote by $Int(T)$ the set of all interior points of T; T is called fuzzy open set if $Int(T) = T$.

A fuzzy normed linear space $(X, N, *)$ is called fuzzy compact if every fuzzy open cover of X has a finite sub-cover. A fuzzy normed linear space $(X, N, *)$ is called fuzzy sequentially compact if every sequence of points of X has a subsequence convergent to a point of X.

3. Main results

Definition 3.1. A function $\varphi : (0, \infty) \to (0, \infty)$ will be called comparison function if $\varphi^n(t) \to \infty$, as $n \to \infty$ for all $t > 0$, where φ^n stands for the n^{th} iterate of φ.

Remark 3.2. Other comparison functions were considered by numerous authors (see for instance [7, 25]).

Example 3.3. For $c \in (0, 1)$, the mapping $\varphi : (0, \infty) \to (0, \infty)$ defined by $\varphi(t) = \frac{t}{c}$ is a comparison function.

Definition 3.4. Let $(X, N_1, *_1), (X, N_2, *_2)$ be two fuzzy normed linear spaces. A mapping $T : (X, N_1, *_1) \to (X, N_2, *_2)$ is said to be φ-metric-contraction if there exists a comparison function $\varphi : (0, \infty) \to (0, \infty)$ such that
$$N_2(Tx - Ty, t) \geqslant N_1(x - y, \varphi(t)), \forall x, y \in X, \forall t > 0 \, .$$
A mapping $T : (X, N_1, *_1) \to (X, N_2, *_2)$ is said to be φ-norm-contraction if there exists a comparison function $\varphi : (0, \infty) \to (0, \infty)$ such that
$$N_2(Tx, t) \geqslant N_1(x, \varphi(t)), \forall x \in X, \forall t > 0 \, .$$

The next example shows that the two notions are different from one another.

Example 3.5. Let $\varphi : (0, \infty) \to (0, \infty), \varphi(t) = \frac{t}{1/2}$ be a comparison function and $(X, N, \wedge)$ be a fuzzy normed linear space, where $X = \mathbb{R}$ and N is defined by
$$N(x, t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$
Then

1. $T : X \to X$ defined by $T(x) = \frac{1}{2}x + \frac{1}{2}$ is a φ-metric-contraction but T is not a φ-norm-contraction;

2. $T : X \to X$ defined by $T(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 1) \cup (2, \infty), \\ x-1, & \text{if } x \in [1,2] \end{cases}$ is a φ-norm-contraction, but T is not a φ-metric-contraction.

Proof. 1. For all $x, y \in X, t > 0$, we have that

$$N(Tx - Ty, t) = N\left(\frac{1}{2}(x-y), t\right) = N\left(x - y, \frac{t}{1/2}\right) = N\left(x - y, \varphi(t)\right).$$

Thus T is a φ-metric-contraction.

On the other hand, for $x_0 = \frac{1}{2}$ and $t > 0$ we have that

$$N(Tx_0, t) = N(3/4, t) = \frac{t}{t + 3/4} < \frac{\frac{t}{1/2}}{\frac{t}{1/2} + 1/2} = N\left(x_0, \frac{t}{1/2}\right) = N\left(x_0, \varphi(t)\right).$$

Thus T is not a φ-norm-contraction.

2. If $x \in (-\infty, 1) \cup (2, \infty)$, then $N(Tx, t) = N(0, t) = 1 \geqslant N(x, \varphi(t))$ for all $t > 0$. If $x \in [1, 2]$, then

$$N(Tx, t) = N(x - 1, t) = \frac{t}{t + x - 1} \geqslant \frac{\frac{t}{1/2}}{\frac{t}{1/2} + x} = N\left(x, \varphi(t)\right), \quad \forall t > 0.$$

Thus T is a φ-norm-contraction.

On the other hand, for $x = 2, y = 1$, and $t > 0$, we have that

$$N(Tx - Ty, t) = N(1, t) = \frac{t}{t+1} < \frac{\frac{t}{1/2}}{\frac{t}{1/2} + 1} = N\left(x - y, \varphi(t)\right).$$

Hence T is not a φ-metric-contraction. □

Remark 3.6. If $T(0) = 0$, then any φ-metric-contraction T is a φ-norm-contraction.

