Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 5803–5812
Research Article

ISSN : 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Approximation with modified Phillips operators

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Communicated by D. Baleanu

Abstract

In the present paper, we study modified Phillips operators in simultaneous approximation. The operators discussed here are important as they have link with the well-known Szász operators. We estimate some direct results for the operators.

Keywords: Szász operators, Phillips operators, simultaneous approximation, modulus of continuity, moment generating function, asymptotic formula, error estimation.
2010 MSC: 41A25, 41A30.

1. Introduction

In order to generalize the well-known Bernstein polynomials to the positive real axis Szász [16] introduced the following operators

$$S_\alpha(f,x) = \sum_{k=0}^{\infty} e^{-\alpha x} \frac{(\alpha x)^k}{k!} f\left(\frac{k}{\alpha}\right), x \in [0,\infty).$$

These operators are linear positive operators and play an important role in the theory of approximation. Recently, Gupta [6] discussed some approximation properties of the operators $S_\alpha(f,x)$. Four years later Phillips [15] proposed a generalization of the Szász operator in the following form:

$$L_\alpha(f,x) = e^{-\alpha x} f(0) + \alpha \sum_{k=1}^{\infty} s_{\alpha,k}(x) \int_0^\infty s_{\alpha,k-1}(t) f(t)dt, x \in [0,\infty),$$

where $s_{\alpha,k}(x) = e^{-\alpha x}\dfrac{(\alpha x)^k}{k!}$. Later Mazhar and Totik [12], Finta and Gupta [1], Gupta and Srivastava [8], Govil et al. [4], Heilmann and Tachev [10], Gupta [5], Tachev [17], etc. discussed several approximation properties of the operators $L_\alpha(f,x)$. Recently, in order to generalize the Phillips operators, based on the parameter $\rho > 0$, Păltănea in [13] proposed the following operators

$$L_\alpha^\rho(f,x) = \int_0^\infty k_\alpha^\rho(x,t) f(t)dt = e^{-\alpha x} f(0) + \sum_{k=1}^{\infty} s_{\alpha,k}(x) \int_0^\infty \theta_{\alpha,k}^\rho(t) f(t)dt, x \in [0,\infty), \tag{1.1}$$

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doi:10.22436/jnsa.010.11.18

Received 2017-05-31

where with $\delta(t)$ being Dirac delta function the kernel is given as

$$k_\alpha^\rho(x,t) = \sum_{k=1}^{\infty} s_{\alpha,k}(x)\theta_{\alpha,k}^\rho(t) + \delta(t)e^{-\alpha x}$$

and the basis functions are defined as

$$s_{\alpha,k}(x) = e^{-\alpha x}\frac{(\alpha x)^k}{k!}, \theta_{\alpha,k}^\rho(t) = \frac{\alpha\rho}{\Gamma(k\rho)}e^{-\alpha\rho t}(\alpha\rho t)^{k\rho-1},$$

and $f : I \to \mathbb{R}$, where $I = [0,\infty)$ is integrable function for which the above series and integrals are convergent.

Let us denote

$$W = \{f : I \to \mathbb{R}, f \text{ is integrable and there exist } M > 0, q \geqslant 0 : |f(t)| \leqslant Me^{qt}, t \geqslant 0\}.$$

It was observed by Păltănea [13] that for any $\alpha > 0, f \in W$ and any $b > 0$, there is $\rho_0 > 0$ such that $L_\alpha^\rho(f,x)$ exists for all $\rho > \rho_0$ and

$$\lim_{\rho\to\infty} L_\alpha^\rho(f,x) = S_\alpha(f,x)$$

uniformly for $x \in [0,b]$. The operators (1.1) preserve constant as well as linear functions. As a special case if $\rho = 1$ these operators reduce to the well-known Phillips operators. Also, it was proved in [13] that the operators $L_\alpha^\rho(f,x)$ preserve convexity of higher order and have the property of simultaneous approximation on compact sets. Very recently Gal and Gupta [2] established some results in complex domain of these operators.

The present article is the extension of the previous work of [13]. Here the aim of the present paper is to study some direct results in simultaneous approximation, which include asymptotic formula and an error estimate in terms of the modulus of continuity.

2. Basic results

In the sequel, we need the following basic lemmas:

Lemma 2.1 ([1]). *For $m \in \mathbb{N}^0$ if we define*

$$\mu_{\alpha,m}(x) = \sum_{k=0}^{\infty} s_{\alpha,k}(x)\left(\frac{k}{\alpha} - x\right)^m,$$

then there holds the recurrence relation

$$\mu_{\alpha,m+1}(x) = x[\mu'_{\alpha,m}(x) + m\mu_{\alpha,m-1}(x)].$$

Further $\mu_{\alpha,m}(x) = O(\alpha^{-[(m+1)/2]})$, where $[s]$ denotes the integral part of s.

