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Solvability of second-order m-point difference equation boundary value problems on infinite intervals

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Abstract

In this paper, we study second-order m-point difference boundary value problems on infinite intervals

$$\begin{cases} \Delta^2 x(k-1) + f(k, x(k), \Delta x(k-1)) = 0, \ k \in N, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \ \lim_{k\to\infty} \Delta x(k) = 0, \end{cases}$$

where $N = \{1, 2, \cdots\}$, $f : N \times R^2 \to R$ is continuous, $\alpha_i \in R$, $\sum_{i=1}^{m-2} \alpha_i \neq 1$, $\eta_i \in N$, $0 < \eta_1 < \eta_2 < \cdots < \infty$ and

$$\Delta x(k) = x(k+1) - x(k),$$

the nonlinear term is dependent in a difference of lower order on infinite intervals. By using Leray-Schauder continuation theorem, the existence of solutions are investigated. Finally, we give one example to demonstrate the use of the main result.

Keywords: Difference equation, boundary value problem, Leray-Schauder continuation theorem, infinite interval.
2010 MSC: 34B15, 34B16, 34B18, 34G20.

1. Introduction

Boundary value problems on infinite intervals originated in the field of the applied mathematics and physics. In recent years, boundary value problems (BVPs) on infinite intervals have received much attention mainly due to their important applications in the study of plasma physics, in analyzing the heat transfer in radial flow between circular disks, in the study of the unsteady flow of a gas through semi-infinite porous medium, and in an analysis of the mass transfer on a rotating disk in non-Newtonian fluid, see [13, 20] and references therein. Some works and various techniques dealing with this kind of boundary value problems, such as different kinds of fixed point theorem, upper and lower solution

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techniques, topological degree theorem and coincidence degree theorem are used to discuss second-order or n-th order boundary value problems on a half-line (Dirichlet problems, periodic, impulsive system, time delay and so on), see [6, 8–11, 14, 16, 19] and the references therein.

In fact, differential equations in actual production and scientific research are extremely complex. In many cases, it is difficult to get the analytic solutions and even to get the analytical expressions. To solve the problem, we need to consider its approximate solutions or study on the properties of solutions. It requires the discretization of differential equations, so difference equations is very important and practical. Therefore, many scholars devote to study difference equation boundary value problems. Recently, boundary value problems for difference equations on infinite intervals have been considered widely, such as modern medical biology mathematics, economics, physics, chemistry and so on, and there are some excellent results on the existence of solutions, see [1, 2, 4, 5, 7, 12, 17, 18] and the references therein. However, to our knowledge, the theory of difference equation boundary value problems on infinite interval is rather less, there are still lots of work and research that should be done.

In [3], Agarwal and Regan studied the existence of non-negative solutions for second order difference boundary value problems on infinite intervals

$$\begin{cases} \Delta^2 x(i-1) + f(i, x(i)) = 0, \\ x(0) = 0, \quad \lim\limits_{i\to\infty} x(i) = 0. \end{cases}$$

In [15], Lian et al. used the Schauder fixed point theorem and upper and lower solution technique to study unbounded positive solutions for second-order discrete boundary value problems on infinite intervals

$$\begin{cases} -\Delta^2 x_{k-1} = f(k, x_k, \Delta x_{k-1}), \quad k \in N, \\ x_0 - a\Delta x_0 = B, \quad \Delta x_\infty = C. \end{cases}$$

Motivated by the work above, in this paper, we consider the existence of solutions for second-order m-point difference boundary value problems on infinite intervals

$$\begin{cases} \Delta^2 x(k-1) + f(k, x(k), \Delta x(k-1)) = 0, \ k \in N, \\ x(0) = \sum\limits_{i=1}^{m-2} \alpha_i x(\eta_i), \ \lim\limits_{k\to\infty} \Delta x(k) = 0, \end{cases} \tag{1.1}$$

where $N = \{1, 2, \cdots\}$, $f : N \times R^2 \to R$ is continuous, $\alpha_i \in R$, $\sum\limits_{i=1}^{m-2} \alpha_i \neq 1$, α_i have the same signal, $\eta_i \in N$, $0 < \eta_1 < \eta_2 < \cdots < \infty$ and $\Delta x(k) = x(k+1) - x(k)$.

