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Some fixed point theorems for θ-ϕ C-contractions

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Abstract

In this paper, we introduce the notion of θ-ϕ C-contraction and establish some fixed point and coupled fixed point theorems for these mappings in the setting of complete metric spaces and ordered metric spaces. The results presented in the paper improve and extend some well-known results. Also, we give an example to illustrate them. ©2017 All rights reserved.

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1. Introduction and preliminaries

As is well-known, Banach contraction principle is one of the famous results in fixed point theory. The generalizations of this result have been established in various settings (see [1, 11, 15, 22, 24] and the references therein). In the meantime, a large number of contractive definitions have been put forward. One of them is the C-contraction introduced by Chatterjea [4].

Definition 1.1 ([4])**.** Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is said to be a C-contraction if there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$, the following inequality holds:

$$d(Tx, Ty) \leqslant \alpha(d(x, Ty) + d(y, Tx)).$$

Using the C-contraction, Chatterjea [4] obtained a fixed point theorem that each C-contraction has a unique fixed point in a complete metric space (see Corollary 2.3).

Later, Choudhury [5] introduced the weakly C-contraction as follows.

Definition 1.2 ([5])**.** Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is said to be a weakly C-contraction if for all $x, y \in X$, the following inequality holds:

$$d(Tx, Ty) \leqslant \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \psi(d(x, Ty), d(y, Tx)),$$

where $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

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Using the weakly C-contraction, Choudhury [5] obtained the fixed point theorem that each weakly C-contraction has a unique point in a complete metric space (see Corollary 2.4).

Harjani et al. [10] studied weakly C-contraction in ordered metric space. They obtained the following theorem.

Theorem 1.3 ([10]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$d(Tx, Ty) \leqslant \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \psi(d(x, Ty), d(y, Tx))$$

for every comparative x and y, where $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

In 2011, Shatanawi [23] generalized weakly C-contraction with the help of altering distance functions in metric spaces and in ordered metric spaces, and then obtained some fixed point theorems.

Bhaskar and Lakshmikantham [9] introduced the notion of the mixed monotone property and the coupled fixed point of a mapping F from $X \times X$ into X and studied coupled fixed points of such mappings in partially ordered metric spaces. Since then, many authors established coupled fixed point results (see [6, 8, 16, 17, 20, 23] and the references therein).

Definition 1.4 ([9]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ be a mapping. We say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 1.5 ([9]). An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \longrightarrow X$ if $x = F(x, y)$ and $y = F(y, x)$.

Bhaskar and Lakshmikantham [9] proved the following result.

Theorem 1.6 ([9]). *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in (0, 1)$ for $x, y, u, v \in X$ with $x \preceq u,\ y \succeq v$,*

$$d(F(x, y), F(u, v)) \leqslant \frac{k}{2}[d(x, u) + d(y, v)].$$

If there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(x_0, y_0)$, then F has a coupled fixed point.

In 2014, Jleli and Samet [12] introduced a new type of contraction called θ-contraction. Just recently, Zheng et al. [25] introduced the notion of $\theta - \phi$ contraction which generalized $\theta-$contraction and other contractions (see [12, 25] and the references therein).

According to [12, 25], we denote by Θ the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;

(Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;

(Θ_3) θ is continuous on $(0, \infty)$.

And we denote by Φ [25] the set of functions $\phi : [1,\infty) \to [1,\infty)$ satisfying the following conditions:

(Φ_1) $\phi : [1,\infty) \to [1,\infty)$ is non-decreasing;

(Φ_2) for each $t > 1, \lim_{n\to\infty} \phi^n(t) = 1$;

(Φ_3) ϕ is continuous on $[1,\infty)$.

Zheng et al. [25] obtained the following result.

Theorem 1.7 ([25]). *Suppose (X, d) is a complete metric space and suppose $T : X \to X$ is a $\theta - \phi$ Suzuki contraction, i.e, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$, $Tx \neq Ty$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \theta(d(Tx, Ty)) \leqslant \phi[\theta(N(x, y))],$$

where

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

As pointed out in [25], Theorem 1.7 improved and extended the corresponding results of Banach, Samet [21], Jleli and Samet [12], Kannan [13], Dugundji-Granas [7], Boyd-Wong [2], Matkowski [14], Browder [3] and so on.