Lemma 2.2 ([14]). *For fixed $\alpha > 0$ and $\rho > 0$, if we denote $T_{\alpha,m}^\rho(x) = L_\alpha^\rho(e_m,x), e_m(t) = t^m, m = 1,2,\cdots$ for $m \in \mathbb{N}_0$ and $x \geqslant 0$, then*

$$T_{\alpha,m}^\rho(x) = \left(x + \frac{m-1}{\alpha\rho}\right)T_{\alpha,m-1}^\rho(x) + \frac{x}{\alpha}[T_{\alpha,m-1}^\rho(x)]'.$$

Further,

$$T_{\alpha,m}^\rho(x) = x^m + \frac{(\rho+1)}{2\alpha\rho}\cdot m(m-1)x^{m-1} + \frac{(\rho+1)}{24(\alpha\rho)^2}\cdot m(m-1)(m-2)[(3m-5)\rho + 3m - 1]x^{m-2} + \cdots.$$

Remark 2.3. We may note here that the moment generating function of the operators (1.1) is given by $L_\alpha^\rho(e^{At},x)$ and we can find the moments in alternate form as

$$
\begin{aligned}
L_\alpha^\rho(e^{At},x) &= \exp\left(\frac{\alpha x[(\alpha\rho)^\rho-(\alpha\rho-A)^\rho]}{(\alpha\rho\rho-A)^\rho}\right)\\
&= 1+xA+\frac{1}{2}\left(\frac{(1+\rho)x}{\alpha\rho}+x^2\right)A^2+\frac{\left(2x+3\rho x+\rho^2x+3\alpha\rho x^2+3\alpha\rho^2x^2+\alpha^2\rho^2x^3\right)A^3}{6\alpha^2\rho^2}\\
&\quad+\frac{\left(6x+11\rho x+6\rho^2x+\rho^3x+11\alpha\rho x^2+18\alpha\rho^2x^2+7\alpha\rho^3x^2+6\alpha^2\rho^2x^3+6\alpha^2\rho^3x^3+\alpha^3\rho^3x^4\right)A^4}{24\alpha^3\rho^3}\\
&\quad+O(A^5).
\end{aligned}
$$

From the above expansion, we observe that the m-th order moment is given by

$$L_\alpha^\rho(e_m,x)=\left[\frac{\partial^m}{\partial A^m}L_\alpha^\rho(e^{At},x)\right]_{A=0}.$$

Lemma 2.4. *For fixed $\alpha>0$ and $\rho>0$, if we denote $U_{\alpha,m}^\rho(x)=L_\alpha^\rho((t-x)^m,x)$ for $m\in\mathbb{N}$ and $x\geqslant 0$, then*

$$\alpha U_{\alpha,m+1}^\rho(x)=x\left[[U_{\alpha,m}^\rho(x)]'+\frac{m(1+\rho)}{\rho}U_{\alpha,m-1}^\rho(x)\right]+\frac{m}{\rho}U_{\alpha,m}^\rho(x).$$

Proof. Using $xs'_{\alpha,k}(x)=(k-\alpha x)s_{\alpha,k}(x)$ and $(t\theta_{\alpha,k}^\rho(t))'=\rho(k-\alpha t)\theta_{\alpha,k}^\rho(t)$, we have

$$
\begin{aligned}
x[U_{\alpha,m}^\rho(x)]' &= \sum_{k=1}^{\infty}xs'_{\alpha,k}(x)\int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^m dt+\alpha e^{-\alpha x}(-x)^{m+1}+m(-x)^m e^{-\alpha x}\\
&\quad -mx\sum_{k=1}^{\infty}s_{\alpha,k}(x)\int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^{m-1}dt\\
&= \sum_{k=1}^{\infty}(k-\alpha x)s_{\alpha,k}(x)\int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^m dt+\alpha e^{-\alpha x}(-x)^{m+1}-mxU_{\alpha,m-1}^\rho(x)\\
&= \sum_{k=1}^{\infty}s_{\alpha,k}(x)\int_0^\infty [(k-\alpha t)+\alpha(t-x)]\theta_{\alpha,k}^\rho(t)(t-x)^m dt\\
&\quad +\alpha e^{-\alpha x}(-x)^{m+1}-mxU_{\alpha,m-1}^\rho(x)\\
&= \frac{1}{\rho}\sum_{k=1}^{\infty}s_{\alpha,k}(x)\int_0^\infty [t\theta_{\alpha,k}^\rho(t)]'(t-x)^m dt+\alpha U_{\alpha,m+1}^\rho(x)-mxU_{\alpha,m-1}^\rho(x).
\end{aligned}
$$