We set

$$P = \sum_{j=1}^{\infty} p(j), \quad P_1 = \sum_{j=1}^{\infty} jp(j), \quad Q = \sum_{j=1}^{\infty} q(j),$$

and we suppose $\alpha_i, i = 1, 2, \cdots, m-2$ are the same signal in this paper and we always assume $\alpha = \sum\limits_{i=1}^{m-2} \alpha_i$.

In this paper, we always assume the following conditions hold:

(C_1) $f : N \times R^2 \to R$ is continuous. For each $r > 0$, there exists $\varphi_r(k) \in l^1$ with $k\varphi_r(k) \in l^1$, $\varphi_r(k) > 0$ such that $\max\{|u|, |v|\} \leqslant r$ implies $|f(k, u, v)| \leqslant \varphi_r(k)$, for each $k \in N$.

(C_2) $f : N \times R^2 \to R$ is continuous, there exist $p(k), q(k), r(k) \in l^1$ with $kp(k), kq(k), kr(k) \in l^1$, such that for each $k \in N$, $(u, v) \in R^2$ implies $|f(k, u, v)| \leqslant p(k)|u| + q(k)|v| + r(k)$.

We deal with the existence of solutions for BVP (1.1) by using the Lerday-Schauder continuation theorem and obtain the result which extends and improves the known results.

2. Preliminary results

Let N_0 be the set of all nonnegative integers and S be the space of sequence, that is $x \in S$, $x = \{x(k)\}_{k\in N_0}$. If $x(k) \leqslant y(k)$ for all $k \in N_0$, we call $x \leqslant y$. Consider the space

$$S_\infty = \{x \in S : \lim_{k\to\infty} x(k) \text{ and } \lim_{k\to\infty} \Delta x(k) \text{ exist }\},$$

with the norm $\|x\| = \max\{\|x\|_\infty, \|\Delta x\|_\infty\}$, where $\|.\|_\infty$ is supremum norm on the infinite intervals. Obviously, $(S_\infty, \|.\|_\infty)$ is a Banach space. In addition, this paper also involves space l^1 and for all $x \in l^1$, we have $\|x\|_{l^1} = \sum_{k=1}^{\infty} |x(k)|$.

Lemma 2.1. *Let $\nu = \{\nu(k)\}_{k\in N}$, $\sum_{k=1}^{\infty} \nu(k) < \infty$ and $\sum_{k=1}^{\infty} k\nu(k) < \infty$, then the BVP*

$$\begin{cases} \Delta^2 x(k-1) + \nu(k) = 0, & k \in N, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), & \lim_{k\to\infty} \Delta x(k) = 0, \end{cases} \tag{2.1}$$

has a unique solution. Moreover, this unique solution can be expressed in the form

$$x(k) = \sum_{j=1}^{\infty} G(k,j)\nu(j),$$

where $G(k,j)$ is defined by

$$G(k,j) = \frac{1}{\Lambda}\begin{cases} j, & j \leqslant \eta_1, j \leqslant k, \\ \sum_{i=1}^{m-2} \alpha_i j + \Lambda k, & j \leqslant \eta_1, j > k, \\ \sum_{a=1}^{i} \alpha_a \eta_a + \sum_{a=i+1}^{m-2} \alpha_a j + \Lambda j, & 0 < \eta_i < j \leqslant \eta_{i+1},\ j \leqslant k,\ i = 1, 2, \cdots, m-3, \\ \sum_{a=1}^{i} \alpha_a \eta_a + \sum_{a=i+1}^{m-2} \alpha_a j + \Lambda k, & 0 < \eta_i < j \leqslant \eta_{i+1},\ j > k,\ i = 1, 2, \cdots, m-3, \\ \sum_{i=1}^{m-2} \alpha_i \eta_i + \Lambda j, & \eta_{m-2} < j,\ j \leqslant k, \\ \sum_{i=1}^{m-2} \alpha_i \eta_i + \Lambda k, & \eta_{m-2} < j,\ j > k. \end{cases}$$

Here $\Lambda = 1 - \sum_{i=1}^{m-2} \alpha_i$.