Inspired by [25], we introduce the notion of θ-ϕ C-contraction. The purpose of this paper is to prove some fixed point and coupled fixed point theorems for θ-ϕ C-contraction in the setting of complete metric spaces and ordered metric spaces. The results presented in the paper improve and extend the corresponding results of Chatterjea [4], Choudhury [5], Harjani et al. [10], Bhaskar and Lakshmikantham [9]. Also, we give an example to illustrate them.

We give some lemmas that will be used in the paper.

Lemma 1.8 ([25]). *If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for each $t > 1$.*

Lemma 1.9. *Suppose $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. Let $\phi_1(t) = \inf\{\psi(s, 2t - s) : 0 \leqslant s \leqslant t\}$, then $\phi_1(t)$ is continuous on $[0,\infty)$ and $\phi_1(t) = 0$ if and only if $t = 0$.*

Proof. Let $A_t = \{(x, y) : 0 \leqslant x \leqslant t, x + y = 2t\}$, then A_t is an arc of R^2. Since $\psi(x, y)$ is continuous and A_t is a connected compact subset of R^2, then $\psi(A_t)$ is an arc of R, that is to say, $\psi(A_t)$ is a finite closed interval. Thus,

$$\phi_1(t) = \inf\{\psi(s, 2t - s) : 0 \leqslant s \leqslant t\} = \inf \psi(A_t) = \min \psi(A_t). \tag{1.1}$$

Suppose $t_0 \in [0, +\infty)$ is an arbitrary point. Let $B_{t_0} = \{(x, y) : 0 \leqslant x \leqslant 2t_0 + 1, 0 \leqslant y \leqslant 2t_0 + 1\}$, then $\psi(x, y)$ is uniform continuous on B_{t_0}. So, for each $\varepsilon > 0$, there exists $\delta > 0$ with $\delta < 1$ such that $(u_1, v_1), (u_2, v_2) \in B_{t_0}$,

$$\sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2} < \delta \Longrightarrow |\psi(u_1, v_1) - \psi(u_2, v_2)| < \varepsilon.$$

Now, let $|t - t_0| < \frac{\delta}{2}$, then for each $(x, y) \in A_t$, there exists $(x_0, y_0) \in A_{t_0}$ such that

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Therefore, $|\psi(x, y) - \psi(x_0, y_0)| < \varepsilon$. That is, $\psi(x, y) > \psi(x_0, y_0) - \varepsilon \geqslant \phi_1(t_0) - \varepsilon$. Thus, $\phi_1(t) > \phi_1(t_0) - \varepsilon$.

Similarly, we can get $\phi_1(t_0) > \phi_1(t) - \varepsilon$. Then, we have $|\psi(t) - \psi(t_0)| < \varepsilon$, which shows that $\psi(t)$ is continuous at t_0. Since t_0 is an arbitrary point, then $\phi_1(t)$ is continuous on $[0,\infty)$.

If $t = 0$, then $A_t = \{(0, 0)\}$. Therefore, $\phi_1(t) = 0$.

If $\phi_1(t) = 0$, by (1.1), there exists $(x, y) \in A_t$ such that $\psi(x, y) = \phi_1(t) = 0$. Then $(x, y) = (0, 0)$, and $t = \frac{x+y}{2} = 0$. □

Lemma 1.10. *Let ψ, ϕ_1 be as in Lemma 1.9 and define $\phi(t) = \frac{t}{e^{\phi_1(2\ln t)}}$ for $t \in [1, +\infty)$, then $\phi \in \Phi$.*

Lemma 1.11 ([18, 19])**.** *Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

If $\{y_n\}$ is not a Cauchy sequence in (X, d), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}, \{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences tend to ε^+ when $k \to \infty$:

$$d(y_{m(k)}, y_{n(k)}), d(y_{m(k)}, y_{n(k)+1}), d(y_{m(k)-1}, y_{n(k)}), d(y_{m(k)-1}, y_{n(k)+1}).$$

2. θ-ϕ C-contractions

Based on the functions $\theta \in \Theta$ and $\phi \in \Phi$, we give the following definition.

