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The necessary and sufficient conditions of Hyers-Ulam stability for a class of parabolic equation

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Abstract

The aim of this paper is to consider the Hyers-Ulam stability of a class of parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - a^2\Delta u + b\cdot\nabla u + cu = 0, & (x,t)\in\mathbb{R}^n\times(0,+\infty), \\ u(x,0)=\varphi(x), & x\in\mathbb{R}^n. \end{cases}$$

We conclude that

(i) it is Hyers-Ulam stable on any finite interval;

(ii) if $c \neq 0$, it is Hyers-Ulam stable on the semi-infinite interval;

(iii) if $c = 0$, it is not Hyers-Ulam stable on the semi-infinite interval by using Fourier transformation.

Furthermore, our results can be applied to the mean square Hyers-Ulam stability of parabolic equations driven by an n-dimensional Brownian motion.

Keywords: Hyers-Ulam stability, parabolic equation, stochastic differential equations.
2010 MSC: 35A24, 35A30, 35A35.

1. Introduction

The notion of Hyers-Ulam stability arose for the stability question of functional equations posed on by Ulam in 1940. Hyers answered it in the upcoming years [6, 16]. Theories of the stability for functional equations have received a lot of attention in the past two decades.

In 1993, Obloza introduced the stability to study approximate solutions of differential equations [10, 11]. Since then, there have been many papers dealing with the Hyers-Ulam stability for ordinary differential equations [1–3, 8, 13–15, 17–19] and partial differential equations [4, 9]. Lungu and Popa [9] investigated the sufficient conditions of Hyers-Ulam stability for the following first order linear partial differential equation

$$p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} = p(x,y)r(x)u + f(x,y).$$

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András and Mészáros [4] considered the sufficient conditions of Hyers-Ulam stability for the following elliptic partial differential equation

$$\begin{cases} \Delta u = f(x, u(x)) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Parabolic equations play an important role in physics, chemistry and biology. There have been many studies on the kind of equation. However, to the best knowledge of the authors, there exists no work in the literature which discusses the Hyers-Ulam stability of parabolic equations. Motivated by the above work [4, 9], in this paper, we will consider the Hyers-Ulam stability for the following parabolic equation

$$\frac{\partial u}{\partial t} - a^2 \Delta u + b \cdot \nabla u + cu = 0, \quad (x, t) \in \mathbb{R}^n \times (0, +\infty), \tag{1.1}$$

$$\text{subject to } \ u(x, 0) = \varphi(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $a^2 > 0$, $c \geqslant 0$, b is a given constant vector field in $\mathbb{R}^n$, $\varphi \in C(\mathbb{R}^n)$ is bounded, $x = (x_1, x_2, \cdots, x_n)$, $\Delta u = \sum\limits_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.

Moreover, uncertainty is involved in all kinds of natural phenomena, and stochastic models are more suitable for uncertainty phenomena. Therefore, it is important to generalize the results to perturbation cases. We will also consider the mean square Hyers-Ulam stability of (1.1) subject to (1.2) with a stochastic term.

The novelty of our work is the following:

(i) the necessary and sufficient conditions of Hyers-Ulam stability are considered;

(ii) the stochastic approximate solutions of (1.1) are considered and the mean square errors are studied.

The paper is organized as follows. Some background material is given in Section 2. The Hyers-Ulam stability of (1.1) is considered in Section 3. The mean square Hyers-Ulam stability of (1.1) is considered in Section 4.

2. Preliminary

Throughout the paper, all random variables and processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ adopted to $\mathcal{F}_t, t \geqslant 0$, filtration $\mathcal{F}_t, t \geqslant 0$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. B is a vector of n independent Brownian motions B_i $(i = 1, 2, \cdots, n)$ adopted to $\mathcal{F}_t$ and independent of $\mathcal{F}_0$. For the details of it, we refer the reader to reference [12]. Moreover, we also note that solutions of (1.1) subject to (1.2) denote classical solutions defined in $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$, where $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$ denotes the function space having continuous derivatives up to order two in the space variable x and having a continuous derivative with respect to the time variable t; $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$, in the context.

