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Existence of solutions for nonlinear Caputo-Hadamard fractional differential equations via the method of upper and lower solutions

Yunru Bai[a,b], Hua Kong[b,*]

[a] *Institute of Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Lojasiewicza 6, 30-348 Krakow, Poland.*

[b] *Data Recovery Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, China.*

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Abstract

The purpose of this paper is devoted to consider the existence of solutions for a class of nonlinear Caputo-Hadamard fractional differential equations with integral terms ((CHFDE), for short). Firstly, by applying the semi-group property of Hadamard fractional integral operator, a necessary condition of solvability for (CHFDE) is established. Then, under the suitable conditions, we prove the solution set of (CHFDE) is nonempty by using the method of upper and lower solutions, and Arzelà-Ascoli theorem. Finally, we present several numerical examples to explicate the main results.

Keywords: Caputo-Hadamard derivative, fractional differential equations, upper and lower solutions, monotone sequences, Arzelà-Ascoli theorem.
2010 MSC: 49J53, 49K40, 90C33, 90C46.

1. Introduction

The fractional calculus ((FC), for short) deals with extensions of derivatives and integrals to non-integer orders, which was started to be considered deeply as a powerful tool to reveal the hidden aspects of the dynamics of the complex or hyper complex systems (see [1, 4, 8, 10, 14, 15, 17, 18, 24–26]). In the last thirty years or so, the theory of (FC) has produced an abundance of important results both in pure and applied mathematics as well as in other fields as for example: physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. (see [11–13, 22, 28–31]), as it is allowed mathematical formulations for new classes of interesting problems (see [3, 5, 6, 19, 20, 23, 32]).

To our knowledge, the fractional integral and derivative operators are usually studied in the sense of Riemann-Liouville, Caputo or Grünwald-Letnikov. However, Hadamard fractional derivative operator was introduced and studied in order to consider the problems, which is different completely to the ones in the sense of Riemann-Liouville, Caputo and Grünwald-Letnikov (see [2, 7, 9, 10, 16, 21]). In the present

*Corresponding author
Email addresses: yunrubai@163.com (Yunru Bai), konghua2008@126.com (Hua Kong)

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paper, we shall consider the following fractional nonlinear differential system involving Hadamard fractional integral term

$$\begin{cases} {}^{C}D^{\alpha}_{a+}x(t) = f(t, x(t), I^{\alpha}_{a+}x(t)), \ t \in [a, b], \\ x(a) = x_a, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha}_{a+}$ and I^{α}_{a+} stand for the Caputo-Hadamard derivative and Hadamard integral operators (see Definitions 2.1 and 2.4, below), respectively, $f\colon [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $x_a \in \mathbb{R}$, and $0 < a < b < \infty$.

The rest of paper is structured as follows. Section 2 contains basic definitions and results needed in the sequel. Section 3 is devoted to present the main results describing the existence of solutions for Caputo-Hadamard fractional differential equation (1.1). In Section 4, we show several numerical examples to explicate our results.

2. Preliminaries

In this section we collect a few notions and results to be used later in the paper. In the sequel, we denote $AC(a, b; \mathbb{R})$ the function space, by $\mathbb{R}$-valued absolutely continuous functions on $[a, b]$. In the begin, we recall fractional operators in the sense of Hadamard.

Definition 2.1 ([7, 10])**.** Let $0 < a < b < \infty$ and $x\colon [a, b] \to \mathbb{R}$. The Hadamard fractional integral of order $\alpha > 0$ of x is defined by

$$I^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{x(s)}{s}\, ds \quad \text{for } t \in [a, b],$$

where Γ stands for the well-known Gamma function by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt.$$

Definition 2.2 ([7, 10])**.** Let $0 < a < b < \infty$ and $x\colon [a, b] \to \mathbb{R}$. The Hadamard fractional derivative of order $\alpha \in (0, 1]$ of x is defined by

$$D^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(1-\alpha)} t \frac{d}{dt} \int_a^t \left(\ln \frac{t}{s}\right)^{-\alpha} \frac{x(s)}{s}\, ds \quad \text{for } t \in [a, b].$$

Obviously, we can obtain

$$I^{\alpha}_{a+}\left(\ln \frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\ln \frac{t}{a}\right)^{\beta+\alpha-1}, \quad D^{\alpha}_{a+}\left(\ln \frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\ln \frac{t}{a}\right)^{\beta-\alpha-1}$$

for $t \in [a, b]$.