Definition 2.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a θ-ϕ C-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$

$$\theta(d(Tx, Ty)) \leqslant \phi[\theta(\frac{d(x, Ty) + d(y, Tx)}{2})]. \tag{2.1}$$

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a θ-ϕ C-contraction with $\theta \in \Theta$ and $\phi \in \Phi$. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

Proof. Let $x_0 \in X$ be an arbitrary point. We define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in N$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in N$, then $x^* = x_{n_0}$ is a fixed point for T and the proof is done. In the next, we assume that $x_{n+1} \neq x_n$ for all $n \in N$. Then $d(x_{n+1}, x_n) > 0$ for all $n \in N$. For the sake of simplicity, take $d_n = d(x_n, x_{n+1})$.

Applying the inequality (2.1) with $x = x_{n-1}$, $y = x_n$, we obtain

$$\begin{aligned}
\theta(d_n) &= \theta(d(x_n, x_{n+1})) \\
&= \theta(d(Tx_{n-1}, Tx_n)) \\
&\leqslant \phi[\theta(\frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2})] \\
&\leqslant \phi[\theta(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2})] \\
&= \phi[\theta(\frac{d_{n-1} + d_n}{2})] < \theta(\frac{d_{n-1} + d_n}{2}).
\end{aligned}$$

So, $d_n < \frac{d_{n-1}+d_n}{2}$, that is $d_n < d_{n-1}$. Therefore, $\{d_n\}$ is a decreasing and bounded from below, thus converging to some $r \in R^+$. If $r > 0$, taking limit in the above inequality $\theta(d_n) \leqslant \phi[\theta(\frac{d_{n-1}+d_n}{2})]$, we have $\theta(r) \leqslant \phi(\theta(r))$. By Lemma 1.8, $\phi(\theta(r)) < \theta(r)$, then $\theta(r) \leqslant \phi(\theta(r)) < \theta(r)$, which yields a contradiction. Thus $r = 0$, that is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

In what follows, we shall prove that $\{x_n\}$ is a Cauchy sequence in X .

Suppose, on the contrary, $\{x_n\}$ is not a Cauchy sequence. By Lemma 1.11, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$, $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences tend to ε^+ when $k \to \infty$:

$$d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{n(k)+1}), d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{n(k)+1}).$$

By (2.1), we have

$$\begin{aligned}\theta(d(x_{m(k)},x_{n(k)+1})) &= \theta(d(Tx_{m(k)-1},Tx_{n(k)}))\\ &\leqslant \phi[\theta(\frac{d(x_{m(k)-1},Tx_{n(k)})+d(x_{n(k)},Tx_{m(k)-1})}{2})]\\ &= \phi[\theta(\frac{d(x_{m(k)-1},x_{n(k)+1})+d(x_{n(k)},x_{m(k)})}{2})].\end{aligned}$$

By (Θ_3), (Φ_3) and Lemma 1.11, passing to limit as $k\to\infty$, we get $\theta(\varepsilon)\leqslant\phi[\theta(\varepsilon)]$.

By Lemma 1.8, $\phi[\theta(\varepsilon)]<\theta(\varepsilon)$, then $\theta(\varepsilon)\leqslant\phi[\theta(\varepsilon)]<\theta(\varepsilon)$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X.

Due to the completeness of (X,d), $\{x_n\}$ converges to some point $x^*\in X$.

By (2.1),

$$\theta(d(Tx^*,Tx_{n-1}))\leqslant\phi[\theta(\frac{d(x^*,Tx_{n-1})+d(x_{n-1},Tx^*)}{2})]=\phi[\theta(\frac{d(x^*,x_n)+d(x_{n-1},Tx^*)}{2})].$$

Passing to limit as $n\to\infty$, then we get

$$\theta(d(Tx^*,x^*))\leqslant\phi[\theta(\frac{d(x^*,Tx^*)+0}{2})]=\phi[\theta(\frac{d(x^*,Tx^*)}{2})]\leqslant\theta(\frac{d(x^*,Tx^*)}{2}),$$

which yields $d(x^*,Tx^*)=0$ from the very definition of ϕ and θ. So, x^* is a fixed point of T.