Now we introduce the fundamental definitions and two lemmas, which are used later in the paper.

Definition 2.1. Let $I \subset [0, \infty)$ be an interval, K is a positive constant. Assume that for any function $u_\varepsilon \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$ with $u_\varepsilon(x, 0) = \varphi(x)$ satisfying the differential inequality

$$\left|\frac{\partial}{\partial t} u - a^2 \Delta u + b \cdot \nabla u + cu\right| \leqslant \varepsilon$$

for all $(x, t) \in \mathbb{R}^n \times I$ and some $\varepsilon > 0$, there exists a solution $u_0 \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$ of (1.1) subject to (1.2) such that $|u_\varepsilon(x, t) - u_0(x, t)| \leqslant K\varepsilon$ for any $(x, t) \in \mathbb{R}^n \times I$. Then we say that (1.1) is Hyers-Ulam stable on $\mathbb{R}^n \times I$, K is the Hyers-Ulam constant.

Definition 2.2 ([12]). An n-dimensional Itô processes or stochastic integral x is a vector of n stochastic process x_i $(i = 1, 2, \cdots, n)$ on $(\Omega, \mathcal{F}, P)$ adopted to $\mathcal{F}_t, t \geqslant 0$ which can be written in the form

$$x_i(t) = x_i(0) + \int_0^t U_i(s)ds + V_i(s)dB_i(s), \quad i = 1, 2, \cdots, n, \tag{2.1}$$

where $U_i, V_i \in \mathcal{L}_2$, note also that $\mathcal{L}_2$ is a space of stochastic processes defined by

$$\mathcal{L}_2 = \left\{ u \Big| \int_0^T u(t)dt \text{ exists a.s.}, \mathbb{E}[\int_0^T u^2(t)dt] < \infty, \text{ for every } T > 0 \right\}.$$

As a shorthand notation, we will write (2.1) as

$$dx(t) = U(t)dt + V(t)dB(t),$$

or

$$\frac{dx}{dt} = U + V\frac{dB}{dt},$$

where $x = (x_1, x_2, \cdots, x_n)$, $U = (U_1, U_2, \cdots, U_n)$, $V = (V_1, V_2, \cdots, V_n)$, $B = (B_1, B_2, \cdots, B_n)$.

Definition 2.3. Let $I \subset [0, \infty)$ be a real interval, K is a positive constant, $\sigma : \mathbb{R}^n \times I \to \mathbb{R}^n$,

$$\frac{\partial u}{\partial t} - a^2 \Delta u + b \cdot \nabla u + cu = \sigma \cdot \frac{dB}{dt}.$$

Assume that for any stochastic process $u_\varepsilon(x, \cdot) \in \mathcal{L}_2$ with $u(x, 0) = \varphi(x)$ satisfying the inequality

$$|\sigma(x, t)| \leqslant \varepsilon$$

for all $(x, t) \in \mathbb{R}^n \times I$ and some $\varepsilon > 0$, there exists a solution $u_0 \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$ of (1.1) subject to (1.2) satisfying $\mathbb{E}|u_\varepsilon(t, x) - u_0(t, x)|^2 \leqslant K^2\varepsilon$ for any $(x, t) \in \mathbb{R}^n \times I$. Then we say that (1.1) is mean square Hyers-Ulam stable on $\mathbb{R}^n \times I$, K is the mean square Hyers-Ulam constant.

We will use the Itô formula and Itô isometry [12] in the paper as follows.

Lemma 2.4. *Let y, z be two one-dimensional Itô processes, then yz is also a one-dimensional Itô process and*

$$dyz = ydz + zdy + dydz, \tag{2.2}$$

where $dydz$ is computed using the rules $dtdt = dtdB_i(t) = dB_i(t)dt = 0$, $dB_i(t)dB_j(t) = 0$ for all $i \neq j$ and $(dB_i(t))^2 = dt$, $i, j = 1, 2, \cdots, n$.