Proof. Since $\sum_{k=1}^{\infty} \nu(k) < \infty$ and $\sum_{k=1}^{\infty} k\nu(k) < \infty$, we obtain that

$$x(k) = x(0) + \sum_{i=1}^{k} \sum_{j=i}^{\infty} \nu(j). \tag{2.2}$$

Since $x(0) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i)$, from (2.2), we obtain that

$$x(0) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i}\Bigg[\sum_{i=1}^{m-2} \alpha_i \sum_{j=1}^{\eta_1} j\nu(j) + \sum_{i=1}^{m-3} \sum_{j=\eta_i+1}^{\eta_{i+1}} (\sum_{a=i+1}^{m-2} \alpha_a j + \sum_{a=1}^{i} \alpha_a \eta_a)\nu(j) + \sum_{i=1}^{m-2} \alpha_i \eta_i \sum_{j=\eta_{m-2}+1}^{\infty} \nu(j)\Bigg].$$

The unique solution of (2.1) can be stated by

$$x(k)=\frac{1}{1-\sum\limits_{i=1}^{m-2}\alpha_i}\Big[\sum_{i=1}^{m-2}\alpha_i\sum_{j=1}^{\eta_1}jv(j)+\sum_{i=1}^{m-3}\sum_{j=\eta_i+1}^{\eta_{i+1}}(\sum_{a=i+1}^{m-2}\alpha_a j+\sum_{a=1}^{i}\alpha_a\eta_a)v(j)+\sum_{i=1}^{m-2}\alpha_i\eta_i\sum_{j=\eta_{m-2}+1}^{\infty}v(j)\Big]$$
$$+\sum_{j=1}^{k}jv(j)+\sum_{j=k+1}^{\infty}kv(j).$$

For $0\leqslant k\leqslant\eta_1$, the unique solution of (2.1) can be stated by

$$x(k)=\frac{1}{1-\sum\limits_{i=1}^{m-2}\alpha_i}\sum_{j=1}^{k}jv(j)+\sum_{j=k+1}^{\eta_1}\left(\frac{\sum\limits_{i=1}^{m-2}\alpha_i j}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+k\right)v(j)$$
$$+\sum_{i=1}^{m-3}\sum_{j=\eta_i+1}^{\eta_{i+1}}\left(\frac{\sum\limits_{a=i+1}^{m-2}\alpha_a j+\sum\limits_{a=1}^{i}\alpha_a\eta_a}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+k\right)v(j)+\sum_{j=\eta_{m-2}+1}^{\infty}\left(\frac{\sum\limits_{i=1}^{m-2}\alpha_i\eta_i}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+k\right)v(j).$$

For $\eta_i<k\leqslant\eta_{i+1}$, $1\leqslant i\leqslant m-3$, the unique solution of (2.1) can be stated by

$$x(k)=\frac{1}{1-\sum\limits_{i=1}^{m-2}\alpha_i}\sum_{j=1}^{\eta_1}jv(j)+\sum_{b=1}^{i-1}\sum_{j=\eta_b+1}^{\eta_{b+1}}\left(\frac{\sum\limits_{a=b+1}^{m-2}\alpha_a j+\sum\limits_{a=1}^{b}\alpha_a\eta_a}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+j\right)v(j)$$
$$+\sum_{j=\eta_i+1}^{k}\left(\frac{\sum\limits_{a=i+1}^{m-2}\alpha_a j+\sum\limits_{a=1}^{i}\alpha_a\eta_a}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+j\right)v(j)+\sum_{j=k+1}^{\eta_{i+1}}\left(\frac{\sum\limits_{a=i+1}^{m-2}\alpha_a j+\sum\limits_{a=1}^{i}\alpha_a\eta_a}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+k\right)v(j)$$
$$+\sum_{b=i+1}^{m-3}\sum_{j=\eta_b+1}^{\eta_{b+1}}\left(\frac{\sum\limits_{a=1}^{b}\alpha_a\eta_a+\sum\limits_{a=b+1}^{m-2}\alpha_a j}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+k\right)v(j)+\sum_{j=\eta_{m-2}+1}^{\infty}\left(\frac{\sum\limits_{i=1}^{m-2}\alpha_i\eta_i}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+k\right)v(j).$$