Now, we will prove that T has at most one fixed point. Suppose, on the contrary, that there exists another distinct fixed point y^* of T such that $Tx^*=x^*\neq Ty^*=y^*$, then by (2.1), we get

$$\theta(d(x^*,y^*))=\theta(d(Tx^*,Ty^*))\leqslant\phi[\theta(\frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2})]=\phi[\theta(d(x^*,y^*))]<\theta(d(x^*,y^*)),$$

which is a contradiction. Therefore, the fixed point of T is unique. □

The following corollary is Chatterjea's Theorem [4].

Corollary 2.3. *Let (X,d) be a complete metric space and $T:X\to X$ be a C-contraction mapping, that is, there exists $\alpha\in[0,\frac{1}{2})$ such that for all $x,y\in X$*

$$d(Tx,Ty)\leqslant\alpha(d(x,Ty)+d(y,Tx)).$$

Then T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T^nx\}$ converges to x^*.

Proof. If $\alpha=0$, it is easy to prove. So we suppose $\alpha\in(0,\frac{1}{2})$. Let

$$\theta(t)=e^t$$

for all $t\in[0,+\infty)$, and

$$\phi(t)=t^{2\alpha}$$

for all $t\in[1,+\infty)$. Obviously, $\theta\in\Theta,\phi\in\Phi$.

In what follows, we prove that T is a θ-ϕ C-contraction.

$$\begin{aligned}\phi(\theta(\frac{d(x,Ty)+d(y,Tx)}{2}))&=(e^{\frac{d(x,Ty)+d(y,Tx)}{2}})^{2\alpha}\\ &=e^{\alpha(d(x,Ty)+d(y,Tx))}\\ &\geqslant e^{d(Tx,Ty)}\\ &=\theta(d(Tx,Ty)).\end{aligned}$$

Thus, T is a θ-ϕ C-contraction. Therefore, from Theorem 2.2, T has a unique fixed point $x^*\in X$ and for every $x\in X$ the sequence $\{T^nx\}$ converges to x^*. □

The following corollary is Choudhury's Theorem [5].

Corollary 2.4. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a weakly C-contraction mapping, that is, for all* $x, y \in X$*, the following inequality holds:*

$$d(Tx, Ty) \leqslant \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \psi(d(x, Ty), d(y, Tx)),$$

where $\psi : [0, +\infty)^2 \to [0, +\infty)$ *is a continuous function such that* $\psi(x, y) = 0$ *if and only if* $x = y = 0$*. Then* T *has a unique fixed point* $x^* \in X$ *and for every* $x \in X$ *the sequence* $\{T^n x\}$ *converges to* x^**.*

Proof. Let

$$\theta(t) = e^t, \quad \phi_1(t) = \inf\{\psi(s, 2t - s) : 0 \leqslant s \leqslant t\}$$

for all $t \in [0, +\infty)$. Obviously, $\theta \in \Theta$, and let

$$\phi(t) = \frac{t}{e^{\phi_1(2 \ln t)}}$$

for $t \in [1, +\infty)$, then $\phi \in \Phi$ by Lemma 1.9 and Lemma 1.10.

In what follows, we prove that T is a θ-ϕ C-contraction.

$$\begin{aligned}
\phi(\theta(\frac{d(x, Ty) + d(y, Tx)}{2})) &= \phi(e^{\frac{d(x,Ty)+d(y,Tx)}{2}}) \\
&= \frac{e^{\frac{d(x,Ty)+d(y,Tx)}{2}}}{e^{\phi_1(d(x,Ty)+d(y,Tx))}} \\
&\geqslant \frac{e^{\frac{d(x,Ty)+d(y,Tx)}{2}}}{e^{\psi(d(x,Ty),d(y,Tx))}} \\
&= e^{[\frac{d(x,Ty)+d(y,Tx)}{2} - \psi(d(x,Ty),d(y,Tx))]} \\
&\geqslant e^{d(Tx,Ty)} \\
&= \theta(d(Tx, Ty)).
\end{aligned}$$

Thus, T is a θ-ϕ C-contraction. Therefore, from Theorem 2.2, T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*. □

Now, we give an example to illustrate our results.

Example 2.5. Consider the sequence $\{S_n\}_{n \in N}$ as follows:

$$S_1 = 1, \quad S_2 = 1 + 2, \cdots,$$

$$S_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \cdots.$$

Let $X = \{S_n : n \in N\}$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space.

