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Weighted Simpson type inequalities for h-convex functions

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Abstract

In this paper we establish some weighted Simpson type inequalities for functions whose derivatives in absolute value are h-convex.

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1. Introduction

The Simpson inequality states that if f exists and is bounded on (a, b), then

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant \frac{1}{2880}\left\|f^{(4)}\right\|_\infty \cdot (b-a)^4,$$

where

$$\left\|f^{(4)}\right\|_\infty := \sup_{t\in(a,b)} | f^{(4)}(t) | < \infty.$$

In [3], Dragomir et al. proved the following inequality.

Theorem 1.1. *Suppose* $f : [a, b] \to R$ *is a differentiable mapping whose derivative is continuous on* (a, b) *and* $f' \in L([a, b])$*. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{3} \parallel f' \parallel_1,$$

where

$$\parallel f' \parallel_1 = \int_a^b | f'(x) |\,dx.$$

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In [9], Sarikaya et al. obtained inequalities for differentiable convex mappings. The main inequality is as follows.

Theorem 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L([a,b])$, where $a,b \in I^0$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}\right\}, \tag{1.1}$$

where

$$\frac{1}{p}+\frac{1}{q}=1.$$

In [10], Sarikaya et al. obtained the following inequality for s-convex functions.

Theorem 1.3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0such that $f' \in L([a,b])$, where $a,b \in I^0$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q < 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(b)|^q+|f'\left(\frac{a+b}{2}\right)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'\left(\frac{a+b}{2}\right)|^q}{4}\right)^{\frac{1}{q}}\right\}, \tag{1.2}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For recent refinements, counterparts, generalizations, and inequalities of Simpson type, see [1–7, 9, 10] and [11, 12].

In 2007, Varošanec in [13] introduced a large class of functions, the so-called h-convex functions. This class contains several well-known classes of functions such as non-negative convex functions, s-convex in the second sense, Godunova Levin functions and P-functions. This class is defined in the following way: a function $f : I \to \mathbb{R}$, $\emptyset \neq I \subset \mathbb{R}$ being an interval is called h-convex, if

$$f(tx+(1-t)y) \leqslant h(t)f(x)+h(1-t)f(y),$$

holds for all $x, y \in I$, $t \in (0,1)$, where $h : J \to \mathbb{R}$, $h \neq 0$ and J is an interval, $(0,1) \subseteq J$.

In [8], Sarikaya et al. proved that for h-convex function the following variant of the Hadamard inequality is fulfilled:

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant [f(a)+f(b)]\cdot\int_0^1 h(t)\,dt. \tag{1.3}$$

The main purpose of the present paper is to establish new weighted Simpson type inequalities for functions whose derivatives in absolute value are h-convex.

2. Main result

In order to prove our main theorems, we need the following lemma.

Lemma 2.1. *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$ and let $w:[a,b]\to R$ be symmetric mapping to $\frac{a+b}{2}$. If f', $w\in L([a,b])$, then the following indentity holds:*

$$\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx$$
$$=\frac{b-a}{2}\left\{\int_0^1\left[\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right]f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right.$$
$$\left.+\int_0^1\left[\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^t w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right\}.$$

Proof. By integration by parts and changing the variables, we get

$$I_1=\int_0^1\left[\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right]f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$
$$=\frac{2}{b-a}\left[\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right]f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Big|_0^1$$
$$-\frac{1}{b-a}\int_0^1 w\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$
$$=\frac{2}{b-a}\left[\frac{1}{6}f(b)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right]\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds$$
$$-\frac{1}{b-a}\int_0^1 w\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$
$$=\frac{4}{(b-a)^2}\left[\frac{1}{6}f(b)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right]\int_{\frac{a+b}{2}}^b w(x)\,dx-\frac{2}{(b-a)^2}\int_{\frac{a+b}{2}}^b w(x)f(x)\,dx.$$

and similarly

$$I_2=\int_0^1\left[\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^t w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$
$$=-\frac{2}{b-a}\left[\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^t w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right]f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\Big|_0^1$$
$$-\frac{1}{b-a}\int_0^1 w\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$
$$=-\frac{2}{b-a}\left[-\frac{1}{6}f(a)-\frac{1}{3}f\left(\frac{a+b}{2}\right)\right]\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds$$

$$-\frac{1}{b-a}\int_0^1 w\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$=\frac{4}{(b-a)^2}\left[\frac{1}{6}f(a)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right]\int_a^{\frac{a+b}{2}} w(x)\,dx-\frac{2}{(b-a)^2}\int_a^{\frac{a+b}{2}} w(x)f(x)\,dx.$$