For $\eta_{m-2}<k<\infty$, the unique solution of (2.1) can be stated by

$$x(k)=\frac{1}{1-\sum\limits_{i=1}^{m-2}\alpha_i}\sum_{j=1}^{\eta_1}jv(j)+\sum_{b=1}^{m-3}\sum_{j=\eta_b+1}^{\eta_{b+1}}\left(\frac{\sum\limits_{a=b+1}^{m-2}\alpha_a j+\sum\limits_{a=1}^{b}\alpha_a\eta_a}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+j\right)v(j)$$
$$+\sum_{j=\eta_{m-2}+1}^{k}\left(\frac{\sum\limits_{i=1}^{m-2}\alpha_i\eta_i}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+j\right)v(j)+\sum_{j=k+1}^{\infty}\left(\frac{\sum\limits_{i=1}^{m-2}\alpha_i\eta_i}{1-\sum\limits_{i=1}^{m-2}\alpha_i}+k\right)v(j).$$

We note $\Lambda=1-\sum\limits_{i=1}^{m-2}\alpha_i$, and

$$G(k,j)=\frac{1}{\Lambda}\begin{cases} j, & j\leqslant \eta_1,\ \ j\leqslant k,\\ \sum\limits_{i=1}^{m-2}\alpha_i j+\Lambda k, & j\leqslant \eta_1,\ \ j>k,\\ \sum\limits_{a=1}^{i}\alpha_a\eta_a+\sum\limits_{a=i+1}^{m-2}\alpha_a j+\Lambda j, & 0<\eta_i<j\leqslant \eta_{i+1},\ \ j\leqslant k,\ \ i=1,2,\cdots,m-3,\\ \sum\limits_{a=1}^{i}\alpha_a\eta_a+\sum\limits_{a=i+1}^{m-2}\alpha_a j+\Lambda k, & 0<\eta_i<j\leqslant \eta_{i+1},\ \ j>k,\ \ i=1,2,\cdots,m-3,\\ \sum\limits_{i=1}^{m-2}\alpha_i\eta_i+\Lambda j, & \eta_{m-2}<j,\ \ j\leqslant k,\\ \sum\limits_{i=1}^{m-2}\alpha_i\eta_i+\Lambda k, & \eta_{m-2}<j,\ \ j>k. \end{cases}$$

Therefore, the unique solution of (2.1) is $x(k)=\sum\limits_{j=1}^{\infty}G(k,j)v(j)$, which completes the proof. □

Remark 2.2. Obviously $G(k,j)$ satisfies the properties of a Green function, so we call $G(k,j)$ the Green function of the corresponding homogeneous multipoint BVP of (2.1) on infinite intervals.

Lemma 2.3. *For all* $k,j\in N$, *it holds that*

$$|G(k,j)|\leqslant\begin{cases} j, & \sum\limits_{i=1}^{m-2}\alpha_i<0,\\ \frac{j}{\Lambda}, & 0\leqslant\sum\limits_{i=1}^{m-2}\alpha_i<1,\\ \max\left\{\frac{\sum_{i=1}^{m-2}\alpha_i j}{-\Lambda},\frac{\sum_{i=1}^{m-2}\alpha_i\eta_{m-2}}{-\Lambda}\right\}, & \sum\limits_{i=1}^{m-2}\alpha_i>1. \end{cases}$$

Proof. For each $j\in N$, $G(k,j)$ is nondecreasing in k, we have

$$\begin{aligned} &\min\left\{\frac{\sum_{i=1}^{m-2}\alpha_i j}{\Lambda},\frac{\sum_{a=1}^{i}\alpha_a\eta_a+\sum_{a=i+1}^{m-2}\alpha_a j}{\Lambda},\frac{\sum_{i=1}^{m-2}\alpha_i\eta_i}{\Lambda}\right\}\\ &\leqslant G(k,j)\leqslant G(j,j)\\ &=\frac{1}{\Lambda}\begin{cases} j, & j\leqslant\eta_1,\\ \sum\limits_{a=1}^{i}\alpha_a\eta_a+\sum\limits_{a=i+1}^{m-2}\alpha_a j+\Lambda j, & 0<\eta_i<j\leqslant\eta_{i+1},\ \ i=1,2,\cdots,m-3,\\ \sum\limits_{i=1}^{m-2}\alpha_i\eta_i+\Lambda j, & \eta_{m-2}<j. \end{cases} \end{aligned}$$