Define the mapping $T : X \to X$ by $TS_1 = S_1$ and $TS_n = S_{n-1}$ for every $n > 1$.

Firstly, we observe that the Banach contraction principle cannot be applied since

$$\lim_{n\to\infty} \frac{d(TS_n, TS_1)}{d(S_n, S_1)} = \lim_{n\to\infty} \frac{S_{n-1} - 1}{S_n - 1} = \lim_{n\to\infty} \frac{n^2 - n - 1}{n^2 + n - 1} = 1.$$

Secondly, T is not a C-contraction map. In fact, suppose that there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, Ty) + \alpha d(y, Tx)$$

for all $x, y \in X$.

Now, let $x = S_n,\ y = S_1$, by the above inequality we know that

$$\frac{n(n-1)}{2} - 1 = d(Tx, Ty) \leqslant \alpha d(x, Ty) + \alpha d(y, Tx) = \alpha d(S_n, TS_1) + \alpha d(S_1, TS_n) = \alpha(n^2 - 2)$$

for all $n \in N$, that is, $\alpha \geqslant \frac{\frac{n(n-1)}{2}-1}{n^2-2}$, passing to limit as $n \to \infty$, we obtain $\alpha \geqslant \frac{1}{2}$, which yields a contradiction. So T is not a C-contraction map. Now, let the function $\theta : (0, \infty) \to (1, \infty)$ defined by

$$\theta(t) = e^t,$$

and $\phi : [1, \infty) \to [1, \infty)$ defined by

$$\phi(t) = \begin{cases} 1, & \text{if } 1 \leqslant t \leqslant 2, \\ t - 1, & \text{if } t \geqslant 2. \end{cases}$$

Obviously, $\theta \in \Theta,\ \phi \in \Phi$.

In what follows, we prove that T is a θ-ϕ C-contraction.

We consider two cases.

Case 1. $x = S_n,\ y = S_m,\ n > m > 1$.

In this case, we have

$$\begin{aligned} d(Tx, Ty) &= d(TS_n, TS_m) = S_{n-1} - S_{m-1}, \\ d(x, Ty) &= d(S_n, S_{m-1}) = S_n - S_{m-1} = n + S_{n-1} - S_{m-1}, \\ d(y, Tx) &= d(S_m, S_{n-1}) = S_{n-1} - S_m = S_{n-1} - S_{m-1} - m, \end{aligned}$$

$$\begin{aligned} \phi(\theta(\frac{d(x, Ty) + d(y, Tx)}{2})) &= \phi(\theta(S_{n-1} - S_{m-1} + \frac{n-m}{2})) \\ &= e^{S_{n-1} - S_{m-1} + \frac{n-m}{2}} - 1 \\ &= e^{\frac{n-m}{2}} e^{S_{n-1} - S_{m-1}} - 1 \\ &> e^{S_{n-1} - S_{m-1}} \\ &= \theta(d(Tx, Ty)). \end{aligned}$$

Case 2. $x = S_n,\ y = S_1,\ n > 1$.

In this case, we have

$$\begin{aligned} d(Tx, Ty) &= d(TS_n, TS_1) = S_{n-1} - S_1, \\ d(x, Ty) &= d(S_n, S_1) = S_n - S_1 = n + S_{n-1} - S_1, \\ d(y, Tx) &= d(S_1, S_{n-1}) = S_{n-1} - S_1, \end{aligned}$$

$$\begin{aligned} \phi(\theta(\frac{d(x, Ty) + d(y, Tx)}{2})) &= \phi(\theta(S_{n-1} - S_1 + \frac{n}{2})) \\ &= e^{S_{n-1} - S_1 + \frac{n}{2}} - 1 \\ &= e^{\frac{n}{2}} e^{S_{n-1} - S_1} - 1 \\ &> e^{S_{n-1} - S_1} \\ &= \theta(d(Tx, Ty)). \end{aligned}$$

Therefore, we have for all $x, y \in X$

$$\theta(d(Tx, Ty)) \leqslant \phi[\theta(\frac{d(x, Ty) + d(y, Tx)}{2})].$$

So all the hypotheses of Theorem 2.2 are satisfied, and T has a fixed point. In this example $x = S_1$ is the fixed point.