Now, we review some significant properties for fractional Hadamard integral and derivative operators in which their proof can be found in [7, 10].

Lemma 2.3 ([7, 10])**.** *Let α, β be such that $\alpha > 0$ and $\beta > 0$.*

(i) *If $1 \leqslant p < \infty$, then for $x \in L^p(a, b; \mathbb{R})$, we have*

$$I^{\beta}_{a+} I^{\alpha}_{a+} x(t) = I^{\alpha+\beta}_{a+} x(t) \quad \textit{for } t \in [a, b].$$

(ii) *If $1 \leqslant p < \infty$ and $\alpha > \beta$, then for $x \in L^p(a, b; \mathbb{R})$, we have*

$$D^{\beta}_{a+} I^{\alpha}_{a+} x(t) = I^{\alpha-\beta}_{a+} x(t) \quad \textit{for } t \in [a, b].$$

Definition 2.4. Let $0 < a < b < \infty$ and $x\colon [a, b] \to \mathbb{R}$. The Caputo-Hadamard fractional derivative of order $\alpha \in (0, 1]$ of x is defined by

$${}^{C}D^{\alpha}_{a+}x(t) = D^{\alpha}_{a+}[x(t) - x(a)] \quad \text{for } t \in [a, b].$$

Remark 2.5. It is obvious that if $x \in AC(a,b;\mathbb{R})$, then Caputo-Hadamard fractional derivative, Definition 2.4, has the following equivalent formulation

$$^{C}D^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{-\alpha} x'(s)\,ds \quad \text{for } t \in [a,b].$$

We conclude this section by recalling the following component properties for Caputo-Hadamard fractional operators.

Lemma 2.6 ([7, 10])**.** *Let $\alpha > 0$ be such that $n = [\alpha]+1$.*

(i) *If $x \in C(a,b;\mathbb{R})$, then*

$$^{C}D^{\alpha}_{a+}(I^{\alpha}_{a+}x(t)) = x(t) \quad \textit{for } t \in [a,b].$$

(ii) *If $x \in AC(a,b;\mathbb{R})$, then*

$$I^{\alpha}_{a+}(^{C}D^{\alpha}_{a+}x(t)) = x(t) - x(a) \quad \textit{for } t \in [a,b].$$

3. Main results

In this section, we focus our attention on the existence of solutions for fractional nonlinear differential system (1.1).

Theorem 3.1. *Let $f\colon [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that $x \in C(a,b;\mathbb{R})$ is a solution of the following integral equation*

$$x(t) = x_a + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s),I^{\alpha}_{a+}x(s))}{s}\,ds \quad \textit{for } t \in [a,b], \tag{3.1}$$

then it also resolves the fractional nonlinear differential equation (1.1).

Proof. Assume that $x \in C(a,b;\mathbb{R})$ is a solution of the integral equation (3.1). Obviously, we obtain $x(a) = x_a$ and $t \mapsto I^{\alpha}_{a+}x(t) \in C(a,b;\mathbb{R})$. The continuity of f and definition of Hadamard integral I^{α}_{a+} guarantee that $t \mapsto f(t,x(t),I^{\alpha}_{a+}x(t))$ is continuous as well and

$$I^{\alpha}_{a+}f(t,x(t),I^{\alpha}_{a+}x(t))\big|_{t=a} = 0.$$

Since, $t \mapsto I^{\alpha}_{a+}f(t,x(t),I^{\alpha}_{a+}x(t))$ is continuous, then we have x is differential for a.e. $t \in (a,b)$, see (3.1), i.e., $x \in AC(a,b;\mathbb{R})$. From Lemma 2.6, we have

$$^{C}D^{\alpha}_{a+}I^{\alpha}_{a+}f(t,x(t),I^{\alpha}_{a+}x(t)) = f(t,x(t),I^{\alpha}_{a+}x(t)) \quad \text{for } t \in [a,b].$$