Further, we have

$$\begin{aligned} &\frac{\sum_{i=1}^{m-2}\alpha_i j}{\Lambda}\leqslant G(k,j)\leqslant j, && \sum_{i=1}^{m-2}\alpha_i<0,\\ &0<\min\left\{\frac{\sum_{i=1}^{m-2}\alpha_i j}{\Lambda},\frac{\sum_{i=1}^{m-2}\alpha_i\eta_1}{\Lambda}\right\}\leqslant G(k,j)\leqslant\frac{j}{\Lambda}, && 0\leqslant\sum_{i=1}^{m-2}\alpha_i<1,\\ &\min\left\{\frac{\sum_{i=1}^{m-2}\alpha_i j}{\Lambda},\frac{\sum_{i=1}^{m-2}\alpha_i\eta_{m-2}}{\Lambda}\right\}\leqslant G(k,j)\leqslant j, && \sum_{i=1}^{m-2}\alpha_i>1. \end{aligned}$$

Therefore, this completes the proof. □

Lemma 2.4. *For the Green function* $G(k, j)$, *it holds that*

$$\lim_{k\to\infty} G(k,j) = \overline{G}(j) = \frac{1}{\Lambda}\begin{cases} j, & j \leqslant \eta_1, \\ \sum_{a=1}^{i} \alpha_a\eta_a + \sum_{a=i+1}^{m-2} \alpha_a j + \Lambda j, & 0 < \eta_i < j \leqslant \eta_{i+1}, \quad i = 1, 2, \cdots, m-3, \\ \sum_{i=1}^{m-2} \alpha_i\eta_i + \Lambda j, & \eta_{m-2} < j. \end{cases}$$

Theorem 2.5 ([10])**.** *Let*

$$M \subset S_\infty = \{x \in S : \lim_{k\to\infty} x(k) \ \textit{and} \ \lim_{k\to\infty} \Delta x(k) \ \textit{exist}\}.$$

If M *is uniformly bounded, and uniformly convergent on infinite interval, then* M *is relatively compact.*

3. Main result

Consider the space

$$X = \left\{x \in S_\infty : \ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \ \lim_{k\to\infty} \Delta x(k) = 0\right\},$$

and define the operator $T : X \times [0, 1] \to X$ by

$$T(x, \lambda)(k) = \lambda \sum_{j=1}^{\infty} G(k, j)f(j, x(j), \Delta x(j-1)), \quad k \in \mathbb{N}.$$

The main result of this paper is the following.

Lemma 3.1. *Let* (C_1) *hold. Then, for each* $\lambda \in [0, 1]$, $T(x, \lambda)$ *is completely continuous in* X.

Proof. First, we show T is well-defined. For each $x \in X$, then there exists $r > 0$ such that $\|x\| < r$. For each $\lambda \in [0, 1]$, it holds that

$$\begin{aligned} T(x, \lambda)(k) &= \lambda \sum_{j=1}^{\infty} G(k, j)f(j, x(j), \Delta x(j-1)) \\ &\leqslant \sum_{j=1}^{\infty} |G(k, j)f(j, x(j), \Delta x(j-1))| \\ &\leqslant \sum_{j=1}^{\infty} |G(k, j)|\varphi_r(j) < \infty, \quad k \in \mathbb{N}. \end{aligned}$$

By the definition of T, we have

$$\begin{aligned} |\Delta T(x, \lambda)(k)| &= |(T(x, \lambda)(k+1) - T(x, \lambda)(k))| \\ &= \left|\lambda \sum_{j=1}^{\infty} G(k+1, j)f(j, x(j), \Delta x(j-1)) - \lambda \sum_{j=1}^{\infty} G(k, j)f(j, x(j), \Delta x(j-1))\right| \\ &\leqslant \lambda \sum_{j=1}^{\infty} |G(k+1, j) - G(k, j)|\varphi_r(j) \\ &< \infty, \quad k \in \mathbb{N}. \end{aligned}$$

Therefore, $Tx \in S_\infty$.