Since $w(x)$ is symmetric to $\frac{a+b}{2}$, we have

$$\int_a^{\frac{a+b}{2}} w(x)\,dx=\int_{\frac{a+b}{2}}^{b} w(x)\,dx=\frac{1}{2}\int_a^b w(x)\,dx.$$

Thus, we can write

$$\frac{b-a}{2}(I_1+I_2)=\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx+\frac{1}{b-a}\int_a^b w(x)f(x)\,dx,$$

which completes the proof. □

Throughout this paper, let $\| w \|_{[a,b],\infty}=\sup_{x\in[a,b]} | w(x) |$, for the continuous function $w:[a,b]\to R$. Now, we are ready to state and prove our results.

Theorem 2.2. *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) and $f' \in L([a,b])$ with $a<b$ and $w:[a,b]\to R$ be continuous and symmetric to $\frac{a+b}{2}$. If $| f' |$ is h-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right|\\&\leqslant (b-a)\| w \|_{[a,b],\infty}\left[| f'(a) |+| f'(b) |\right]\int_0^1 h(t)\,dt.\end{aligned}\tag{2.1}$$

Proof. From Lemma 2.1 and since $| f' |$ is h-convex on $[a,b]$ we have

$$\begin{aligned}&\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right|\\
&\leqslant\frac{b-a}{2}\left\{\int_0^1 \left|\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right|\cdot\left| f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right| dt\right.\\
&\left.\quad+\int_0^1 \left|\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right|\cdot\left| f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right| dt\right\}\\
&\leqslant\frac{b-a}{2}\| w \|_{[a,b],\infty}\left\{\int_0^1 \left|\frac{1}{2}\int_0^t ds-\frac{1}{3}\int_0^1 ds\right|\cdot\left| f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right| dt\right.\end{aligned}$$

$$+\int_0^1 \left| \frac{1}{3}\int_0^1 ds - \frac{1}{2}\int_0^t ds \right| \cdot \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right| dt \Bigg\}$$

$$\leqslant \frac{b-a}{2} \parallel w \parallel_{[a,b],\infty} \left\{ \int_0^1 \left(h\left(\frac{1-t}{2}\right) | f'(a) | + h\left(\frac{1+t}{2}\right) | f'(b) | \right) dt \right.$$

$$\left. + \int_0^1 \left(h\left(\frac{1+t}{2}\right) | f'(a) | + h\left(\frac{1-t}{2}\right) | f'(b) | \right) dt \right\}$$

$$= (b-a) \parallel w \parallel_{[a,b],\infty} \left[| f'(a) | + | f'(b) | \right] \int_0^1 h(t)\, dt,$$

where

$$\parallel w \parallel_{[a,\frac{a+b}{2}],\infty} = \parallel w \parallel_{[\frac{b+a}{2},b],\infty} = \parallel w \parallel_{[a,b],\infty},$$

and

$$\left| \frac{1}{2}\int_0^t ds - \frac{1}{3}\int_0^1 ds \right| = \left| \frac{1}{3}\int_0^1 ds - \frac{1}{2}\int_0^t ds \right| \leqslant \frac{1}{3}$$

for each $t \in [0,1]$. This completes the proof. □

Corollary 2.3. *In Theorem 2.2, if we take* $h(t) = t$*, then inequality (2.1) becomes the following inequality for convex functions:*

$$\left| \frac{1}{6(b-a)} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] \int_a^b w(x)\, dx - \frac{1}{b-a}\int_a^b w(x) f(x)\, dx \right|$$
$$\leqslant \frac{b-a}{2} \parallel w \parallel_{[a,b],\infty} \left[| f'(a) | + | f'(b) | \right].$$

Corollary 2.4. *Suppose* $h(t) = t^s$*,* $s \in (0,1]$ *in Theorem 2.2, we have the following inequality for s-convex functions:*

$$\left| \frac{1}{6(b-a)} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] \int_a^b w(x)\, dx - \frac{1}{b-a}\int_a^b w(x) f(x)\, dx \right|$$
$$\leqslant \frac{b-a}{(s+1)} \parallel w \parallel_{[a,b],\infty} \left[| f'(a) | + | f'(b) | \right].$$