Integrating by parts the last integral, we have

$$
\begin{aligned}
x[U_{\alpha,m}^\rho(x)]' &= -\frac{m}{\rho}\sum_{k=1}^{\infty}s_{\alpha,k}(x)\int_0^\infty t\theta_{\alpha,k}^\rho(t)(t-x)^{m-1}dt+\alpha U_{\alpha,m+1}^\rho(x)-mxU_{\alpha,m-1}^\rho(x)\\
&= -\frac{m}{\rho}\left[U_{\alpha,m}^\rho(x)-(-x)^m e^{-\alpha x}\right]\\
&\quad +-\frac{m}{\rho}\left[xU_{\alpha,m-1}^\rho(x)-x(-x)^{m-1}e^{-\alpha x}\right]-mxU_{\alpha,m-1}^\rho(x)+\alpha U_{\alpha,m+1}^\rho(x).
\end{aligned}
$$

Thus, we get

$$\alpha U_{\alpha,m+1}^\rho(x)=x\left[[U_{\alpha,m}^\rho(x)]'+\frac{m(1+\rho)}{\rho}U_{\alpha,m-1}^\rho(x)\right]+\frac{m}{\rho}U_{\alpha,m}^\rho(x).\qquad\square$$

Remark 2.5. From Lemma 2.4, it can easily be seen that

$$U_{\alpha,0}^\rho(x)=1,\quad U_{\alpha,1}^\rho(x)=0,\quad U_{\alpha,2}^\rho(x)=\frac{(\rho+1)x}{\alpha\rho},\quad U_{\alpha,3}^\rho(x)=\frac{(\rho+1)(\rho+2)x}{\alpha^2\rho^2},$$

and

$$U^{\rho}_{\alpha,4}(x) = \frac{3(\rho+1)^2x^2}{\alpha^2\rho^2} + \frac{(\rho+1)(\rho+2)(\rho+3)x}{\alpha^3\rho^3}.$$

Lemma 2.6. *There exist the polynomials $q_{i,j,r}(x)$ independent of α and k such that*

$$x^r\frac{d^r}{dx^r}s_{\alpha,k}(x) = \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha^i(k-\alpha x)^j q_{i,j,r}(x)s_{\alpha,k}(x).$$

3. Simultaneous approximation

In this section we establish a Voronovskaja type asymptotic formula and error estimation in simultaneous approximation.

For $\gamma > 0$, we denote the class of functions as

$$C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leqslant Mt^\gamma, M > 0\}.$$

The norm-$\|\cdot\|_\gamma$ on this class of functions is defined as

$$\|f\|_\gamma = \sup_{x\in[0,\infty)} |f(t)|t^{-\gamma}.$$

Theorem 3.1. *Let $f \in C_\gamma[0,\infty)$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then for $\gamma > r+2$, we have*

$$\lim_{\alpha\to\infty} \alpha\left(\left(\frac{d^r}{dw^r}L^{\rho}_{\alpha}(f,w)\right)_{w=x} - f^{(r)}(x)\right) = \frac{(\rho+1)r}{2\rho}f^{(r+1)}(x) + \frac{(\rho+1)x}{2\rho}f^{(r+2)}(x). \tag{3.1}$$

Further, if $f^{(r+2)}$ is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then the limit in (3.1) holds uniformly in $[a,b]$.

Proof. From the Taylor's theorem, we may write

$$f(t) = \sum_{\nu=0}^{r+2}\frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu + \psi(t,x)(t-x)^{r+2}, \quad t \in [0,\infty), \tag{3.2}$$

where the function $\psi(t,x) \to 0$ as $t \to x$. From equation (3.2), we obtain

$$\begin{aligned}\left(\frac{d^r}{dw^r}L^{\rho}_{\alpha}(f(t),w)\right)_{w=x} &= \sum_{\nu=0}^{r+2}\frac{f^{(\nu)}(x)}{\nu!}\left(\frac{d^r}{dw^r}L^{\rho}_{\alpha}((t-x)^\nu,w)\right)_{w=x}\\ &\quad + \left(\frac{d^r}{dw^r}L^{\rho}_{\alpha}(\psi(t,x)(t-x)^{r+2},w)\right)_{w=x}\\ &= \sum_{\nu=0}^{r+2}\frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{dw^r}L^{\rho}_{\alpha}(t^j,w)\right)_{w=x}\\ &\quad + \left(\frac{d^r}{dw^r}L^{\rho}_{\alpha}(\psi(t,x)(t-x)^{r+2},w)\right)_{w=x}\\ &:= I_1 + I_2, \quad \text{(say)}.\end{aligned}$$