Indeed, by $N_2(Tx - Ty, t) \geqslant N_1(x - y, \varphi(t))$ for all $x, y \in X$ and $t > 0$, in particular, for $y = 0$, we obtain the desired result.

Remark 3.7. Let $\varphi : (0, \infty) \to (0, \infty)$ be a comparison function and let $(X, N_1, *_1)$, $(X, N_2, *_2)$ be two fuzzy normed linear spaces. If $S, T : (X, N_1, *_1) \to (X, N_2, *_2)$ satisfy

$$N_2(Sx - Ty, t) \geqslant N_1(x - y, \varphi(t)), \ \forall x, y \in X, \ \forall t > 0,$$

then $S = T$ and S, T are φ-metric-contractions.

Indeed, for $y = x$, we obtain that

$$N_2((S - T)x, t) \geqslant N_1(0, \varphi(t)) = 1, \ \forall x \in X, \ \forall t > 0.$$

Thus $(S - T)(x) = 0, \ \forall x \in X$, i.e., $S = T$.

Remark 3.8. Let $\varphi : (0, \infty) \to (0, \infty)$ be a comparison function and let $(X, N_1, *_1)$, $(X, N_2, *_2)$ be two fuzzy normed linear spaces. If $S, T : (X, N_1, *_1) \to (X, N_2, *_2)$ satisfy

$$N_2((S - T)x, t) \geqslant N_1(x, \varphi(t)), \ \forall x \in X, \ \forall t > 0,$$

then $S(0) = T(0)$.

Indeed, $N_2((S - T)(0), t) \geqslant N_1(0, \varphi(t)) = 1, \ \forall t > 0$. Thus $(S - T)(0) = 0$.

The first result states that every iterate of a φ-norm-contraction has a unique fixed point.

Theorem 3.9. *Let $(X, N, *)$ be a fuzzy normed linear space and $T : (X, N, *) \to (X, N, *)$. If there exists $q \in \mathbb{N}^*$ such that T^q is a φ-norm-contraction, then T has 0 as a unique fixed point and*

$$\lim_{n\to\infty} T^n(x) = 0, \ \forall x \in X .$$

Proof. By $N(T^q x, t) \geqslant N(x, \varphi(t))$ for all $x \in X$ and $t > 0$, it results that $N(T^q(0), t) \geqslant N(0, \varphi(t)) = 1$ for all $t > 0$. Thus $T^q(0) = 0$.

We show that T^q has 0 as a unique fixed point. We assume that $x_0 \neq 0$ is a fixed point for T^q. As $x_0 \neq 0$, there exists $s > 0$ such that $N(x_0, s) = a < 1$. Then

$$a = N(x_0, s) = N(T^{nq}(x_0), s) \geqslant N(T^{(n-1)q}(x_0), \varphi(s)) \geqslant \cdots \geqslant N(x_0, \varphi^n(s)) \to 1,$$

as $n \to \infty$. Thus $a = 1$, which is a contradiction. Therefore T^q has 0 as a unique fixed point.

We prove now that 0 is a fixed point for T. Let $z = T(0)$. Then

$$z = T(0) = T(T^q(0)) = T^{q+1}(0) = T^q(T(0)) = T^q(z).$$

Thus z is a fixed point for T^q. Hence $z = 0$ and therefore $T(0) = 0$.

Now we show the uniqueness. We assume that $y \in X$ is a fixed point for T. Thus $T^q y = y$. As T^q has 0 as a unique fixed point, we obtain that $y = 0$.