On the other hand, Remark 2.5 reveals

$$^{C}D^{\alpha}_{a+}[x(t)-x_a] = \frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{-\alpha}[x(s)-x_a]'\,ds = \frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{-\alpha} x'(s)\,ds = {}^{C}D^{\alpha}_{a+}x(t)$$

for $t \in [a,b]$.

By all above, we conclude that $x \in C(a,b;\mathbb{R})$ is a solution of fractional nonlinear differential equation (1.1), which completes the proof of the theorem. □

Now, we introduce the concept of upper and lower solutions for integral equation (3.1), which plays a remarkable role in our work.

Definition 3.2. Let $(\underline{x},\overline{x}) \in C(a,b;\mathbb{R})\times C(a,b;\mathbb{R})$. A pair of functions $(\underline{x},\overline{x})$ is called to be upper and lower solutions of fractional integral equation (3.1), respectively, if

$$\underline{x}(t) \leqslant x_a + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,\underline{x}(s),I^{\alpha}_{a+}\underline{x}(s))}{s}\,ds \quad \text{for all } t \in [a,b],$$

and

$$\overline{x}(t) \geqslant x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, \overline{x}(s), I_{a^+}^{\alpha} \overline{x}(s))}{s} \, ds \quad \text{for all } t \in [a, b].$$

Let $(\underline{x}, \overline{x})$ be a pair of upper and lower solutions of fractional integral equation (3.1). In the sequel, we denote an admissible set of solutions for fractional integral equation (3.1) governed by a pair of upper and lower solutions $(\underline{x}, \overline{x})$ as follows

$$U_{(\underline{x},\overline{x})} := \left\{ x \in C(a, b; \mathbb{R}) : \underline{x}(t) \leqslant x(t) \leqslant \overline{x}(t),\ t \in [a, b] \text{ and } x \text{ is a solution of (3.1)} \right\}.$$

Theorem 3.3. *Let* $f \in C([a, b] \times \mathbb{R}^2; \mathbb{R})$*. Assume that* $(\underline{x}, \overline{x}) \in C(a, b; \mathbb{R}) \times C(a, b; \mathbb{R})$ *is a pair of upper and lower solutions of fractional integral equation* (3.1) *with* $\underline{x}(t) \leqslant \overline{x}(t)$ *for* $t \in [a, b]$*. If* $(x, y) \mapsto f(t, x, y)$ *is nondecreasing, that is*

$$f(t, x_1, y_1) \leqslant f(t, x_2, y_2) \ \textit{for}\ x_1 \leqslant x_2 \ \textit{and}\ y_1 \leqslant y_2,$$

then there exist maximal and minimal solutions x_M*,* $x_L \in U_{(\underline{x},\overline{x})}$ *in* $U_{(\underline{x},\overline{x})}$*, i.e., for each* $x \in U_{(\underline{x},\overline{x})}$ *one has*

$$x_L(t) \leqslant x(t) \leqslant x_M(t) \quad \textit{for}\ t \in [0, T].$$

Proof. To this end, we construct two sequences $\{y_n\}$ and $\{z_n\}$ as follows

$$\begin{cases} y_0 = \underline{x}, \\ y_{n+1}(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_n(s), I_{a^+}^{\alpha} y_n(s))}{s} \, ds,\ t \in [a, b] \text{ and } n = 0, 1, \ldots, \end{cases}$$

and

$$\begin{cases} z_0 = \overline{x}, \\ z_{n+1}(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, z_n(s), I_{a^+}^{\alpha} z_n(s))}{s} \, ds,\ t \in [a, b] \text{ and } n = 0, 1, \ldots. \end{cases}$$

We now split the proof in three parts.