Now, we estimate I_1.

$$I_1 = \sum_{\nu=0}^{r-1}\frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{dw^r}L^{\rho}_{\alpha}(t^j,w)\right)_{w=x}$$

$$
\begin{aligned}
&+\frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r}\binom{r}{j}(-x)^{r-j}\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^j,w)\right)_{w=x}\\
&+\frac{f^{(r+1)}(x)}{(r+1)!}\sum_{j=0}^{r+1}\binom{r+1}{j}(-x)^{r+1-j}\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^j,w)\right)_{w=x}\\
&+\frac{f^{(r+2)}(x)}{(r+2)!}\sum_{j=0}^{r+2}\binom{r+2}{j}(-x)^{r+2-j}\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^j,w)\right)_{w=x}\\
=&\frac{f^{(r)}(x)}{r!}\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^r,w)\right)_{w=x}\\
&+\frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x)\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^r,w)\right)_{w=x}+\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^{r+1},w)\right)_{w=x}\right]\\
&+\frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+2)(r+1)}{2}x^2\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^r,w)\right)_{w=x}\right.\\
&\left.+(r+2)(-x)\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^{r+1},w)\right)_{w=x}+\left(\frac{d^r}{dw^r}L_\alpha^\rho(t^{r+2},w)\right)_{w=x}\right].
\end{aligned}
$$

Applying Lemma 2.2, we have

$$
\begin{aligned}
I_1=&\frac{f^{(r)}(x)}{r!}r!+\frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x).r!+(r+1)!x+\frac{\rho+1}{2\alpha\rho}(r+1)r.r!\right]\\
&+\frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+2)(r+1)}{2}x^2.r!+(r+2)(-x)\left((r+1)!x+\frac{\rho+1}{2\alpha\rho}(r+1)r.r!\right)\right.\\
&+\frac{(r+2)!}{2}x^2+(r+1)(r+2)\frac{(\rho+1)}{2\alpha\rho}(r+1)!x\\
&\left.+\frac{(\rho+1)}{24(\alpha\rho)^2}.(r+2)(r+1)r[(3r+1)\rho+3r+5]r!\right].
\end{aligned}
$$

Thus

$$
\lim_{\alpha\to\infty}\alpha\left(\left(\frac{d^r}{dw^r}L_\alpha^\rho(f,w)\right)_{w=x}-f^{(r)}(x)\right)=\left[f^{(r+1)}(x)\frac{(\rho+1)r}{2\rho}+f^{(r+2)}(x)\frac{(\rho+1)x}{2\rho}+\lim_{\alpha\to\infty}\alpha I_2\right].
$$

In order to complete the proof, it is sufficient to show that $\lim\limits_{\alpha\to\infty}\alpha I_2\to 0$. We proceed as follows.

Next, in view of Lemma 2.6, we have

$$
\begin{aligned}
|I_2|\leqslant&\sum_{k=1}^{\infty}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}\alpha^i|k-\alpha x|^j\frac{|q_{i,j,r}(x)|}{x^r}s_{\alpha,k}(x)\int_0^\infty\theta_{\alpha,k}^\rho(t)|\psi(t,x)|.|t-x|^{r+2}dt\\
&+|\psi(0,x)(-x)^{r+2}|\left(\frac{d^r}{d\omega^r}s_{\alpha,0}(\omega)\right)_{\omega=x}=\Delta_1+\Delta_2.
\end{aligned}
\tag{3.3}
$$

Now, we estimate Δ_1. Since $\psi(t,x)\to 0$ as $t\to x$, for a given $\epsilon>0$ there exists a $\delta>0$ such that $|\psi(t,x)|<\epsilon$ whenever $|t-x|<\delta$. For $|t-x|\geqslant\delta$, we have $|(t-x)^{r+2}\psi(t,x)|\leqslant M|t-x|^\gamma$, for some $M>0$. Thus, from equation (3.3) we may write

$$
\begin{aligned}
|\Delta_1|\leqslant&\sum_{k=1}^{\infty}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}\alpha^i|k-\alpha x|^j\frac{|q_{i,j,r}(x)|}{x^r}s_{\alpha,k}(x)\left(\epsilon\int_{|t-x|<\delta}\theta_{\alpha,k}^\rho(t)|t-x|^{r+2}dt\right.\\
&\left.+M\int_{|t-x|\geqslant\delta}\theta_{\alpha,k}^\rho(t)|t-x|^\gamma dt\right):=J_1+J_2.
\end{aligned}
$$