Let $x \in X$ be arbitrary. Then

$$N(T^{nq}x, t) \geqslant N(T^{(n-1)q}x, \varphi(t)) \geqslant \cdots \geqslant N(x, \varphi^n(t)) \to 1, \text{ as } n \to \infty.$$

Thus $T^{nq}x \to 0$, as $n \to \infty$. If $m \in \mathbb{N}$ is arbitrary, then $m = nq + r$ and

$$T^m x = T^{nq+r}x = T^{nq}(T^r x) \to 0, \text{ as } n \to \infty,$$

which finishes the proof. □

Theorem 3.10 (A fuzzy version of Nemytzki-Edelstein's theorem)**.** *Let $(X, N, *)$ be a fuzzy sequentially compact normed linear space and $T : X \to X$ be a φ-metric-contraction. Then T has a unique fixed point x^* and*

$$\lim_{n\to\infty} T^n(x) = x^*, \ \forall x \in X.$$

Proof. Let $x \in X$ be arbitrary and $x_n = T^n(x)$. As $(X, N, *)$ is fuzzy sequentially compact, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x^* \in X$, namely

$$\lim_{k\to\infty} N(x_{n_k} - x^*, t) = 1, \ \forall t > 0.$$

Therefore

$$\begin{aligned} N(T(x_{n_k}) - x_{n_k}, t) &= N(T(T^{n_k}(x)) - T^{n_k}(x), t) \\ &= N(T^{n_k+1}(x) - T^{n_k}(x), t) \\ &\geqslant N(T^{n_k}(x) - T^{n_k-1}(x), \varphi(t)) \geqslant \cdots \geqslant N(Tx - x, \varphi^{n_k}(t)) \to 1, \text{ as } k \to \infty, \ \forall t > 0. \end{aligned}$$

On the other hand

$$N(T(x_{n_k}) - T(x^*), t) \geqslant N(x_{n_k} - x^*, \varphi(t)) \to 1, \text{ as } k \to \infty, \ \forall t > 0.$$

Finally, we have that

$$\begin{aligned} N(T(x^*) - x^*, t) &= N(T(x^*) - T(x_{n_k}) + T(x_{n_k}) - x_{n_k} + x_{n_k} - x^*, t) \\ &\geqslant N\left(T(x^*) - T(x_{n_k}), \frac{t}{3}\right) * N\left(T(x_{n_k}) - x_{n_k}, \frac{t}{3}\right) * N\left(x_{n_k} - x^*, \frac{t}{3}\right) \to 1, \end{aligned}$$

as $k \to \infty$ for all $t > 0$. Thus $T(x^*) = x^*$, i.e., x^* is a fixed point for T. We show that $x_n \to x^*$, as $n \to \infty$. Indeed,

$$\begin{aligned} N(x_n - x^*, t) &= N(T^n(x) - T(x^*), t) \\ &\geqslant N(T^{n-1}(x) - x^*, \varphi(t)) \geqslant \cdots \geqslant N(T(x) - x^*, \varphi^{n-1}(t)) \to 1, \text{ as } n \to \infty, \ \forall t > 0. \end{aligned}$$

Now we prove the uniqueness. We assume that there exist $x, y \in X, x \neq y$ such that $T(x) = x, T(y) = y$. Therefore, there exists $s > 0$ such that $N(x - y, s) = a < 1$. Consequently,

$$\begin{aligned} a = N(x - y, s) &= N(T^n(x) - T^n(y), s) \\ &\geqslant N(T^{n-1}(x) - T^{n-1}(y), \varphi(s)) \geqslant \cdots \geqslant N(x - y, \varphi^n(s)) \to 1, \text{ as } n \to \infty. \end{aligned}$$

Thus $a = 1$, which is contradiction. Hence T has a unique fixed point. □

In [15] a positive fuzzy metric is introduced, which was necessary for proving the space endowed with this metric to become a Hausdorff fuzzy metric space. In 2009, Sadeqi and Kia [28], proved the same result within the framework of fuzzy normed linear spaces where the fuzzy norm satisfies (N6) and (N7). Later in 2014, Nadaban and Dzitac [22] proved that a fuzzy normed linear space $(X, N, *)$ as in our setting is Hausdorff if $\sup_{x \in (0;1)} x * x = 1$.

Our result is valid in more general conditions and it is crucial for the uniqueness of a convergent sequence limit.