Remark 2.5. If we set $h(t) = t$ in the proof of Theorem 2.2, then using the fact that

$$\int_0^1 \left| \frac{1}{2}\int_0^t ds - \frac{1}{3}\int_0^1 ds \right| dt = \int_0^1 \left| \frac{1}{3}\int_0^1 ds - \frac{1}{2}\int_0^t ds \right| dt = \int_0^1 \left| \frac{1}{2}t - \frac{1}{3} \right| dt = \int_0^1 \left| \frac{1}{3} - \frac{1}{2}t \right| dt = \frac{5}{36},$$

we obtain the following inequality:

$$\left| \frac{1}{6(b-a)} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] \int_a^b w(x)\, dx - \frac{1}{b-a}\int_a^b w(x) f(x)\, dx \right|$$
$$\leqslant \frac{5}{72} \parallel w \parallel_{[a,b],\infty} \left[| f'(a) | + | f'(b) | \right].$$

Theorem 2.6. *Let* $f : [a, b] \to R$ *be a differentiable mapping on* (a, b) *and* $f' \in L([a, b])$ *with* $a < b$ *and* $w : [a, b] \to R$ *be continuous and symmetric to* $\frac{a+b}{2}$. *If* $|f'|^q$ *is* h*-convex on* $[a, b]$ *and* $q > 1$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right| \\
&\leqslant \frac{b-a}{12}\,\|w\|_{[a,b],\infty}\cdot\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\cdot 2^{\frac{1}{q}}\left\{\left(|f'(a)|^q\int_0^{\frac{1}{2}}h(t)\,dt\right.\right. \\
&\left.\left.+|f'(b)|^q\int_{\frac{1}{2}}^1 h(t)\,dt\right)^{\frac{1}{q}}+\left(|f'(a)|^q\int_{\frac{1}{2}}^1 h(t)\,dt+|f'(b)|^q\int_0^{\frac{1}{2}}h(t)\,dt\right)^{\frac{1}{q}}\right\},
\end{aligned} \tag{2.2}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and using the Hölder's integrals inequality and the h-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right| \\
&\leqslant \frac{(b-a)}{2}\left\{\left(\int_0^1\left|\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right|^p dt\right)^{\frac{1}{p}}\right. \\
&\times\left(\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left|\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^t w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right|^p dt\right)^{\frac{1}{p}} \\
&\left.\times\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
&\leqslant \frac{b-a}{2}\,\|w\|_{[a,b],\infty}\cdot\left\{\left(\int_0^1\left|\frac{1}{2}t-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\cdot\left(\int_0^1\left(h\left(\frac{1-t}{2}\right)|f'(a)|^q+h\left(\frac{1+t}{2}\right)|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\frac{1}{2}t-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\cdot\left(\int_0^1\left(h\left(\frac{1+t}{2}\right)|f'(a)|^q+h\left(\frac{1-t}{2}\right)|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right\} \\
&=\frac{b-a}{12}\,\|w\|_{[a,b],\infty}\cdot\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\cdot 2^{\frac{1}{q}}\left\{\left(|f'(a)|^q\int_0^{\frac{1}{2}}h(t)\,dt+|f'(b)|^q\int_{\frac{1}{2}}^1 h(t)\,dt\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(|f'(a)|^q\int_{\frac{1}{2}}^{1}h(t)\,dt+|f'(b)|^q\int_{0}^{\frac{1}{2}}h(t)\,dt\right)\Bigg\},$$

where

$$\int_0^1 \left|\frac{1}{2}t-\frac{1}{3}\right|^p dt=\int_0^1 \left|\frac{1}{3}-\frac{1}{2}t\right|^p dt=\frac{2+2^{p+2}}{(p+1)\cdot 6^{p+1}},$$

which completes the proof. □

Corollary 2.7. *If we set* $h(t)=t$ *in Theorem* 2.6*, we obtain the inequality for convex functions:*

$$\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right|$$
$$\leqslant \frac{b-a}{12}\,\|w\|_{[a,b],\infty}\left(\frac{1+2^{p+1}}{3(p+1)}\right)\left(\frac{1}{4}\right)^{\frac{1}{q}}\left\{\left(|f'(a)|^q+3|f'(b)|^q\right)^{\frac{1}{q}}+\left(3|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right\}.$$

Corollary 2.8. *If we set* $h(t)=t$, $s\in(0,1]$ *in Theorem* 2.6*, we obtain the inequality for s-convex functions:*

$$\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right|$$
$$\leqslant \frac{b-a}{12}\,\|w\|_{[a,b],\infty}\left(\frac{1+2^{p+1}}{3(p+1)}\right)\left(\frac{1}{4}\right)^{\frac{1}{q}}\cdot\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left\{\left(|f'(a)|^q\left(\frac{1}{2}\right)^{s+1}+|f'(b)|^q\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f'(a)|^q\left(1-\left(\frac{1}{2}\right)^{s+1}\right)+|f'(b)|^q\left(\frac{1}{2}\right)^{s+1}\right)^{\frac{1}{q}}\right\}.$$

Remark 2.9. If we set $h(t)=t$ and $w(x)=1$ for each $x\in[a,b]$ in Theorem 2.6, then inequality (2.2) reduces to the inequality (1.1).