Let $K = \sup\limits_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \dfrac{|q_{i,j,r}(x)|}{x^r}$. Using Schwarz inequality, Lemma 2.1, and Lemma 2.2, we have

$$\begin{aligned} J_1 &= \epsilon\ K \sum_{k=1}^{\infty} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha^i |k-\alpha x|^j s_{\alpha,k}(x) \left(\int_0^\infty \theta^{\rho}_{\alpha,k}(t)dt \right)^{1/2} \left(\int_0^\infty \theta^{\rho}_{\alpha,k}(t)|t-x|^{2r+4}dt \right)^{1/2} \\ &\leqslant \epsilon\ K \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha^{i+j} \left(\sum_{k=0}^{\infty} s_{\alpha,k}(x) \left(\frac{k}{\alpha}-x\right)^{2j} - x^{2j}s_{\alpha,0}(x) \right)^{1/2} \left(L^{\rho}_{\alpha}((t-x)^{2r+4},x) - x^{2r+4}s_{\alpha,0}(x) \right)^{1/2} \\ &= \epsilon \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha^{i+j} \left\{ O\left(\frac{1}{\alpha^j}\right) + O\left(\frac{1}{\alpha^s}\right) \right\}^{1/2} \left\{ O\left(\frac{1}{\alpha^{r+2}}\right) + O\left(\frac{1}{\alpha^p}\right) \right\}^{1/2} \quad \text{for any } s,p>0. \end{aligned}$$

Choose s and p such that $s > j$ and $p > r+2$

$$J_1 \leqslant \epsilon \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha^{i+j} O\left(\frac{1}{\alpha^{j/2}}\right) O\left(\frac{1}{\alpha^{r/2+1}}\right) = \epsilon \cdot O(\alpha^{-1}).$$

Since $\epsilon > 0$ is arbitrary, $\alpha J_1 \to 0$ as $\alpha \to \infty$.

Again, using Schwarz inequality, Lemma 2.1, and Lemma 2.4, we obtain

$$\begin{aligned} J_2 &\leqslant M_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha^{i+j} \left(\sum_{k=0}^{\infty} \left(\frac{k}{\alpha}-x\right)^{2j} s_{\alpha,k}(x) - x^{2j}s_{\alpha,0}(x) \right)^{1/2} \\ &\quad \times \left(\sum_{k=1}^{\infty} s_{\alpha,k}(x) \int_{|t-x|\geqslant\delta} \theta^{\rho}_{\alpha,k}(t)(t-x)^{2\gamma}dt \right)^{1/2} \\ &\leqslant M_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha^{i+j} \left\{ O\left(\frac{1}{\alpha^j}\right) + O\left(\frac{1}{\alpha^p}\right) \right\}^{1/2} \left\{ O\left(\frac{1}{\alpha^\gamma}\right) \right\}^{1/2} \quad \text{for any } p>0. \end{aligned}$$

Choose p such that $p > j$

$$J_2 \leqslant M_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha^{i+j} O\left(\frac{1}{\alpha^{j/2}}\right) O\left(\frac{1}{\alpha^{\gamma/2}}\right) = M_1 O\left(\frac{1}{\alpha^{(\gamma-r)/2}}\right),$$

which implies that $\alpha J_2 \to 0$, as $\alpha \to \infty$ on choosing $\gamma > r+2$.

From the above estimates of J_1 and J_2, $\alpha\Delta_1 \to 0$, as $\alpha \to \infty$.

Next, we estimate Δ_2.

$$|\Delta_2| = |\psi(0,x)(-x)^r| \left(\frac{d^r}{d\omega^r} s_{\alpha,0}(\omega) \right)_{\omega=x}.$$

Since $|\psi(0,x)(-x)^r| < N_1$ for some $N_1 > 0$, we get $\left(\frac{d^r}{d\omega^r} s_{n,0}(\omega) \right)_{\omega=x} = \left[\frac{d^r}{d\omega^r} \left(e^{-\alpha w} \right) \right]_{\omega=x} \to 0$ as $\alpha \to \infty$, which yields that $\Delta_2 \to 0$ as $\alpha \to \infty$. By combining the estimates of Δ_1 and Δ_2, we obtain $\alpha I_2 \to 0$ as $\alpha \to \infty$. Thus, from the estimates of I_1 and I_2, the required result follows.