Proposition 3.11. *If $(X, N, *)$ is a fuzzy normed linear space, then $(X, \mathcal{T}_N)$ is Hausdorff.*

Proof. Let $x, y \in X, x \neq y$. Then there exists $t > 0$ such that $N(x - y, t) = a < 1$. Let $\epsilon > 0$ such that $0 < a + \epsilon < 1$. By [8] we obtain that there exist $\alpha, \beta \in (0, 1)$ such that $\alpha * \beta = a + \epsilon$. Let $r_1 = \max\{\alpha, \beta\}$. Then $r_1 * r_1 \geqslant \alpha * \beta = a + \epsilon > a$. We show that

$$B\left(x, 1 - r_1, \frac{t}{2}\right) \cap B\left(y, 1 - r_1, \frac{t}{2}\right) = \emptyset.$$

Indeed, if we suppose that there exists $z \in B\left(x, 1 - r_1, \frac{t}{2}\right) \cap B\left(y, 1 - r_1, \frac{t}{2}\right)$, we obtain that

$$N\left(x - z, \frac{t}{2}\right) > r_1, \quad N\left(y - z, \frac{t}{2}\right) > r_1.$$

Thus

$$a = N(x - y, t) \geqslant N\left(x - z, \frac{t}{2}\right) * N\left(z - y, \frac{t}{2}\right) \geqslant r_1 * r_1 > a,$$

which is a contradiction. □

Theorem 3.12 (A fuzzy version of Maia's theorem)**.** *Let $(X, N_1, *_1), (X, N_2, *_2)$ be fuzzy normed linear spaces. We suppose that:*

1. $N_2(x, t) \geqslant N_1(x, t), \ \forall x \in X, \ \forall t > 0$;
2. $T : (X, N_2, *_2) \to (X, N_2, *_2)$ *is fuzzy continuous;*
3. $T : (X, N_1, *_1) \to (X, N_1, *_1)$ *is φ-metric-contraction;*
4. $(X, N_2, *_2)$ *is complete.*

Then

i) T *has a unique fixed point* $x^* \in X$;
ii) $\{T^n(x)\}_{n \in \mathbb{N}}$ *is convergent in* $(X, N_2, *_2)$ *to* x^* *for all* $x \in X$.

Proof. Let $x \in X$ be fixed. First we prove that $\{T^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, N_1, *_1)$. Indeed,

$$N_1(T^{n+p}(x) - T^n(x), t) \geqslant \cdots \geqslant N_1(T^p(x) - x, \varphi^n(t)) \to 1, \text{ as } n \to \infty, \ \forall t > 0.$$

Using 1, we obtain that

$$N_2(T^{n+p}(x) - T^n(x), t) \geqslant N_1(T^{n+p}(x) - T^n(x), t) \to 1, \text{ as } n \to \infty, \ \forall t > 0.$$

Thus $\{T^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, N_2, *_2)$. As $(X, N_2, *_2)$ is a fuzzy Banach space, there exists $x^* \in X$ such that $T^n(x) \to x^*$, as $n \to \infty$.

Since $T : (X, N_2, *_2) \to (X, N_2, *_2)$ is fuzzy continuous, we obtain that $T^{n+1}(x) \to T(x^*)$. Thus $T(x^*) = x^*$, namely x^* is a fixed point for T.

Like in the proof of Theorem 3.10, the uniqueness follows. □

Theorem 3.13. *Let $(X, N, *)$ be a fuzzy normed linear space, let φ be a comparison function and $f : [0, 1] \to [0, 1]$ with the following properties:*

1. *f is a nondecreasing function;*
2. *$f(1) = 1$;*
3. *for any sequence (s_n), $\lim_{n\to\infty} f^n(s_n) = 1$ whenever $\lim_{n\to\infty} s_n = 1$.*

If $T : X \to X$ satisfies $N(Tx, t) \geqslant f(N(x, \varphi(t)))$, then T has a unique fixed point.

Proof. For $x = 0$ we obtain that $N(T(0), t) = 1$ for all $t > 0$. Hence $T(0) = 0$.