Step 1. Sequences $\{y_n\}$ and $\{z_n\}$ satisfy the following relation:

$$\underline{x}(t) = y_0(t) \leqslant y_1(t) \leqslant y_2(t) \leqslant \ldots \leqslant y_n(t) \leqslant \ldots \leqslant z_n(t) \leqslant \ldots \leqslant z_1(t) \leqslant z_0(t) = \overline{x}(t) \tag{3.2}$$

for $t \in [0, T]$.

First, we shall prove that sequence $\{y_n\}$ is nondecreasing and

$$y_n(t) \leqslant z_0(t),\ t \in [a, b] \quad \text{for all } n \in \mathbb{N}.$$

According to the assumptions, we can know $\underline{x}(t) = y_0(t) \leqslant \overline{x}(t) = z_0(t)$ for $t \in [a, b]$ and

$$y_1(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_0(s), I_{a^+}^{\alpha} y_0(s))}{s} \, ds \geqslant y_0(t) \quad \text{for } t \in [a, b].$$

Since $(x, y) \mapsto f(t, x, y)$ is nondecreasing, then it is obvious that

$$f(s, y_0(s), I_{a^+}^{\alpha} y_0(s)) \leqslant f(s, z_0(s), I_{a^+}^{\alpha} z_0(s))$$

for $s \in [0, T]$. This deduces

$$y_1(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_0(s), I_{a^+}^{\alpha} y_0(s))}{s} \, ds$$

$$\leqslant x_a + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s, z_0(s), I_{a^+}^{\alpha} z_0(s))}{s}\, ds \leqslant z_0(t) \quad \text{for } t \in [a, b].$$

Therefore, we assume inductively

$$y_{n-1}(t) \leqslant y_n(t) \leqslant z_0(t) \quad \text{for } t \in [a, b].$$

By virtue of definition of $\{y_n\}$, we have

$$y_n(t) = x_a + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_{n-1}(s), I_{a^+}^{\alpha} y_{n-1}(s))}{s}\, ds,$$
$$y_{n+1}(t) = x_a + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_n(s), I_{a^+}^{\alpha} y_n(s))}{s}\, ds$$

for $t \in [0, T]$. Using the monotonicity of f, we readily obtain

$$y_n(t) \leqslant y_{n+1}(t) \leqslant z_0(t) \quad \text{for } t \in [a, b].$$

Further, we will show that

$$y_n(t) \leqslant z_n(t) \quad \text{for } t \in [a, b] \text{ and } n \in \mathbb{N}.$$

For $n = 0$, it is clear that $\underline{x}(t) = y_0(t) \leqslant z_0(t) = \overline{x}(t)$ for $t \in [a, b]$. Now, we also suppose inductively

$$y_n(t) \leqslant z_n(t), \ t \in [a, b].$$

Analogically, we easily conclude from the monotonicity of f with respect to the second and the third variables that

$$y_{n+1}(t) \leqslant z_{n+1}(t), \ t \in [a, b].$$

Also, we have that the sequence $\{z_n\}$ is nonincreasing.

Step 2. Sequences $\{y_n\}$ and $\{z_n\}$ are both relatively compact in $C(a, b; \mathbb{R})$.

Because f is continuous and $\underline{x}, \overline{x} \in C(a, b; \mathbb{R})$, from Step 1, we have $\{y_n\}$ and $\{z_n\}$ belong to $C(a, b; \mathbb{R})$ as well. It follows from (3.2) that $\{y_n\}$ and $\{z_n\}$ are uniformly bounded. On the other hand, for any $t_1, t_2 \in [a, b]$, without loss of generality, let $t_1 \leqslant t_2$, we have