Remark 2.10. If we set $h(t)=t^s$, $s\in(0,1]$ and $w(x)=1$ for each $x\in[a,b]$ in Theorem 2.6, then inequality (2.2) reduces to the inequality [10, Eq. (2.9)].

Theorem 2.11. *Let* $f:[a,b]\to R$ *be a differentiable mapping on* (a,b) *and* $f'\in L([a,b])$ *with* $a<b$ *and* $w:[a,b]\to R$ *be continuous and symmetric to* $\frac{a+b}{2}$*. If* $|f'|^q$ *is* h*-convex on* $[a,b]$ *and* $q\geqslant 1$*, then the following inequality holds:*

$$\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right|$$
$$\leqslant \frac{b-a}{12}\,\|w\|_{[a,b],\infty}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\cdot\left(\frac{2}{3}\right)^{\frac{1}{q}}\left\{\left(|f'(a)|^q\int_0^{\frac{1}{2}}h(t)\,dt+|f'(b)|^q\int_{\frac{1}{2}}^{1}h(t)\,dt\right)^{\frac{1}{q}}\right. \tag{2.3}$$
$$\left.+\left(|f'(a)|^q\int_{\frac{1}{2}}^{1}h(t)\,dt+|f'(b)|^q\int_0^{\frac{1}{2}}h(t)\,dt\right)^{\frac{1}{q}}\right\}.$$

Proof. From Lemma 2.1 and the power mean inequality, we have that the following inequality holds:

$$\begin{aligned}
&\left| \frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx \right| \\
&\leqslant \frac{b-a}{2}\left\{\left(\int_0^1 \left|\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right| dt\right)^{\frac{1}{1-q}}\right.\\
&\times\left(\int_0^1 \left|\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right|\left| f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 \left|\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^t w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right| dt\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_0^1 \left|\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^t w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right|\left| f'\left(\frac{1+t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By the h-convexity of $|f'|^q$ and using the facts that

$$\left|\frac{1}{2}\int_0^t ds-\frac{1}{3}\int_0^1 ds\right|=\left|\frac{1}{3}\int_0^1 ds-\frac{1}{2}\int_0^t ds\right|=\left|\frac{1}{2}t-\frac{1}{3}\right|\leqslant\frac{1}{3}$$

for all $t\in[0,1]$ we have

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right|\cdot\left| f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\\
&\qquad\leqslant \| w\|_{[a,b],\infty}\frac{1}{3}\left(|f'(a)|^q\int_0^1 h\left(\frac{1-t}{2}\right)dt+|f'(b)|^q\int_0^1 h\left(\frac{1+t}{2}\right)dt\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^t w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right|\cdot\left| f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\\
&\qquad\leqslant \| w\|_{[a,b],\infty}\frac{1}{3}\left(|f'(a)|^q\int_0^1 h\left(\frac{1+t}{2}\right)dt+|f'(b)|^q\int_0^1 h\left(\frac{1-t}{2}\right)dt\right).
\end{aligned}$$

Using the last two inequalities we obtain

$$\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right|$$

$$\leqslant \frac{b-a}{2}\parallel w\parallel_{[a,b],\infty}\cdot\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\cdot\left(\frac{2}{3}\right)^{\frac{1}{q}}\cdot\left\{\left(|f^{'}(a)|^{q}\int_{0}^{\frac{1}{2}}h(t)\,dt+|f^{'}(b)|^{q}\int_{\frac{1}{2}}^{1}h(t)\,dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f^{'}(a)|^{q}\int_{\frac{1}{2}}^{1}h(t)\,dt+|f^{'}(b)|^{q}\int_{0}^{\frac{1}{2}}h(t)\,dt\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Corollary 2.12. *In Theorem 2.11, if we take* $h(t)=t$*, then inequality (2.3) becomes the following inequality for convex functions:*

$$\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_{a}^{b}w(x)\,dx-\frac{1}{b-a}\int_{a}^{b}w(x)f(x)\,dx\right|$$
$$\leqslant\frac{b-a}{8}\parallel w\parallel_{[a,b],\infty}\cdot\left(\frac{5}{9}\right)^{1-\frac{1}{q}}\cdot\left(\frac{1}{3}\right)^{\frac{1}{q}}\cdot\left\{\left(|f^{'}(a)|^{q}+3|f^{'}(b)|^{q}\right)^{\frac{1}{q}}+\left(3|f^{'}(a)|^{q}+|f^{'}(b)|^{q}\right)^{\frac{1}{q}}\right\}.$$