We assume that $x^* \neq 0$ is a fixed point for T. Then there exists $s > 0$ such that $N(x^*, s) = a < 1$. We have that

$$a = N(x^*, s) = N(T^n x^*, s) \geqslant f(N(T^{n-1}x^*, \varphi(s))) \geqslant \cdots \geqslant f^n(N(x^*, \varphi^n(s))) \to 1,$$

as $n \to \infty$, which is a contradiction. □

Theorem 3.14. *Let $(X, N, *)$ be a fuzzy Banach space, φ a comparison function, and $T : X \to X$ be a fuzzy continuous mapping such that*

$$N(T^2(x) - T(y), t) \geqslant N(T(x) - y, \varphi(t)), \ \forall x, y \in X, \ \forall t > 0.$$

Then T has a unique fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n(x) = x^*$, $\forall x \in X$.

Proof. Let $x \in X$ be fixed. Then

$$\begin{aligned} N(T^{n+p}(x) - T^n(x), t) &\geqslant N(T^{n+p-1}(x) - T^{n-1}(x), \varphi(t)) \\ &\geqslant \cdots \geqslant N(T^p(x) - x, \varphi^n(t)) \to 1, \text{ as } n \to \infty, \ \forall t > 0. \end{aligned}$$

Thus $\{T^n(x)\}$ is a Cauchy sequence. As $(X, N, *)$ is complete, we obtain that there exists $x^* \in X$ such that $T^n(x) \to x^*$, as $n \to \infty$. As T is fuzzy continuous, we have that $T^{n+1}(x) \to T(x^*)$. Thus $T(x^*) = x^*$.

We assume now that $x, y \in X, x \neq y$ are fixed points for T. Then there exists $s > 0$ such that $a = N(x - y, s) < 1$. We have that

$$a = N(x - y, s) = N(T^{n+1}x - T^n y, s) \geqslant \cdots \geqslant N(Tx - y, \varphi^n(s)) \to 1,$$

as $n \to \infty$, which is a contradiction. □

Theorem 3.15. *Let $(X, N, \wedge)$ be a fuzzy Banach space, φ_1 a comparison function, and $T : X \to X$ be a fuzzy continuous mapping such that*

$$N(T(x) - T(y), t) \geqslant N(T(x) - x, \varphi_1(t)) \wedge N(T(y) - y, \varphi_2(t)), \ \forall x, y \in X, \ \forall t > 0,$$

where $\varphi_2 : (0, \infty) \to (0, \infty)$ satisfies $N(T^2(x) - T(x), \varphi_2(t)) > N(T^2(x) - T(x), t)$ for all $x \in X$ and $t > 0$.

Then T has a unique fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n(x) = x^*$ for all $x \in X$.

Proof. Let $x \in X$ be arbitrary. For $y = T(x)$ we obtain that

$$N(T(x) - T^2(x), t) \geqslant N(T(x) - x, \varphi_1(t)) \wedge N(T^2(x) - T(x), \varphi_2(t)).$$

But $N(T^2(x) - T(x), \varphi_2(t)) > N(T^2(x) - T(x), t)$. Hence $N(T(x) - T^2(x), t) \geqslant N(T(x) - x, \varphi_1(t))$. Thus

$$\begin{aligned} N(T^{n+p}(x) - T^n(x), t) &\geqslant N(T^{n+p-1}(x) - T^{n-1}(x), \varphi_1(t)) \\ &\geqslant \cdots \geqslant N(T^p(x) - x, \varphi_1^n(t)) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Thus $\{T^n(x)\}$ is a Cauchy sequence. As $(X, N, \wedge)$ is complete, we obtain that there exists $x^* \in X$ such that $T^n(x) \to x^*$, as $n \to \infty$. As T is fuzzy continuous, we have that $T^{n+1}(x) \to T(x^*)$. Thus $T(x^*) = x^*$.