Lemma 2.5. *Let $y \in \mathcal{L}_2$, then*

$$\mathbb{E}\Big| \int_0^T y(t)dB_i(t)\Big|^2 = \mathbb{E}\Big| \int_0^T y^2(t)dt\Big|, \tag{2.3}$$

where $\mathbb{E}$ denotes expectation, $i = 1, 2, \cdots, n$.

3. Hyers-Ulam stability of (1.1)

In the section, we first consider the Hyers-Ulam stability of (1.1) by using Fourier Transformation. The Fourier transform of u with respect to $x \in \mathbb{R}^n$ is denoted by

$$F\{u(x, t)\} = \hat{u}(\xi, t),$$

and is defined by

$$F\{u(x,t)\} = \hat{u}(\xi,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-i\xi \cdot x} u(x,t)dx, \tag{3.1}$$

provided the integral

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-i\xi \cdot x} u(x,t)dx,$$

exists, where $\xi = (\xi_1, \xi_2 \cdots \xi_n)$ is the n-dimensional transform vector and $\xi \cdot x = \sum_{i=1}^{n} \xi_i x_i$. The inverse Fourier transform, denoted by $F^{-1}\{\hat{u}(\xi,t)\} = u(x,t)$ and is defined by

$$F^{-1}\{\hat{u}(\xi,t)\} = u(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{i\xi \cdot x} \hat{u}(\xi,t)d\xi, \tag{3.2}$$

provided the integral

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{i\xi \cdot x} \hat{u}(\xi,t)d\xi,$$

exists. For functions in bounded solution class of (1.1) subject to (1.2), the Fourier transform always exists. For more details on the Fourier transform formula, we refer the reader to reference [5].

Now we introduce the function

$$\epsilon := \frac{\partial u}{\partial t} - a^2 \Delta u + b \cdot \nabla u + cu. \tag{3.3}$$

Theorem 3.1. *Assume that $\beta > 0$, $\beta = +\infty$ is allowed, if and only if $c \neq 0$, $\varphi \in C(\mathbb{R}^n)$ is bounded. For every $u_\varepsilon \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$ satisfying*

$$\begin{cases} |\epsilon(t,x)| \leqslant \varepsilon, & (x,t) \in \mathbb{R}^n \times [0,\beta], \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.4}$$

there exists a solution $u_0 \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$ of (1.1) subject to (1.2) satisfying

$$|u_\varepsilon(x,t) - u_0(x,t)| \leqslant \frac{1}{c}\varepsilon, \text{ for every } (x,t) \in \mathbb{R}^n \times [0,\beta], \text{ when } c \neq 0,$$

$$|u_\varepsilon(x,t) - u_0(x,t)| \leqslant \beta\varepsilon, \text{ for every } (x,t) \in \mathbb{R}^n \times [0,\beta], \text{ when } c = 0.$$

That is, (1.1) is Hyers-Ulam stable on $\mathbb{R}^n \times [0,\beta]$ and $\frac{1}{c}$ is the Hyers-Ulam constant, when $c \neq 0$; β is the Hyers-Ulam constant, when $c = 0$.

Proof. First, we would like to compute the classic solution of (1.1) subject to (1.2). Apply the Fourier transform with respect to the space variable x defined by (3.1) to (1.1) and (1.2). By

$$F\{\Delta u\} = (i|\xi|)^2 \hat{u}, \ F\{\frac{\partial u}{\partial x}\} = i\xi\hat{u},$$

(1.1) and (1.2) reduce to

$$\begin{cases} \frac{d}{dt}\hat{u}(\xi,t) + (a^2|\xi|^2 + ib \cdot \xi + c)\hat{u}(\xi,t) = 0, \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi). \end{cases} \tag{3.5}$$

Thus, the solution of this transformed problem (3.5) is

$$\hat{u}_0 = \hat{\varphi}(\xi) \exp(-(a^2|\xi|^2 + ib \cdot \xi + c)t). \tag{3.6}$$