Obviously, $T(x,\lambda)(0) = \sum_{i=1}^{m-2} \alpha_i T(x,\lambda)(\eta_i)$, and notice that

$$\lim_{k\to\infty} \Delta T(x,\lambda)(k) = \lim_{k\to\infty} \lambda \sum_{j=k+1}^{\infty} f(j,x(j),\Delta x(j-1)) = 0,$$

so we can get $Tx \in X$.

Second, we claim that $T(x,\lambda)$ is completely continuous in X, that is, for each $\lambda \in (0,1)$, $T(x,\lambda)$ is continuous in X and maps a bounded subset of X into a relatively compact set.

For each $x_n \in X$, $x_n \to x$ as $n \to \infty$. Next, we prove that for each $\lambda \in (0,1)$, $T(x_n,\lambda) \to T(x,\lambda)$ as $n \to \infty$ in X. By condition (C_1), we have

$$\begin{aligned}\Big|\sum_{j=1}^{\infty} G(k,j)f(j,x_n(j),\Delta x_n(j-1)) - \sum_{j=1}^{\infty} G(k,j)f(j,x(j),\Delta x(j-1))\Big| &\leqslant \Big|\sum_{j=1}^{\infty} G(k,j)(f(j,x_n(j),\Delta x_n(j-1)) - f(j,x(j),\Delta x(j-1)))\Big| \\ &\leqslant 2\sum_{j=1}^{\infty} |\overline{G}(j)|\varphi_{r_0}(j), \quad \text{as } n\to\infty,\end{aligned}$$

where $r_0 > 0$ is a real number, such that $\max_{n\in N}\{\|x_n\|, \|x\|\} = r_0$, we have

$$|T(x_n,\lambda)(k) - T(x,\lambda)(k)| \leqslant \sum_{j=1}^{\infty} |G(k,j)||f(j,x_n(j),\Delta x_n(j-1)) - f(j,x(j),\Delta x(j-1))| \to 0, \quad \text{as } n\to\infty,$$

and

$$|\Delta T(x_n,\lambda)(k) - \Delta T(x,\lambda)(k)| \leqslant \sum_{j=1}^{\infty} |f(j,x_n(j),\Delta x_n(j-1)) - f(j,x(j),\Delta x(j-1))| \to 0, \quad \text{as } n\to\infty.$$

Therefore, T is continuous.

Finally, we claim that T is compact set, that is, T maps a bounded subset of X into a relatively compact set. Let $B \subset X$ be a bounded subset. For each $x \in B$, $\|x\| < r$, there exists $r > 0$, we have

$$\begin{aligned}\|T(x,\lambda)(k)\|_\infty = \sup_{k\in N_0} |T(x,\lambda)(k)| = \sup_{k\in N_0}\Big|\lambda \sum_{j=1}^{\infty} G(k,j)f(j,x(j),\Delta x(j-1))\Big| \\ \leqslant \sum_{j=1}^{\infty} |G(k,j)|\varphi_r(k) < \infty, \quad (k\to\infty).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\Delta T(x,\lambda)(k)\|_\infty = \sup_{k\in N_0} |\Delta T(x,\lambda)(k)| = \sup_{k\in N_0}\Big|\sum_{j=k+1}^{\infty} f(j,x(j),\Delta x(j-1))\Big| \\ \leqslant \sup_{k\in N_0} \sum_{j=k+1}^{\infty} \varphi_r(j) < \infty, \quad (k\to\infty).\end{aligned}$$

Therefore, TB is bounded. At the same time, we obtain that

$$\begin{aligned}|(T(x,\lambda)(k) - \lim_{k\to\infty} T(x,\lambda)(k))| = \lambda\Big|\sum_{j=1}^{\infty} (G(k,j) - \overline{G}(j))f(j,x(j),\Delta x(j-1))\Big| \\ \leqslant \sum_{j=1}^{\infty} |G(k,j) - \overline{G}(j)|\varphi_r(j) \to 0, \quad (k\to\infty),\end{aligned}$$

and

$$\begin{aligned}|\Delta(T(x,\lambda)(k)-\lim_{k\to\infty}\Delta T(x,\lambda)(k))| &= |\Delta T(x,\lambda)(k)|\\ &\leqslant \sum_{j=k+1}^{\infty}|f(j,x(j),\Delta x(j-1))|\\ &\leqslant \sum_{j=k+1}^{\infty}\varphi_r(j)\to 0,\quad (k\to\infty).\end{aligned}$$