By using similar proofs in Theorem 2.2, we can obtain the following results in an ordered metric space.

Theorem 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$\theta(d(Tx, Ty)) \leqslant \phi[\theta(\frac{d(x, Ty) + d(y, Tx)}{2})]$$

for every comparative x and y where $\theta \in \Theta$, $\phi \in \Phi$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Theorem 2.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that*

$$\theta(d(Tx, Ty)) \leqslant \phi[\theta(\frac{d(x, Ty) + d(y, Tx)}{2})]$$

for every comparative x and y where $\theta \in \Theta$, $\phi \in \Phi$. Suppose that for a nondecreasing sequence $\{x_n\}$ in X with $x_n \to x$, we have $x_n \preceq x$ for all $n \in N$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Remark 2.8. Theorem 2.6 improves Theorem 1.3, and Theorem 2.7 improves the result of [10].

3. Coupled fixed point

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Assume that for $x, y, u, v \in X$ with $x \preceq u$, $y \succeq v$,*

$$\theta(d(F(x, y), F(u, v))) \leqslant \phi[\theta(\frac{d(x, u) + d(y, v)}{2})]. \tag{3.1}$$

If there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(x_0, y_0)$, then F has a coupled fixed point.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(x_0, y_0)$. Let $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Then $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Again, let $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Then $x_1 \preceq x_2$ and $y_1 \succeq y_2$. Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$, and we have $x_0 \preceq x_1 \preceq x_2 \preceq \cdots$ and $y_0 \succeq y_1 \succeq y_2 \succeq \cdots$.

If $x_{n_0+1} = x_{n_0}$, $y_{n_0+1} = y_{n_0}$ for some $n_0 \in N$, then

$$x_{n_0} = x_{n_0+1} = F(x_{n_0}, y_{n_0}), \quad y_{n_0} = y_{n_0+1} = F(y_{n_0}, x_{n_0}).$$

Thus, (x_{n_0}, y_{n_0}) is a coupled fixed point for F.

Next, we assume that for all $n \in N$, either $x_n \neq F(x_n, y_n)$ or $y_n \neq F(y_n, x_n)$, that is

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) > 0$$

for all $n \in N$.

By (3.1), we have

$$\theta(d(x_{n+1}, x_{n+2})) = \theta(d(F(x_n, y_n), F(x_{n+1}, y_{n+1}))) \leqslant \phi[\theta(\frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2})]. \tag{3.2}$$

Similarly, we have

$$\theta(d(y_{n+1}, y_{n+2})) = \theta(d(F(y_n, x_n), F(y_{n+1}, x_{n+1}))) \leqslant \phi[\theta(\frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2})]. \tag{3.3}$$

Then by (3.2), (3.3) and Lemma 1.8, we have

$$d(x_{n+1}, x_{n+2}) \leqslant \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2}, \tag{3.4}$$

and

$$d(y_{n+1}, y_{n+2}) \leqslant \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2}. \tag{3.5}$$

So, by (3.4) and (3.5), we have

$$d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \leqslant d(x_n, x_{n+1}) + d(y_n, y_{n+1}).$$

Thus, $\{d(x_n, x_{n+1}) + d(y_n, y_{n+1})\}$ is a decreasing sequence and bounded from below. Thus there is $r \geqslant 0$ such that

$$\lim_{n\to\infty}(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) = r.$$

Suppose $r > 0$, then $(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \geqslant r$, so there exist $n_1 < n_2 < \cdots < n_k < \cdots$ such that for each $k \in N$, $d(x_{n_k+1}, x_{n_k+2}) \geqslant \frac{r}{2}$ or $d(y_{n_k+1}, y_{n_k+2}) \geqslant \frac{r}{2}$.

Without loss of generality, we can assume that $d(x_{n_k+1}, x_{n_k+2}) \geqslant \frac{r}{2}$ for $k \in N$, then from (3.4),

$$\theta(\frac{r}{2}) \leqslant \theta(d(x_{n_k+1}, x_{n_k+2})) \leqslant \phi[\theta(\frac{d(x_{n_k}, x_{n_k+1}) + d(y_{n_k}, y_{n_k+1})}{2})].$$

Passing to limit as $k \to \infty$, then we get

$$\theta(\frac{r}{2}) \leqslant \phi[\theta(\frac{r}{2})] < \theta(\frac{r}{2}),$$

which is a contradiction, so $r = 0$. Therefore, we get that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

and

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{3.6}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X.