We assume now that $x, y \in X, x \neq y$ are fixed points for T. Then there exists $s > 0$ such that $a = N(x - y, s) < 1$. We have that

$$a = N(Tx - Ty, s) \geqslant N(T(x) - x, \varphi_1(t)) \wedge N(T(y) - y, \varphi_2(t)) = 1,$$

which is a contradiction. □

Lemma 3.16. *Let f, g be two nondecreasing comparison functions. Then* $\varphi(t) = \min\{f(t), g(t)\}$ *is a comparison function.*

Proof. We note that $f(t) \geqslant t$ for all $t > 0$. Indeed, if we assume that there exists t_0 such that $f(t_0) < t_0$, then $f^n(t_0) < t_0$, which contradicts the fact that $\lim_{n\to\infty} f^n(t_0) = \infty$. Similarly $g(t) \geqslant t$ for all $t > 0$.

Let now $t > 0$ be fixed. As $f^n(t) \to \infty$, for all $M > 0$, there exists $k_0 \in \mathbb{N}^*$ such that $f^{k_0}(t) > M$. Similarly, there exists $k_1 \in \mathbb{N}^*$ such that $g^{k_1}(t) > M$. We remark that $\varphi^{k_0+k_1}(t)$ is a composition of f and g, where f occurs at least k_0 times or g occurs at least k_1 times. Thus $\varphi^{k_0+k_1}(t) \geqslant f^{k_0}(t) > M$ or $\varphi^{k_0+k_1}(t) \geqslant g^{k_1}(t) > M$. Therefore $\lim_{n\to\infty} \varphi^n(t) = \infty$. □

The last theorem is a generalization of Banach's contraction principle for $(X, N, \wedge)$ fuzzy Banach spaces.

Theorem 3.17. *Let* $(X, N, \wedge)$ *be a fuzzy Banach space,* φ_1, φ_2 *be comparison functions, and* $T : X \to X$ *be a fuzzy continuous mapping such that*

$$N(T(x) - T(y), t) \geqslant N(x - y, \varphi_1(t)) \wedge N(y - T(y), \varphi_2(t)),\ \forall x, y \in X,\ \forall t > 0.$$

Then T *has a unique fixed point* $x^* \in X$ *and* $\lim_{n\to\infty} T^n(x) = x^*$ *for all* $x \in X$.

Proof. Let $x_0 \in X$ be arbitrary. For $x = T(x_0), y = x_0$, we have that

$$N(T^2(x_0) - T(x_0), t) \geqslant N(T(x_0) - x_0, \varphi_1(t)) \wedge N(x_0 - T(x_0), \varphi_2(t)).$$

Thus $N(T^2(x_0) - T(x_0), t) = N(T(x_0) - x_0, \varphi(t))$, where $\varphi(t) = \min\{\varphi_1(t), \varphi_2(t)\}$. We note that, according to Lemma 3.16, φ is a comparison function. Thus

$$\begin{aligned} N(T^{n+p}(x) - T^n(x), t) &\geqslant N(T^{n+p-1}(x) - T^{n-1}(x), \varphi(t)) \\ &\geqslant \cdots \geqslant N(T^p(x) - x, \varphi^n(t)) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Hence $\{T^n(x)\}$ is a Cauchy sequence. As $(X, N, \wedge)$ is complete, we obtain that there exists $x^* \in X$ such that $T^n(x) \to x^*$, as $n \to \infty$. As T is fuzzy continuous, we have that $T^{n+1}(x) \to T(x^*)$. Thus $T(x^*) = x^*$.

Now we assume that $x, y \in X, x \neq y$ are fixed points for T. Then there exists $s > 0$ such that $a = N(x - y, s) < 1$. We have that

$$\begin{aligned} a = N(T^n(x) - T^n(y), s) &\geqslant N(T^{n-1}(x) - T^{n-1}(y), \varphi_1(s)) \wedge N(T^{n-1}(y) - T^n(y), \varphi_2(s)) \\ &= N(T^{n-1}(x) - T^{n-1}(y), \varphi_1(s)) \\ &\geqslant N(T^{n-2}(x) - T^{n-2}(y), \varphi_1^2(s)) \geqslant \cdots \geqslant N(x - y, \varphi_1^n(s)) \to 1 \text{ as } n \to \infty, \end{aligned}$$

which is a contradiction. □

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