So, TB is uniformly convergent on infinity. Thus, by Theorem 2.5, $T(\cdot,\lambda): X\times[0,1]\to X$ is completely continuous on infinite intervals. □

Theorem 3.2. *Let* (C_1) *and* (C_2) *hold. Then BVP* (1.1) *has at least one solution provided:*

$$\begin{aligned}&\eta_{m-2}P+P_1+Q<1,\quad \alpha<0,\\ &\frac{\alpha\eta_{m-2}}{1-\alpha}P+P_1+Q<1,\ \ 0\leqslant\alpha<1,\\ &\max\{\frac{\alpha\eta_{m-2}}{\alpha-1}P+P_1+Q,\frac{\alpha\eta_{m-2}}{\alpha-1}P+\frac{\alpha}{\alpha-1}P_1\}<1,\ \ \alpha>1.\end{aligned}$$

Proof. In view of Lemma 2.1, it is clear that $x\in X$ is a solution of the BVP (1.1) if and only if x is a fixed point of $T(\cdot,1)$. Clearly, $T(x,0)=0$ for each $x\in X$. If for each $\lambda\in[0,1]$, the fixed points $T(\cdot,\lambda)$ in X belong to a closed ball of X independent of λ, then the Leray-Schauder continuation theorem completes the proof. We have known $T(\cdot,\lambda)$ is completely continuous by Lemma 3.1. Next, we show that the fixed point of $T(\cdot,\lambda)$ has a priori bound M independently of λ. Assume $x=T(x,\lambda)$ and set

$$Q_1=\sum_{j=1}^{\infty}jq(j),\qquad R=\sum_{j=1}^{\infty}r(j),\qquad R_1=\sum_{j=1}^{\infty}jr(j).$$

Case 1: $(\alpha<0)$. For any $x\in X,\ k\in\mathbb{N}$, we have

$$\begin{aligned}|x(k)|=\left|\sum_{j=1}^{k}\Delta x(j-1)+x(0)\right| &= \left|\sum_{j=1}^{k}\Delta x(j-1)+\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{j=1}^{\eta_i}\Delta x(j-1)\right|\\ &\leqslant k\|\Delta x\|_\infty+\left|\frac{\alpha}{1-\alpha}\right|\eta_{m-2}\|\Delta x\|_\infty\\ &\leqslant (k+\eta_{m-2})\|\Delta x\|_\infty,\quad k\in\mathbb{N},\end{aligned}$$

and so it holds that

$$\begin{aligned}\|\Delta x\|_\infty\leqslant\|\lambda f(j,x(j),\Delta x(j-1))\|_{l^1} &\leqslant \|f(j,x(j),\Delta x(j-1))\|_{l^1}\\ &\leqslant \|p(j)|x(j)|+q(j)|\Delta x(j-1)|+r(j)\|_{l^1}\\ &\leqslant (\eta_{m-2}P+P_1+Q)\|\Delta x\|_\infty+R,\end{aligned}$$

therefore,

$$\|\Delta x\|_\infty\leqslant\frac{R}{1-\eta_{m-2}P-P_1-Q}:=\Delta M_1.$$

At the same time, we have

$$\begin{aligned}|x(k)|\leqslant\left|\lambda\sum_{j=1}^{\infty}G(k,j)f(j,x(j),\Delta x(j-1))\right| &\leqslant \sum_{j=1}^{\infty}|jf(j,x(j),\Delta x(j-1))|\\ &\leqslant \sum_{j=1}^{\infty}\Big|j[p(j)x(j)+q(j)\Delta x(j-1)+r(j)]\Big|\end{aligned}$$

$$\leqslant P_1\|x(j)\|_\infty + Q_1\|\Delta x(j-1)\|_\infty + R_1$$
$$\leqslant P_1\|x\|_\infty + Q_1\Delta M_1 + R_1, \quad k \in N,$$

and so

$$\|x\|_\infty \leqslant \frac{Q_1\Delta M_1 + R_1}{1-P_1} := M_1.$$

Set $M = \max\{M_1, \Delta M_1\}$, which is independent of λ, so $\|x\| \leqslant M$.