Corollary 2.13. *Suppose* $h(t)=t^{s}$, $s\in(0,1]$ *in Theorem 2.11, we have the following inequality for s-convex functions:*

$$\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_{a}^{b}w(x)\,dx-\frac{1}{b-a}\int_{a}^{b}w(x)f(x)\,dx\right|$$
$$\leqslant\frac{b-a}{2}\parallel w\parallel_{[a,b],\infty}\cdot\left(\frac{2}{(s+1)3}\right)^{\frac{1}{q}}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\left(\left(\frac{1}{2}\right)^{s+1}|f^{'}(a)|^{q}+\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f^{'}(b)|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(1-\left(\frac{1}{2}\right)^{s+1}\right)|f^{'}(a)|^{q}+\left(\frac{1}{2}\right)^{s+1}|f^{'}(b)|^{q}\right)^{\frac{1}{q}}\right\}.$$

Theorem 2.14. *Let* $f:[a,b]\to R$ *be a differentiable mapping on* (a,b) *and* $f^{'}\in L([a,b])$ *with* $a<b$ *and* $w:[a,b]\to R$ *be continuous and symmetric to* $\frac{a+b}{2}$. *If* $|f^{'}|^{q}$ *is* h*-convex on* $[a,b]$ *and* $q>1$*, then the following inequality holds:*

$$\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_{a}^{b}w(x)\,dx-\frac{1}{b-a}\int_{a}^{b}w(x)f(x)\,dx\right|$$
$$\leqslant\frac{b-a}{12}\parallel w\parallel_{[a,b],\infty}\cdot\left(\frac{1+2^{p+1}}{(p+1)3}\right)^{\frac{1}{p}}\left(\int_{0}^{1}h(t)\,dt\right)^{\frac{1}{q}}\left\{\left(\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{q}+|f^{'}(b)|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f^{'}(a)|^{q}+\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and using the Hölder's integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right|\\&\leqslant \frac{b-a}{2}\left\{\left(\int_0^1 \left|\frac{1}{2}\int_0^t w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds-\frac{1}{3}\int_0^1 w\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)ds\right|^p dt\right)^{\frac{1}{p}}\right.\\&\quad\times\left(\int_0^1 \left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\\&\quad+\left(\int_0^1 \left|\frac{1}{3}\int_0^1 w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds-\frac{1}{2}\int_0^t w\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)ds\right|^p dt\right)^{\frac{1}{p}}\\&\quad\left.\times\left(\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Since $|f'|^q$ is h-convex, by (1.3) we have

$$\int_0^1 \left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt=\frac{2}{b-a}\int_{\frac{a+b}{2}}^b |f'(x)|^q\,dx\leqslant\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]\int_0^1 h(t)\,dt,$$

and

$$\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt=\frac{2}{b-a}\int_a^{\frac{a+b}{2}} |f'(x)|^q\,dx\leqslant\left[|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]\int_0^1 h(t)\,dt.$$

Therefore we obtain

$$\begin{aligned}&\left|\frac{1}{6(b-a)}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b w(x)f(x)\,dx\right|\\&\leqslant\frac{b-a}{12}\,\|w\|_{[a,b],\infty}\cdot\left(\frac{1+2^{p+1}}{(p+1)3}\right)^{\frac{1}{p}}\cdot\left(\int_0^1 h(t)\,dt\right)^{\frac{1}{q}}\cdot\left\{\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. We also note that

$$\int_0^1 \left|\frac{1}{2}t-\frac{1}{3}\right|^p dt=\int_0^1\left|\frac{1}{3}-\frac{1}{2}t\right|^p dt=\frac{2+2^{p+2}}{(p+1)\cdot 6^{p+1}}^{\frac{1}{p}}.$$

This completes the proof. □

Remark 2.15. If we choose $h(t) = t$ or $h(t) = t^s$, $s \in (0, 1]$ in Theorem 2.14, we obtain the inequalities for convex or s-convex functions respectively.

Remark 2.16. If we choose $h(t) = t$, $s \in (0, 1]$ and $w(x) = 1$ for each $x \in [a, b]$, then we obtain inequality (1.2) of Theorem 1.3.

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