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Asymptotic behavior of solutions to a class of coupled semilinear parabolic systems with gradient terms

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Abstract

This paper concerns the asymptotic behavior of solutions to the Cauchy problem of a class of coupled semilinear parabolic systems with gradient terms. Using the energy comparison method and comparison principle, the blow-up theorem of Fujita type is established and the critical Fujita curve is formulated by spacial dimension, the behavior of the coefficient of the gradient term at infinity.

Keywords: Asymptotic behavior, critical Fujita curve, gradient term.
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1. Introduction

We investigate the asymptotic behavior of solutions to the Cauchy problem of coupled semilinear parabolic systems of the form

$$\frac{\partial u}{\partial t} = \Delta u + b(|x|)x \cdot \nabla u + v^p, \qquad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \Delta v + b(|x|)x \cdot \nabla v + u^q, \qquad x \in \mathbb{R}^n,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^n, \tag{1.3}$$

where $p,\ q > 1$, $0 \leqslant b \in C^1([0,+\infty))$ satisfies

$$\lim_{s\to+\infty} s^2 b(s) = \kappa, \quad (0 \leqslant \kappa \leqslant +\infty), \tag{1.4}$$

and $0 \leqslant u_0,\ v_0 \in L^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ are nontrivial.

It is well-known that the Cauchy problem of the semilinear equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^n,\ t > 0,$$

was investigated by Fujita in [3], where it was proved that the problem does not have any nontrivial global nonnegative solution if $1 < p < p_c = 1 + 2/n$, whereas there exist both nontrivial global (with small initial

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data) and non-global nonnegative (with large initial data) solutions if $p > p_c$. Furthermore, for the critical case $p = p_c$, it was shown in [6, 7, 19] that the problem possesses no nontrivial global nonnegative solution. Later, there have been a number of extensions of Fujita's results in several directions, including similar results for numerous of nonlinear parabolic equations and systems (see, e.g., the survey papers [1, 2, 4, 5, 8–12, 14–18, 20–23] and the references therein).

Among those, there are a few studies on equations and systems with gradient terms. It is Meier [10] who first studied the critical Fujita exponent for the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u + \vec{b}(x) \cdot \nabla u + u^p, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.5}$$

with $\vec{b} \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$. It was shown that

$$p_c = 1 + \frac{1}{\gamma^*},$$

where γ^* is the maximal decay rate for solutions to

$$\frac{\partial w}{\partial t} = \Delta w + \vec{b}(x) \cdot \nabla w, \quad x \in \mathbb{R}^n,\ t > 0.$$

If $\vec{b}$ is constant, it is clear that $\gamma^* = n/2$ and $p_c = 1 + 2/n$. However, for nonconstant $\vec{b} \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, γ^* and p_c are unknown generally. It is noted that [21] studied the case that

$$\vec{b}(s) = \frac{\kappa}{|s|^2}, \quad s \in \mathbb{R}^n, \quad (-\infty < \kappa < +\infty).$$

Since such a function is singular at 0 when $\kappa \neq 0$, the authors considered the following Neumann exterior problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \Delta u + \frac{\kappa}{|x|^2} x \cdot \nabla u + u^p, && x \in \mathbb{R}^n \setminus B_1,\ t > 0,\\
&\frac{\partial u}{\partial \nu}(x,t) = 0, && x \in \partial B_1,\ t > 0,\\
&u(x,0) = u_0(x), && x \in \mathbb{R}^n \setminus \overline{B}_1,
\end{aligned}$$

and showed that its critical Fujita exponent can be formulated as

$$p_c = \begin{cases} 1, & \kappa = +\infty,\\ 1 + 2/(n+\kappa), & -n < \kappa < +\infty,\\ +\infty, & -\infty \leqslant \kappa \leqslant -n, \end{cases} \tag{1.6}$$

where B_1 is the unit ball in $\mathbb{R}^n$ and ν is the unit inner normal vector to ∂B_1. Moreover, [18, 21] also investigated the Neumann exterior problems for the Newtonian and non-Newtonian filtration equations with similar gradient terms. [23] showed that the critical Fujita exponent to the Cauchy problem (1.5) with $\vec{b} = b$ is still (1.6), where b satisfies (1.4). Furthermore, Suzuki [14] considered the Newtonian filtration equation with similar gradient term and got the critical Fujita exponent for some special cases.