Take the inverse Fourier transform defined by (3.2) to (3.6)

$$u_0 = F^{-1}\{\hat{u}(\xi,t)\} = e^{-ct} F^{-1}\{\hat{\varphi}(\xi) \exp(-(a^2|\xi|^2 + ib \cdot \xi)t)\}. \tag{3.7}$$

Applying the Convolution formula to (3.7) implies

$$F^{-1}\{\hat{\varphi}(\xi)\exp(-(a^2|\xi|^2+ib\cdot\xi)t)\}=\frac{1}{(2a\sqrt{\pi t})^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{|x+b-\xi|^2}{4a^2t}}d\xi.$$

So that,

$$u_0=\frac{e^{-ct}}{(2a\sqrt{\pi t})^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{|x+b-\xi|^2}{4a^2t}}d\xi. \tag{3.8}$$

Since $\varphi\in C(\mathbb{R}^n)$ is bounded, $\frac{\partial u_0}{\partial t}$, Δu_0 are uniform convergence with respect to x,t. This implies u_0, $\frac{\partial u_0}{\partial t},\Delta u_0\in C(\mathbb{R}^n\times\mathbb{R}^+)$, $\lim\limits_{t\to 0^+}u_0(x,t)=\varphi(x)$ and u_0 is bounded. So that $u_0\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}^+)$ is the unique bounded solution of (1.1) subject to (1.2). For the details about proof, we refer the reader to reference [7].

Next, we would like to show the Hyers-Ulam stability of (1.1) subject to (1.2). Take the Fourier transformation of (3.3) subject to (1.2) with respect to the time variable x, it is transformed to

$$\begin{cases}\frac{d}{dt}\hat{u}(\xi,t)+(a^2|\xi|^2+ib\cdot\xi+c)\hat{u}(\xi,t)=\hat{\epsilon}(\xi,t),\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi).\end{cases} \tag{3.9}$$

The method of variation of constant gives the unique solution of (3.9)

$$\hat{u}_\varepsilon(\xi,t)=\hat{\varphi}(\xi)\exp(-(a^2|\xi|^2+ib\cdot\xi+c)t)+\int_0^t\hat{\epsilon}(\xi,\tau)\exp(-(a^2|\xi|^2+ib\cdot\xi+c)(t-\tau))d\tau. \tag{3.10}$$

Take the inverse Fourier transform of (3.10) with respect to the variable ξ. Similarly to the proof of (3.8), we can get

$$\begin{aligned}u_\varepsilon(x,t)=&\frac{e^{-ct}}{(2a\sqrt{\pi t})^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{|x+b-\xi|^2}{4a^2t}}d\xi\\&+\int_0^t\frac{e^{-c(t-\tau)}}{(2a\sqrt{\pi(t-\tau)})^2}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\epsilon(\xi,\tau)e^{-\frac{|x+bt-\xi|^2}{4a^2(t-\tau)}}d\xi d\tau.\end{aligned} \tag{3.11}$$

It follows from (3.11) that if $c\neq 0$, then

$$\begin{aligned}|u_\varepsilon(x,t)-u_0(x,t)|&=\left|\int_0^t\frac{e^{-c(t-\tau)}}{(2a\sqrt{\pi(t-\tau)})^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\epsilon(\xi,\tau)e^{-\frac{|x+b-\xi|^2}{4a^2(t-\tau)}}d\xi d\tau\right|\\&\leqslant\varepsilon\left|\int_0^t e^{-c(t-\tau)}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{1}{(2a\sqrt{\pi(t-\tau)})^n}e^{-\frac{|x+bt-\xi|^2}{4a^2(t-\tau)}}d\xi d\tau\right|\\&\leqslant\varepsilon\int_0^t e^{-c(t-\tau)}d\tau=\frac{1}{c}\varepsilon.\end{aligned}$$

If $c=0$, then $|u_\varepsilon(x,t)-u_0(x,t)|\leqslant\beta\varepsilon$. This means that, (1.1) has the Hyers-Ulam stability on $\mathbb{R}^n\times[0,\beta]$, and $\frac{1}{c}$ is the Hyers-Ulam constant, when $c\neq 0$; β is the Hyers-Ulam constant, when $c=0$. The proof is completed. □