Suppose, on the contrary, $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 1.11, there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$, $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{3.7}$$

If there exist $k_1 < k_2 < \cdots < k_s < \cdots$ such that $d(y_{m(k_s)}, y_{n(k_s)}) < \varepsilon$ for each $s \in N$, then

$$\begin{aligned} \theta(d(x_{m(k_s)+1}, x_{n(k_s)+1})) &\leqslant \phi[\theta(\frac{d(x_{m(k_s)}, x_{n(k_s)}) + d(y_{m(k_s)}, y_{n(k_s)})}{2})] \\ &\leqslant \phi[\theta(\frac{d(x_{m(k_s)}, x_{n(k_s)}) + \varepsilon}{2})]. \end{aligned}$$

Passing to limit as $s \to \infty$, then from (3.7) and Lemma 1.8, we get

$$\theta(\varepsilon) \leqslant \phi[\theta(\varepsilon)] < \theta(\varepsilon),$$

which is a contradiction. Therefore, for k large enough, we have

$$d(y_{m(k)}, y_{n(k)}) \geqslant \varepsilon. \tag{3.8}$$

Since

$$\theta(d(y_{m(k)+1}, y_{n(k)+1})) \leqslant \phi[\theta(\frac{d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})}{2})],$$

then we have

$$d(y_{m(k)+1}, y_{n(k)+1}) \leqslant \frac{d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})}{2}.$$

Therefore, by triangle inequality of d,

$$\begin{aligned} d(y_{m(k)}, y_{n(k)}) - d(y_{m(k)}, y_{m(k)+1}) - d(y_{n(k)}, y_{n(k)+1}) \\ \leqslant d(y_{m(k)+1}, y_{n(k)+1}) \leqslant \frac{d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})}{2}. \end{aligned}$$

That is,

$$d(y_{m(k)}, y_{n(k)}) \leqslant d(x_{m(k)}, x_{n(k)}) + 2d(y_{m(k)}, y_{m(k)+1}) + 2d(y_{n(k)}, y_{n(k)+1}). \tag{3.9}$$

Thus, from (3.6), (3.8) and (3.9), we have

$$\lim_{n\to\infty} d(y_{m(k)}, y_{n(k)}) = \varepsilon.$$

Since $\theta(d(x_{m(k)+1}, x_{n(k)+1})) \leqslant \phi[\theta(\frac{d(x_{m(k)}, x_{n(k)}) + d(y_{m(k)}, y_{n(k)})}{2})]$, passing to limit as $k \to \infty$, then we get $\theta(\varepsilon) \leqslant \phi[\theta(\varepsilon)] < \theta(\varepsilon)$, which is a contradiction. Thus we know that $\{x_n\}$ is a Cauchy sequence in X. Similarly, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, then there exist $x, y \in X$ such that $x_n \to x$ and $y_n \to y$. And F is continuous, $x_{n+1} = F(x_n, y_n) \to F(x, y)$ and $y_{n+1} = F(y_n, x_n) \to F(y, x)$. By the uniqueness of limit, we have $x = F(x, y)$ and $y = F(y, x)$. Thus (x, y) is a coupled fixed point of F. □

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X. Assume that for $x, y, u, v \in X$ with $x \preceq u$, $y \succeq v$,*

$$\theta(d(F(x, y), F(u, v))) \leqslant \phi[\theta(\frac{d(x, u) + d(y, v)}{2})].$$

Suppose that for a nondecreasing sequence $\{x_n\}$ in X with $x_n \to x$, we have $x_n \preceq x$ for all $n \in N$ and for a nonincreasing sequence $\{y_n\}$ in X with $y_n \to y$, we have $y_n \preceq y$ for all $n \in N$. If there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(x_0, y_0)$, then F has a coupled fixed point.

Remark 3.3. Take $\theta(t) = e^t, t \in [0, +\infty)$ and $\phi(t) = t^k$, $t \in [1, +\infty)$, where $k \in (0, 1)$ in Theorem 3.1, then we get Theorem 1.6. Also, Theorem 3.2 improves the result of [9].

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