$$\begin{aligned}
|y_{n+1}(t_1) - y_{n+1}(t_2)| &= \frac{1}{\Gamma(\alpha)}\left| \int_a^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \frac{f(s, y_n(s), I_{a^+}^{\alpha} y_n(s))}{s}\, ds - \right.\\
&\quad \left. - \int_a^{t_1} \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \frac{f(s, y_n(s), I_{a^+}^{\alpha} y_n(s))}{s}\, ds \right|\\
&= \frac{1}{\Gamma(\alpha)}\left| \int_a^{t_1} \left[\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right] \frac{f(s, y_n(s), I_{a^+}^{\alpha} y_n(s))}{s}\, ds \right.\\
&\quad \left. + \int_{t_1}^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \frac{f(s, y_n(s), I_{a^+}^{\alpha} y_n(s))}{s}\, ds \right|\\
&\leqslant \frac{M}{\Gamma(\alpha)}\left| \int_a^{t_1} \frac{1}{s}\left[\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right] ds + \int_{t_1}^{t_2} \frac{1}{s}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} ds \right|\\
&\leqslant \frac{M}{\Gamma(1+\alpha)}\left[\left(\ln\frac{t_2}{a}\right)^{\alpha} - \left(\ln\frac{t_1}{a}\right)^{\alpha} + 2\left(\ln\frac{t_2}{t_1}\right)^{\alpha}\right]\\
&\to 0, \quad \text{as } |t_1 - t_2| \to 0,
\end{aligned}$$

where $M > 0$ is a constant independent of n, t_1, and t_2. It implies that $\{y_n\}$ is equicontinuous in $C(a, b; \mathbb{R})$. From Arzelà-Ascoli Theorem (see [27]), we imply that $\{y_n\}$ is relatively compact in $C(a, b; \mathbb{R})$. Similarly, we obtain $\{z_n\}$ is also relatively compact in $C([a, b]; \mathbb{R})$.

Step 3. There exist maximal and minimal solutions in $U_{(\underline{x},\overline{x})}$.

Step 1 and Step 2 indicate that the sequences $\{y_n\}$ and $\{z_n\}$ are both monotone and relatively compact in $C(a,b;\mathbb{R})$. Obviously, there exist continuous functions y and z with $y_n(t) \leqslant y(t) \leqslant z(t) \leqslant z_n(t)$ for all $t \in [a,b]$ and $n \in \mathbb{N}$, such that $\{y_n\}$ and $\{z_n\}$ converge uniformly to y and z in $C(a,b;\mathbb{R})$, respectively. So, y and z are two solutions of (3.1), i.e.,

$$y(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y(s), I_{a+}^{\alpha} y(s))}{s}\, ds,$$
$$z(t) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, z(s), I_{a+}^{\alpha} z(s))}{s}\, ds$$

for $t \in [a,b]$. However, the fact (3.2) ensures that

$$\underline{x}(t) \leqslant y(t) \leqslant z(t) \leqslant \overline{x}(t) \quad \text{for } t \in [a,b].$$

Finally, we shall illustrate that y and z are the minimal and maximal solutions in $U_{(\underline{x},\overline{x})}$, respectively. For any $x \in U_{(\underline{x},\overline{x})}$, then we have

$$\underline{x}(t) \leqslant x(t) \leqslant \overline{x}(t) \quad \text{for } t \in [a,b].$$

Recall that f is nondecreasing with respect to the second and the third arguments, we induct

$$\underline{x}(t) \leqslant y_n(t) \leqslant x(t) \leqslant z_n(t) \leqslant \overline{x}(t) \quad \text{for } t \in [a,b] \quad \text{and } n \in \mathbb{N}.$$

Taking limits as $n \to \infty$ into the above inequality, one implies

$$\underline{x}(t) \leqslant y(t) \leqslant x(t) \leqslant z(t) \leqslant \overline{x}(t) \quad \text{for } t \in [a,b].$$

This means that $x_L = y$ and $x_M = z$ are the minimal and maximal solutions in $U_{(\underline{x},\overline{x})}$, respectively, which completes the proof of the theorem. □

Theorem 3.4. *Assume that hypotheses of Theorem 3.3 are satisfied. Then fractional nonlinear differential equation (1.1) has at least one solution in* $C([a,b];\mathbb{R})$.