Corollary 3.2. *If $c\neq 0$, (1.1) is Hyers-Ulam stable on $\mathbb{R}^n\times[0,+\infty)$ and $\frac{1}{c}$ is the Hyers-Ulam constant.*

Proof. Similarly to the proof of Theorem 3.1, if u_ε satisfies (3.4) on $\mathbb{R}^n\times[0,+\infty)$, then

$$|u_\varepsilon(x,t)-u_0(x,t)|\leqslant\varepsilon\int_0^t e^{-c(t-\tau)}d\tau\leqslant\varepsilon\int_0^{+\infty}e^{-c(t-\tau)}d\tau=\frac{\varepsilon}{c},\ (x,t)\in\mathbb{R}^n\times[0,+\infty).$$

This means that (1.1) has the Hyers-Ulam stability on $\mathbb{R}^n\times[0,\infty)$. $\frac{1}{c}$ is the Hyers-Ulam constant. The proof is completed. □

Corollary 3.3. *If $c = 0$, then (1.1) is not Hyers-Ulam stable on $\mathbb{R}^n \times [0, +\infty)$.*

Proof. Let $\varphi(x) = \exp(\frac{b\cdot x}{a^2})$, $u_0(x,t) = \varphi(x)$, $(x,t) \in \mathbb{R}^n \times [0, +\infty)$, then u_0 is the solution of (1.1) subject to (1.2). Furthermore, let $u_\varepsilon(x,t) = \varphi(x) + \frac{1}{2}\varepsilon t$, $(x,t) \in \mathbb{R}^n \times [0, +\infty)$ then $u_\varepsilon(x, 0) = \varphi(x)$ and u_ε satisfies the differential inequality

$$\left|\frac{\partial}{\partial t}u - a^2\Delta u + b\cdot\nabla u\right| = \frac{1}{2}\varepsilon < \varepsilon$$

for all $(x,t) \in \mathbb{R}^n \times [0, +\infty)$, and

$$|u_\varepsilon(x,t) - u_0(x,t)| = \frac{1}{2}\varepsilon t,$$

is unbounded on $[0, \infty)$. This implies that (1.1) is not Hyers-Ulam stable on $\mathbb{R}^n \times [0, \infty)$. The proof is completed. □

4. Mean square Hyers-Ulam stability of (1.1)

Theorem 4.1. *Assume that $\beta > 0$, $\beta = +\infty$ is allowed, if and only if $c \neq 0$, $\varphi \in C(\mathbb{R}^n)$ is bounded. For every stochastic process $u_\varepsilon(x, \cdot) \in \mathcal{L}_2$ satisfying*

$$\begin{cases} \frac{\partial}{\partial t}u - a^2\Delta u + b\cdot\nabla u + cu = \delta\cdot\frac{dB}{dt}, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $\delta : \mathbb{R}^n \times I \to \mathbb{R}^n$, $|\delta(x,t)| \leqslant \varepsilon$ for $(x,t) \in \mathbb{R}^n \times [0, \beta]$, there exists a solution $u_0 \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$ of (1.1) subject to $u_0(x, 0) = \varphi(x)$ with the property

$$\mathbb{E}|u_\varepsilon(x,t) - u_0(x,t)|^2 \leqslant \frac{1}{c}\varepsilon^2, \textit{ for every } (x,t) \in \mathbb{R}^n \times [0,\beta], \textit{ when } c \neq 0,$$

$$\mathbb{E}|u_\varepsilon(x,t) - u_0(x,t)|^2 \leqslant \beta\varepsilon^2, \textit{ for every } (x,t) \in \mathbb{R}^n \times [0,\beta], \textit{ when } c = 0.$$

That is, (1.1) is mean square Hyers-Ulam stable on $\mathbb{R}^n \times [0, \beta]$ and $\frac{1}{\sqrt{c}}$ is the mean square Hyers-Ulam constant, when $c \neq 0$; $\sqrt{\beta}$ is the mean square Hyers-Ulam constant, when $c = 0$.