To prove the uniformity assertion, it is sufficient to remark that $\delta(\epsilon)$ in the above proof can be chosen to be independent of $x \in [a,b]$ and also that the other estimates hold uniformity in $x \in [a,b]$. This completes the proof. Combining the estimates of I_1 and I_2, we get the required result. □

Theorem 3.2. *Let* $f \in C_\gamma[0,\infty)$ *for some* $\gamma > 0$ *and* $r \leqslant q \leqslant r+2$. *If* $f^{(q)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, *then for* α *sufficiently large, we have*

$$\left\| \left(L_n^\rho(f,.) \right)^{(r)} - f^{(r)} \right\| \leqslant \frac{C_1}{\alpha} \left(\sum_{j=r}^{q} \|f^{(j)}\| \right) + \frac{C_2}{\sqrt{\alpha}} \omega(f^{(r+1)}, \alpha^{-1/2}) + O(\alpha^{-2}),$$

where C_1 *and* C_2 *are absolute constants independent of* f *and* α, *norm is sup-norm on* $[a,b]$ *and* $\omega(f,\delta)$ *is the modulus of continuity of* f *on the interval* $(a-\eta, b+\eta)$.

Proof. Using Taylor's finite expansion of f, we can write

$$f(t) = \sum_{j=0}^{q} \frac{f^{(j)}(x)}{j!}(t-x)^j + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-x)^q \chi(t) + h(t,x)(1-\chi(t)),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of $(a-\eta, b+\eta)$. For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, we have

$$f(t) = \sum_{j=0}^{q} \frac{f^{(j)}(x)}{j!}(t-x)^j + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-\xi)^q.$$

For $t \in (0,\infty) \setminus (a-\eta, b+\eta)$ and $x \in [a,b]$, we set

$$h(t,x) = f(t) - \sum_{j=0}^{q} \frac{f^{(j)}(x)}{j!}(t-x)^j.$$

We have

$$\begin{aligned}
\left(L_n^\rho(f,x) \right)^{(r)} - f^{(r)}(x) &= \left[\sum_{j=0}^{q} \frac{f^{(j)}(x)}{j!} \int_0^\infty [k_\alpha^\rho(x,t)]^{(r)} (t-x)^j dt - f^{(r)}(x) \right] \\
&\quad + \int_0^\infty [k_\alpha^\rho(x,t)]^{(r)} \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-x)^q \chi(t) dt \\
&\quad + \int_0^\infty [k_\alpha^\rho(x,t)]^{(r)} h(t,x)(1-\chi(t)) dt \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Using Lemma 2.2, we have

$$\begin{aligned}
I_1 &= \sum_{j=0}^{q} \frac{f^{(j)}(x)}{j!} \sum_{i=0}^{j} \binom{j}{i} (-x)^{j-i} \frac{d^r}{dx^r} \int_0^\infty [k_\alpha^\rho(x,t)]^{(r)} t^i dt - f^{(r)}(x) \\
&= \sum_{j=0}^{q} \frac{f^{(j)}(x)}{j!} \sum_{i=0}^{j} \binom{j}{i} (-x)^{j-i} \frac{d^r}{dx^r} \left[x^i + \frac{(\rho+1)}{2\alpha\rho}.i(i-1)x^{i-1} \right. \\
&\quad \left. + \frac{(\rho+1)}{24(\alpha\rho)^2}.i(i-1)(i-2)[(3i-5)\rho + 3i - 1]x^{i-2} + \cdots \right] - f^{(r)}(x).
\end{aligned}$$

Hence

$$\|I_1\| \leqslant \frac{C_1}{\alpha} \left(\sum_{j=0}^{q} \|f^{(j)}\| \right) + O(\alpha^{-2}).$$

Next, for $\delta > 0$, we have

$$\begin{aligned} I_2 &= \int_0^\infty |[k_\alpha^\rho(x,t)]^{(r)}| \frac{|f^{(q)}(\xi) - f^{(q)}(x)|}{q!} |t-x|^q \chi(t) dt \\ &\leqslant \frac{\omega(f^{(q)},\delta)}{q!} \int_0^\infty |[k_\alpha^\rho(x,t)]^{(r)}| \left(1 + \frac{|t-x|}{\delta}\right) |t-x|^q dt \\ &\leqslant \frac{\omega(f^{(q)},\delta)}{q!} \left[\sum_{k=1}^\infty |s_{\alpha,k}^{(r)}(x)| \int_0^\infty \theta_{\alpha,k}^\rho(t) \left(|t-x|^q + \delta^{-1}|t-x|^{q+1} \right) dt + \alpha^r e^{-\alpha x} \left(|x|^q + \delta^{-1}|x|^{q+1} \right) \right]. \end{aligned}$$