Proof. By the assumptions and Theorem 3.3, we have $U_{(\underline{x},\overline{x})} \neq \emptyset$, i.e., the solution set of fractional integral equation (3.1) is nonempty in $C(a,b;\mathbb{R})$. This combines with Theorem 3.1 to verify that fractional nonlinear differential equation (1.1) has at least one solution in $C(a,b;\mathbb{R})$, which completes the proof of the theorem. □

4. Examples

In this section, we will apply foregoing theoretical results stated in Section 3 to present two simple examples to explicate the results.

Example 4.1. Consider the following Caputo-Hadamard fractional differential equation

$$\begin{cases} {}^{C}D_{1^+}^{\frac{1}{2}} x(t) = \frac{8}{3\sqrt{\pi}}(\ln t)^{\frac{3}{2}} - (\ln t)^2 + x(t) & \text{for } t \in [1,e], \\ x(1) = 0. \end{cases}$$

Proof. Let $f(t,x(t)) = \frac{8}{3\sqrt{\pi}}(\ln t)^{\frac{3}{2}} - (\ln t)^2 + x(t)$ for $t \in [1,e]$. From Theorem 3.1, we only prove that the following fractional integral equation has at least one solution in $C(1,e;\mathbb{R})$

$$x(t) = I_{1^+}^{\frac{1}{2}} \left(\frac{8}{3\sqrt{\pi}}(\ln t)^{\frac{3}{2}} - (\ln t)^2 + x(t) \right) \quad \text{for } t \in [1,e]. \tag{4.1}$$

Indeed, we can see that $(\underline{x}(t), \overline{x}(t)) = (0, (\ln t)^2 + (\ln t)^3)$ is a pair of upper and lower solutions of (4.1). In addition, f is continuous and nondecreasing with respect to the second argument. We can calculate the sequences $\{y_n\}$ and $\{z_n\}$ by

$$\begin{cases} y_0(t) = \underline{x}(t), \\ y_{n+1}(t) = I_{1^+}^{\frac{1}{2}} f(t, y_n(t)),\ n = 0, 1, \ldots, \end{cases} \qquad \begin{cases} z_0(t) = \overline{x}(t), \\ z_{n+1}(t) = I_{1^+}^{\frac{1}{2}} f(t, z_n(t)),\ \ n = 0, 1, \ldots \end{cases}$$

for $t \in [1, e]$. We are now in a position to apply Theorem 3.3 to conclude $y_n \to y \in C(1, e; \mathbb{R})$ and $z_n \to z \in C(1, e; \mathbb{R})$ as $n \to \infty$. In the meantime, we can get $z(t) = y(t) = (\ln t)^2$ for $t \in [1, e]$. The approximation of sequences $\{y_n\}$ and $\{z_n\}$ to $(\ln t)^2$ is shown in Fig. 1 and Table 1. □