Proof. Take the Fourier transformation of (4.1) with respect to the variable x. Similarly to the proof of Theorem 3.1, we can get

$$\begin{cases} \frac{d}{dt}\hat{u}(\xi,t) + (a^2|\xi|^2 + ib\cdot\xi + c)\hat{u}(\xi,t) = \hat{\delta}(\xi,t)\cdot\frac{dB(t)}{dt}, \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi). \end{cases} \tag{4.2}$$

Multiply the first equation of (4.2) by $\exp((a^2|\xi|^2 + ib\cdot\xi + c)t)$ and change it to

$$\begin{aligned} &\exp((a^2|\xi|^2 + ib\cdot\xi + c)t)d\hat{u}(\xi,t) + (a^2|\xi|^2 + ib\cdot\xi + c)\exp((a^2|\xi|^2 + ib\cdot\xi + c)t)\hat{u}(\xi,t)dt \\ &\quad = \exp((a^2|\xi|^2 + ib\cdot\xi + c)t)\hat{\delta}(\xi,t)\cdot dB(t). \end{aligned}$$

Using Itô formula (2.2), we have

$$\begin{aligned} d[\exp((a^2|\xi|^2 + ib\cdot\xi + c)t)\hat{u}(\xi,t)] &= \hat{u}(\xi,t)\,d\exp((a^2|\xi|^2 + ib\cdot\xi + c)t) \\ &\quad + \exp((a^2|\xi|^2 + ib\cdot\xi + c)t)d\hat{u}(\xi,t) \\ &\quad + d\exp((a^2|\xi|^2 + ib\cdot\xi + c)t)d\hat{u}(\xi,t) \\ &= (a^2|\xi|^2 + ib\cdot\xi + c)\exp((a^2|\xi|^2 + ib\cdot\xi + c)t)\hat{u}(\xi,t)dt \\ &\quad + \exp((a^2|\xi|^2 + ib\cdot\xi + c)t)d\hat{u}(\xi,t) \\ &\quad + [(a^2|\xi|^2 + ib\cdot\xi + c)\exp((a^2|\xi|^2 + ib\cdot\xi + c)t)dt] \\ &\quad \times [-(a^2|\xi|^2 + ib\cdot\xi + c)\hat{u}(\xi,t)dt + \hat{\delta}(\xi,t)\cdot dB(t)] \\ &= \exp((a^2|\xi|^2 + ib\cdot\xi + c)t)d\hat{u}(\xi,t) + (a^2|\xi|^2 \\ &\quad + ib\cdot\xi + c)\exp((a^2\xi^2 + ib\cdot\xi + c)t)\hat{u}(\xi,t)dt. \end{aligned}$$

This implies that (4.2) can be transformed to

$$\begin{cases} d[\exp((a^2|\xi|^2 + ib\cdot\xi + c)t)\hat{u}(\xi,t)] = \exp((a^2|\xi|^2 + ib\cdot\xi + c)t)\hat{\delta}(\xi,t)\cdot dB(t), \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi). \end{cases} \tag{4.3}$$

Thus the solution of (4.3) is

$$\hat{u}_\varepsilon(\xi,t) = \hat{\varphi}(\xi)\exp(-(a^2|\xi|^2 + ib\cdot\xi + c)t) + \int_0^t \hat{\epsilon}(\xi,\tau)\exp(-(a^2|\xi|^2 + ib\cdot\xi + c)(t-\tau))\cdot dB(\tau). \tag{4.4}$$

Take the inverse Fourier transform of (4.4) with respect to the variable ξ. Similarly to the proof of Theorem 3.1, we can get

$$\begin{aligned} u_\varepsilon(x,t) = &\frac{e^{-ct}}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{|x+bt-\xi|^2}{4a^2t}}d\xi \\ &+\int_0^t \frac{e^{-c(t-\tau)}}{2a\sqrt{\pi(t-\tau)}}\left[\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\epsilon(\xi,\tau)e^{-\frac{|x+bt-\xi|^2}{4a^2(t-\tau)}}d\xi\right]\cdot dB(\tau). \end{aligned}$$