To evaluate I_2, we consider the following and applying Lemmas 2.6, 2.1, 2.4 in the next steps,

$$\begin{aligned} &\sum_{k=1}^\infty |s_{\alpha,k}^{(r)}(x)| \int_0^\infty \theta_{\alpha,k}^\rho(t)|t-x|^m dt \\ &\qquad \leqslant \sum_{k=1}^\infty \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} \alpha^i |k-\alpha x|^j \frac{q_{i,j,r}(x)}{x^r} s_{\alpha,k}(x) \int_0^\infty \theta_{\alpha,k}^\rho(t)|t-x|^m dt \\ &\qquad \leqslant C \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} \alpha^i \sum_{k=1}^\infty s_{\alpha,k}(x)|k-\alpha x|^j \int_0^\infty \theta_{\alpha,k}^\rho(t)|t-x|^m dt \\ &\qquad \leqslant C \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} \alpha^i \sum_{k=1}^\infty s_{\alpha,k}(x)|k-\alpha x|^j \left[\left(\int_0^\infty \theta_{\alpha,k}^\rho(t) dt \right)^{1/2} \left(\int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^{2m} dt \right)^{1/2} \right] \\ &\qquad \leqslant C \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} \alpha^i \left(\sum_{k=1}^\infty s_{\alpha,k}(x)(k-\alpha x)^{2j} \right)^{1/2} \left(\sum_{k=1}^\infty s_{\alpha,k}(x) \int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^{2m} dt \right)^{1/2} \\ &\qquad = C \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} \alpha^i \cdot O(\alpha^{(j-m)/2}) = O(\alpha^{(r-m)/2}) \end{aligned}$$

uniformly in x, with $C = \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} \sup_{x\in[a,b]} \frac{|q_{i,j,r}(x)|}{x^r}$.

Thus choosing $\delta = \alpha^{-1/2}$ and using above we get for any $s > 0$

$$\|I_2\| \leqslant \frac{\omega(f^{(q)},\alpha^{-1/2})}{q!} \left[O(\alpha^{(r-q)/2}) + \alpha^{1/2} O(\alpha^{(r-q-1)/2}) + O(\alpha^{-s}) \right] \leqslant C_2 \alpha^{-(q-r)/2} \omega(f^{(q)},\alpha^{-1/2}).$$

We choose a $\delta \in (0,\eta)$. Using Lemma 2.6, we obtain

$$|I_3| \leqslant \sum_{k=1}^\infty \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} \alpha^i |k-\alpha x|^j \frac{|q_{i,j,r}(x)|}{x^r} s_{\alpha,k}(x) \int_{|t-x|\geqslant\delta} \theta_{\alpha,k}^\rho(t)|h(t,x)| dt.$$

If β is any integer greater than or equal to $\{\gamma, q\}$ we can find a constant K such that $|h(t,x)| \leqslant K|t-x|^\beta$ for $|t-x| \geqslant \delta$. Applying Schwarz inequality and Lemma 2.4, we get $I_3 = O(\alpha^{-m})$ for any $m > 0$ uniformly on $[a,b]$. Combining the estimates of I_1, I_2, I_3, we get the desired result. □

For sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,2}$ of 2nd order corresponding to $f \in C_\gamma[0,\infty)$ and $t \in I_i = [a_i, b_i], i = 1,2$ is defined as follows:

$$f_{\eta,2}(t) = \eta^{-2} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} (f(t) - \Delta_h^2 f(t)) dt_1 dt_2,$$

where $h = \frac{t_1+t_2}{2}$ and Δ_h^2 is the second order forward difference operator with step length h. The following properties are satisfied (see [7, 11] and references therein):

(i) $f_{\eta,2}$ has continuous derivatives up to order 2 over I_1;
(ii) $\| f_{\eta,2}^{(r)} \|_{C(I_2)} \leqslant C\eta^{-r}\omega_r(f,\eta,I_2), r = 1,2;$
(iii) $\| f - f_{\eta,2} \|_{C(I_2)} \leqslant C\omega_2(f,\eta,I_1);$
(iv) $\| f_{\eta,2} \|_{C(I_2)} \leqslant C \| f \|_{C(I_1)} \leqslant C\|f\|_\gamma,$

where C is a constant not necessarily the same at each occurrence and is independent of f and η.

Lemma 3.3 ([3]). *Let* $f \in C(I)$. *Then,*

$$\| f_{\eta,2k}^{(i)} \|_{C(I)} \leqslant C_i\{\| f_{\eta,2} \|_{C(I)} + \| f_{\eta,2}^{(2k)} \|_{C(I)}\},\ i = 1,2,\dots,2k-1,$$

where C_i*'s are certain constants independent of* f.