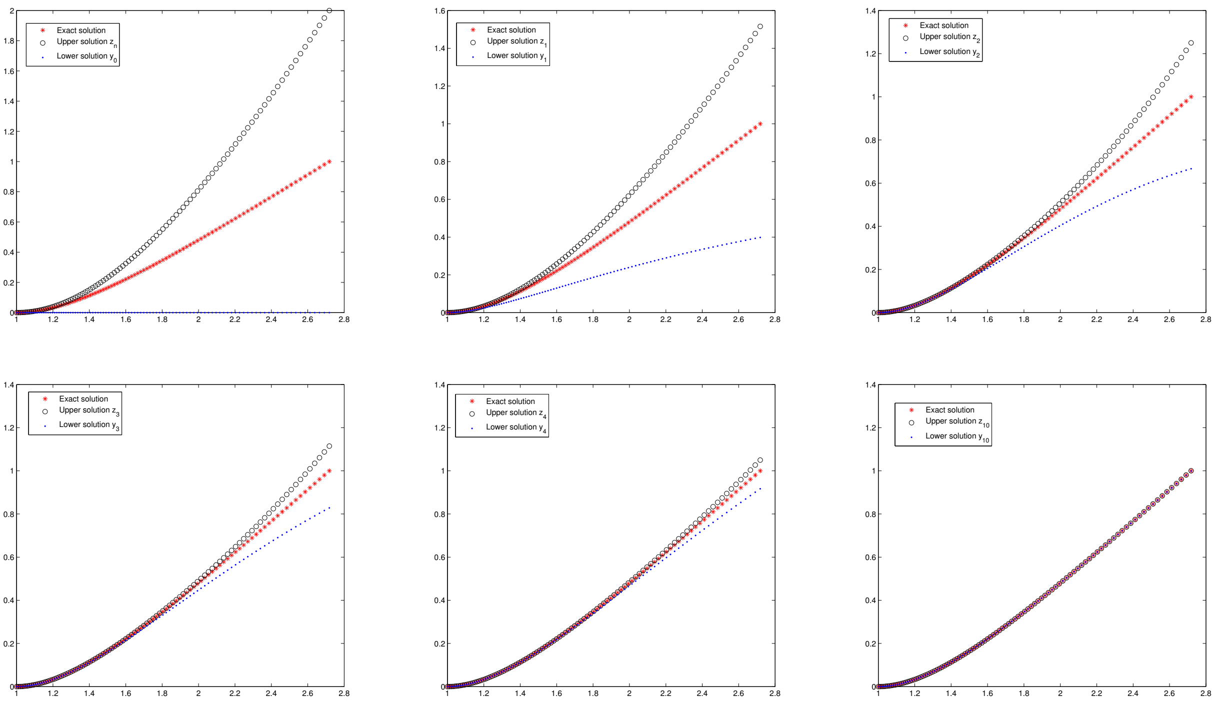

Figure 1: A plot of y_k and z_k, $k = 0, 1, 2, 3, 4, 10$ for Example 4.1.

Table 1: Error analysis.

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{t\in[1,e]} \lvert y_n(t) - x(t)\rvert$	8.33×10^{-2}	1.10×10^{-3}	5.51×10^{-6}	1.46×10^{-8}
$\sup_{t\in[1,e]} \lvert z_n(t) - x(t)\rvert$	5.00×10^{-2}	1.49×10^{-4}	1.65×10^{-6}	3.51×10^{-9}

Example 4.2. Consider the following Caputo-Hadamard fractional differential equation

$$\begin{cases} {}^{C}D_{1^+}^{\frac{1}{2}} x(t) = \frac{2}{\sqrt{\pi}} (\ln t)^{\frac{1}{2}} - \frac{4}{3\sqrt{\pi}} (\ln t)^{\frac{3}{2}} + I_{1^+}^{\frac{1}{2}} x(t) \quad \text{for } t \in [1, e] \\ x(1) = 0. \end{cases}$$

Proof. Let $f(t, x(t), I_{1^+}^{\frac{1}{2}} x(t)) = \frac{2}{\sqrt{\pi}} (\ln t)^{\frac{1}{2}} - \frac{4}{3\sqrt{\pi}} (\ln t)^{\frac{3}{2}} + I_{1^+}^{\frac{1}{2}} x(t)$ for $t \in [1, e]$. Then, the corresponding fractional integral equation is obtained by

$$x(t) = I_{1^+}^{\frac{1}{2}} \left(\frac{2}{\sqrt{\pi}} (\ln t)^{\frac{1}{2}} - \frac{4}{3\sqrt{\pi}} (\ln t)^{\frac{3}{2}} + I_{1^+}^{\frac{1}{2}} x(t) \right) \quad \text{for } t \in [1, e]. \tag{4.2}$$

Obviously, $(\underline{x}(t), \overline{x}(t)) = (0, (\ln t)^2 + \ln t)$ is a pair of upper and lower solutions of (4.2). All conditions in

Theorem 3.3 are verified readily, so, we construct the sequences $\{y_n\}$ and $\{z_n\}$ by

$$\begin{cases} y_0(t) = \underline{x}(t), \\ y_{n+1}(t) = I_{1^+}^{\frac{1}{2}} f(t, y_n(t), I_{1^+}^{\frac{1}{2}} y_n(t)), \end{cases} \qquad \begin{cases} z_0(t) = \overline{x}(t), \\ z_{n+1}(t) = I_{1^+}^{\frac{1}{2}} f(t, z_n(t), I_{1^+}^{\frac{1}{2}} z_n(t)). \end{cases}$$