Case 2: $(0 \leqslant \alpha < 1)$. For any $x \in X$, $k \in N$, we have

$$|x(k)| = \left|\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{j=1}^{\eta_i}\Delta x(j-1) + \sum_{j=1}^{k}\Delta x(j-1)\right| \leqslant (\frac{\alpha\eta_{m-2}}{1-\alpha} + k)\|\Delta x\|_\infty, \quad k \in N.$$

In the same way as for Case 1, we can get

$$\|\Delta x\|_\infty \leqslant \frac{(1-\alpha)R}{(1-\alpha)(1-P_1-Q)-\alpha\eta_{m-2}P} := \Delta M_2,$$
$$\|x\|_\infty \leqslant \frac{Q_1\Delta M_1 + R_1}{1-P_1} := M_2.$$

Set $M = \max\{M_2, \Delta M_2\}$, which is independent of λ and is what we need, so $\|x\| \leqslant M$.

Case 3: $(\alpha > 1)$. For any $x \in X$, $k \in N$, we have

$$|x(k)| = \left|x(0) + \sum_{j=1}^{k}\Delta x(j-1)\right| \leqslant \left(\frac{\alpha\eta_{m-2}}{\alpha-1} + k\right)\|\Delta x\|_\infty, \quad k \in N.$$

Similarly, we obtain

$$\|\Delta x\|_\infty \leqslant \frac{(\alpha-1)R}{(\alpha-1)(1-P_1-Q)-\alpha\eta_{m-2}P} := \Delta M_3,$$

and

$$\begin{aligned}|x(k)| &\leqslant \sum_{j=1}^{\infty}|G(k,j)f(j,x(j),\Delta x(j-1))| \\ &\leqslant \sum_{j=1}^{\infty}\left|\frac{\alpha j}{\alpha-1}f(j,x(j),\Delta x(j-1))\right| + \sum_{j=\eta}^{\infty}\left|\frac{\alpha\eta_{m-2}}{\alpha-1}f(j,x(j),\Delta x(j-1))\right| \\ &\leqslant \frac{\alpha}{\alpha-1}(P_1\|x\|_\infty + Q_1\Delta M_3 + R_1) + \frac{\alpha\eta_{m-2}}{\alpha-1}(P\|x\|_\infty + Q\Delta M_3 + R),\end{aligned}$$

that is,

$$\|x\|_\infty \leqslant \frac{\alpha(Q_1\Delta M_3 + R_1) + \alpha\eta_{m-2}(Q_1\Delta M_3 + R)}{\alpha - 1 - \alpha(P_1 + \eta_{m-2}P)} := M_3.$$

Set $M = \max\{M_3, \Delta M_3\}$ and this is what we need. Hence, BVP (1.1) has at least one solution. □

4. Example

Example 4.1. Consider the following second-order four-point difference equation BVP on infinite intervals

$$\begin{cases} \Delta^2 x(k-1) + \frac{\sin(x(k))}{10^2 k!} + \frac{\arctan(\Delta x(k-1))+1}{3^k} = 0, & k \in N, \\ x(0) = \frac{1}{10}x(10) + \frac{1}{10}x(100), & \lim_{k\to\infty}\Delta x(k) = 0, \end{cases} \tag{4.1}$$

where $m = 4, \alpha_1 = \alpha_2 = \frac{1}{10}$, $\eta_1 = 10$, $\eta_2 = 100$, $f(t,u,v) = \frac{\sin(x(k))}{10^2 k!} + \frac{\arctan(\Delta x(k-1))+1}{3^k}$, and $p(k) =$

$\frac{1}{10^2 k!}$, $q(k) = \frac{1}{3^k}$, $r(k) = \frac{1}{3^k}$.

Obviously, $f(t, u, v) \leqslant p(k)|u| + q(k)|v| + r(k)$. By a simple calculation, we find that (C_1) and (C_2) hold. Since $\alpha = \alpha_1 + \alpha_2 = \frac{1}{5} < 1$, $\frac{\alpha\eta_2}{1-\alpha}P + P_1 + Q \approx 0.96 < 1$. Hence, by Theorem 3.2, we obtain the BVP (4.1) has at least one solution.

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