Theorem 3.4. *Let* $f \in C_\gamma[0,\infty)$ *for some* $\gamma > 0$ *and* $0 < a < a_1 < b_1 < b < \infty$. *Then for* α *sufficiently large, we have*

$$\left\|\left(L_\alpha^{\rho(r)}(f,.)\right) - f^{(r)}\right\|_{C(I_1)} \leqslant K_1\omega_2(f^{(r)},\alpha^{-1/2},I) + K_2\alpha^{-1}\|f\|_\gamma,$$

where $K_1 = K_1(r)$ *and* $K_2 = K_2(r,f)$.

Proof. We can write

$$\begin{aligned}\left\|\left(L_\alpha^{\rho(r)}(f,.)\right) - f^{(r)}\right\|_{C(I_1)} &\leqslant \left\|L_\alpha^{\rho(r)}((f-f_{\eta,2}),.)\right\|_{C(I_1)} + \left\|\left(L_\alpha^{\rho(r)}(f_{\eta,2},.)\right) - f_{\eta,2}^{(r)}\right\|_{C(I_1)} + \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C(I_1)}\\ &= M_1 + M_2 + M_3.\end{aligned}$$

Since $f_{\eta,2}^{(r)} = (f^{(r)})_{\eta,2}$, hence by property (iii) of the Steklov mean, we get

$$M_3 \leqslant K_1\omega_2(f^{(r)},\eta,I).$$

Next, applying Theorem 3.1 and Lemma 3.3, we obtain

$$M_2 \leqslant K_2\alpha^{-1}\sum_{i=r}^{r+2} \| f_{\eta,2}^{(i)} \|_{C(I_1)} \leqslant K_3\alpha^{-1}\{\| f_{\eta,2} \|_{C(I_1)} + \| f_{\eta,2}^{(r+2)} \|_{C(I_1)}\}.$$

By using properties (ii) and (iv) of Steklov mean, we get

$$M_2 \leqslant K_4\alpha^{-1}\{\| f \|_\gamma + \eta^{-2}\omega_2(f^{(r)},\eta,I)\}.$$

Let a^* and b^* be such that $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$ and I^* denote the interval $[a^*,b^*]$.

Now, we estimate M_1. Let $f - f_{\eta,2} \equiv F$. By our hypothesis we can write

$$F(t) = \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!}(t-x)^m + \frac{F^{(r)}(\xi) - F^{(r)}(x)}{r!}(t-x)^r\chi(t) + \theta(t,x)(1-\chi(t)), \tag{3.4}$$

where ξ lies between t and x, and χ denotes the characteristic function of the interval I^*. For $t \in I^*$ and $x \in I_1$, we get

$$F(t) = \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!}(t-x)^m + \frac{F^{(r)}(\xi) - F^{(r)}(x)}{r!}(t-x)^r,$$

and for $t \in [0, \infty) \setminus I^*, x \in I_1$ we set

$$\theta(t,x) = F(t) - \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!}(t-x)^m.$$

Operating $L_\alpha^{\rho(r)}$ on both sides of (3.4), we get three terms J_1, J_2, and J_3, corresponding to three terms in right hand side of (3.4). Using Theorem 3.1, we get

$$|J_1| \leqslant \| f^{(r)} - f_{\eta,2}^{(r)} \|_{C(I_1)} .$$

Next, using Theorem 3.1, we obtain

$$|J_2| \leqslant \frac{2 \| F^{(r)} \|}{r!} L_\alpha^{\rho(r)}((t-x)^r \chi(t), x) \leqslant K_5 \| f^{(r)} - f_{\eta,2}^{(r)} \|_{C(I_*)} .$$

Lastly, we easily have

$$|J_3| = L_\alpha^{\rho(r)}(1 - \chi(t)\theta(t,x), x) = O(\alpha^{-s}) \text{ for any } s > 0.$$

Combining $J_1 - J_3$, and from property (iii) of the Steklov mean, we obtain

$$M_1 \leqslant K_6 \| f^{(r)} - f_{\eta,2}^{(r)} \|_{C(I_*)} \leqslant K_6 \, \omega_2(f^{(r)}, \eta, I).$$

Finally choosing $\eta = \alpha^{-1/2}$, the required result follows at once. □

Remark 3.5. Proceeding along the lines of [9], one can extend the results for Păltănea type operators $L_\alpha^\rho(f,x)$. As the analysis is different, and due to the complicated form of these operators, it may be considered elsewhere.

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