By Itô Isometry (2.3), if $c \neq 0$, we have

$$\begin{aligned} \mathbb{E}|u_\varepsilon(x,t) - u_0(x,t)|^2 &= \mathbb{E}\left|\int_0^t \frac{e^{-c(t-\tau)}}{2a\sqrt{\pi(t-\tau)}}\left[\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\epsilon(\xi,\tau)e^{-\frac{|x+bt-\xi|^2}{4a^2(t-\tau)}}d\xi\right]\cdot dB(\tau)\right|^2 \\ &\leqslant 2\varepsilon^2\int_0^t e^{-2c(t-\tau)}\left|\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{(x+bt-\xi)^2}{4a^2(t-\tau)}}d\xi\right|^2 d\tau \\ &\leqslant 2\varepsilon^2\int_0^t e^{-2c(t-\tau)}d\tau \\ &= \frac{1}{c}\varepsilon^2, \end{aligned}$$

if $c = 0$, we have $\mathbb{E}|u(x,t) - u_0(x,t)|^2 \leqslant \beta\varepsilon^2$. That is, (1.1) is mean square Hyers-Ulam stable on $\mathbb{R}^n \times [0,\beta]$ and $\frac{1}{\sqrt{c}}$ is the mean square Hyers-Ulam constant, when $c \neq 0$; $\sqrt{\beta}$ is the mean square Hyers-Ulam constant, when $c = 0$. The proof is completed. □

Corollary 4.2. *If $c \neq 0$, (1.1) is mean square Hyers-Ulam stable on $\mathbb{R}^n \times [0,+\infty)$ and $\frac{1}{\sqrt{c}}$ is the Hyers-Ulam constant.*

Proof. Similarly to the proof of Theorem 4.1, if u_ε satisfies (4.2) on $\mathbb{R}^n \times [0,+\infty)$, we have

$$\mathbb{E}|u_\varepsilon(x,t) - u_0(x,t)|^2 \leqslant 2\varepsilon^2\int_0^t e^{-2c(t-\tau)}d\tau = \frac{\varepsilon^2}{c},\ (x,t)\in\mathbb{R}^n\times[0,+\infty).$$

This means that (1.1) is Hyers-Ulam stable on $\mathbb{R}^n \times [0,+\infty)$ and $\frac{1}{\sqrt{c}}$ is the Hyers-Ulam constant. The proof is completed. □

Corollary 4.3. *If $c = 0$, then (1.1) is not mean square Hyers-Ulam stable on $\mathbb{R}^n \times [0,+\infty)$.*

Proof. Let $\varphi(x) = \exp(\frac{b\cdot x}{a^2})$, $u_0(x,t) = \varphi(x)$, $(x,t) \in \mathbb{R}^n \times [0,+\infty)$, then u_0 is the solution of (1.1) subject to (1.2). Furthermore, let $u_\varepsilon(x,t) = \varphi(x) + \frac{1}{2}\gamma\cdot B(t)$, $(x,t) \in \mathbb{R}^n \times [0,+\infty)$, where γ is a given constant vector field in $\mathbb{R}^n$ and $|\gamma| = \varepsilon$, then $u_\varepsilon(x,0) = \varphi(x)$ and u_ε satisfies the following

$$\frac{\partial}{\partial t}u - a^2\Delta u + b\cdot\nabla u = \frac{1}{2}\gamma\cdot\frac{dB}{dt},$$

on $\mathbb{R}^n \times [0,+\infty)$, and

$$\mathbb{E}|u_\varepsilon(x,t) - u_0(x,t)|^2 = \frac{1}{4}\varepsilon^2 t^2,$$

is unbounded on $\mathbb{R}^n \times [0,\infty)$. This implies that (1.1) is not Hyers-Ulam stable on $\mathbb{R}^n \times [0,\infty)$. The proof is completed. □

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