Applying Theorem 3.3 again, we have $y_n \to y \in C([1, e]; \mathbb{R})$ and $z_n \to z \in C(1, e; \mathbb{R})$ as $n \to \infty$. Besides, we have that $z(t) = y(t) = \ln t$ for $t \in [1, e]$. Furthermore, we also obtain the approximation results, Fig. 2 and Table 2. □

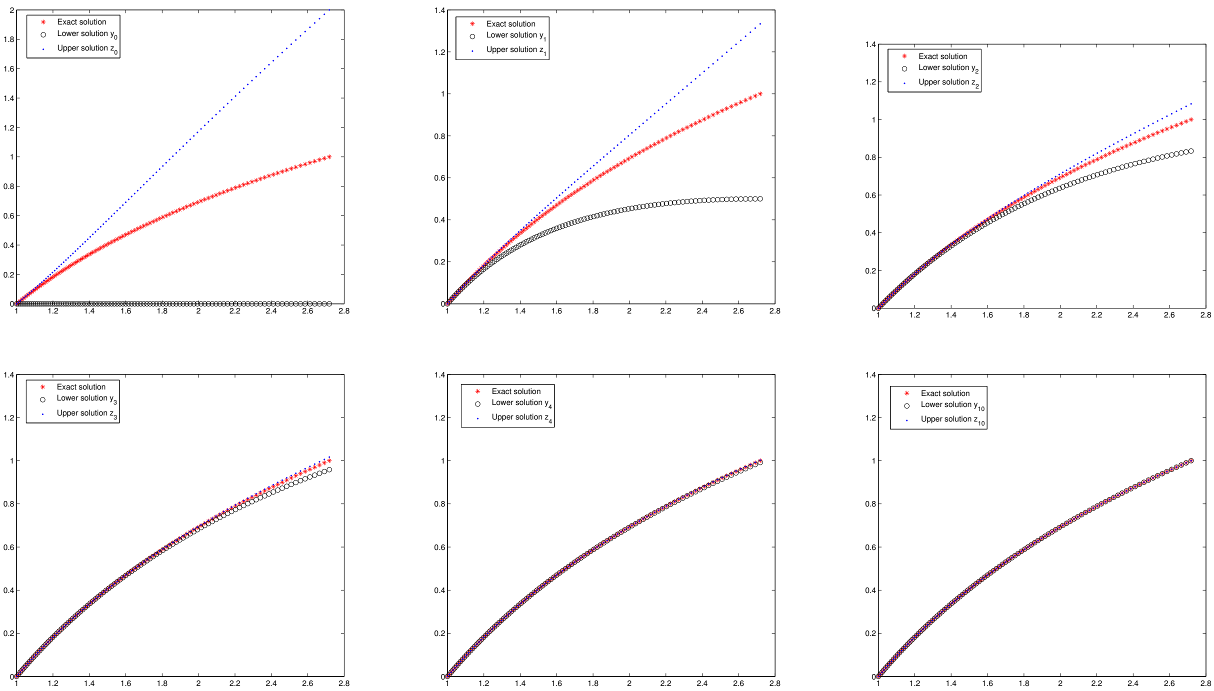

Figure 2: A plot of y_k and z_k, $k = 0, 1, 2, 3, 4, 10$ for Example 4.2.

Table 2: Error analysis.

Absolute Error \ Iterations	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{t \in [1,e]} \lvert y_n(t) - x(t) \rvert$	8.34×10^{-3}	2.6×10^{-7}	7.65×10^{-13}	1.11×10^{-16}
$\sup_{t \in [1,e]} \lvert z_n(t) - x(t) \rvert$	2.78×10^{-3}	5.01×10^{-8}	9.57×10^{-14}	1.14×10^{-16}

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