In 1991, Escobedo and Herrero [2] investigated the coupled semilinear parabolic system (1.1), (1.2), (1.3) in the special case $b \equiv 0$, and formulated the critical Fujita curve as

$$(pq)_c = 1 + \frac{2}{n} \max\{p+1,\ q+1\}. \tag{1.7}$$

The authors in [4] considered the Neumann exterior problem of coupled nonlinear diffusion systems

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\kappa}{|x|^2} x \cdot \nabla u + |x|^\lambda v^p, \qquad x \in \mathbb{R}^n \setminus B_1,\ t > 0,$$

$$\frac{\partial v}{\partial t} = \Delta v + \frac{\kappa}{|x|^2} x \cdot \nabla v + |x|^{\lambda} u^q, \qquad x \in \mathbb{R}^n \setminus B_1,\ t > 0,$$

where $\kappa \in \mathbb{R}$, $\lambda \geqslant 0$, and proved its critical Fujita curve

$$(pq)_c = \begin{cases} 1 + \dfrac{2+\lambda}{n+\kappa} \max\{p+1,\ q+1\}, & \kappa > -n, \\ +\infty, & \kappa \leqslant -n. \end{cases}$$

In the present paper, we prove that the critical Fujita curve to the problem can be formulated as

$$(pq)_c = \begin{cases} 1 + \dfrac{2}{n+\kappa} \max\{p+1,\ q+1\}, & 0 \leqslant \kappa < +\infty, \\ 1, & \kappa = +\infty. \end{cases} \tag{1.8}$$

That is to say, if $1 < pq < (pq)_c$, any nonnegative nontrivial solution blows up in a finite time, whereas if $pq > (pq)_c$, there exist both nontrivial nonnegative global solution (with small initial data) and nonnegative blow-up solution (with large initial data). The difference between (1.8) and (1.7) shows that the gradient term can affect the large time behavior of solutions essentially. The technique used in this paper is mainly inspired by [16, 18, 21, 23]. To prove the blow-up of solutions, we determine the interactions among the diffusion and the gradient by a series of precise energy integral estimates instead of pointwise comparison principle. For the global existence of nontrivial solutions, we construct a nontrivial global supersolution. Note that neither (1.1) nor (1.2) possesses a self-similar construct, we have to seek a complicated supersolution and do some precise calculations.

The paper is organized as follows. Some preliminaries are listed in Section 2, such as the definition of solution, the well-posedness of the problem (1.1)-(1.3). Several useful auxiliary lemmas are illustrated in Section 3. The blow-up theorems of Fujita type for the problem (1.1)-(1.3) are obtained in Section 4.

2. Preliminaries

The subsolutions, supersolutions, as well as solutions to the problem (1.1)-(1.3) are defined as follows.

Definition 2.1. Let $0 < T \leqslant +\infty$. A pair of nonnegative functions (u, v) is called a super (sub) solution to the problem (1.1)-(1.3) in $(0, T)$, if

$$u,\ v \in C([0,T), L^1_{loc}(\mathbb{R}^n)) \cap L^{\infty}_{loc}(0,T; L^{\infty}(\mathbb{R}^n)),$$

and the integral identities

$$\int_0^T \int_{\mathbb{R}^n} u(x,t) \left(\frac{\partial \varphi}{\partial t}(x,t) + \Delta\varphi(x,t) - \operatorname{div}(b(|x|)\varphi(x,t)x) \right) dxdt + \int_0^T \int_{\mathbb{R}^n} v^p(x,t)\varphi(x,t) dxdt + \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)dx \leqslant (\geqslant)\, 0,$$

and

$$\int_0^T \int_{\mathbb{R}^n} v(x,t) \left(\frac{\partial \psi}{\partial t}(x,t) + \Delta\psi(x,t) - \operatorname{div}(b(|x|)\psi(x,t)x) \right) dxdt + \int_0^T \int_{\mathbb{R}^n} u^q(x,t)\psi(x,t) dxdt + \int_{\mathbb{R}^n} v_0(x)\psi(x,0)dx \leqslant (\geqslant)\, 0,$$

are fulfilled for any $0 \leqslant \varphi,\ \psi \in C^{2,1}(\mathbb{R}^n \times [0,T))$ vanishing when t near T or $|x|$ being sufficiently large. (u, v) is called a solution to the problem (1.1)-(1.3) in $(0, T)$, if it is both a supersolution and a subsolution.

Definition 2.2. A solution (u, v) to the problem (1.1)-(1.3) is said to blow up in a finite time $0 < T_* < +\infty$, which is called blow-up time, if

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} + \|v(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \to +\infty, \quad \text{as} \quad t \to T_*^-.$$

Otherwise, (u, v) is said to be global.

We give the existence theorem and the comparison principle to the problem (1.1)-(1.3) as follows without proofs, one can see [13] and the references therein.

Theorem 2.3 (Local existence theorem)**.** *If* $0 \leqslant u_0,\ v_0 \in L^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *then the Cauchy problem* (1.1)-(1.3) *admits at least one solution locally in time.*

Theorem 2.4 (Comparison principle)**.** *Let* (u_1, v_1) *and* (u_2, v_2) *be two solutions to the system* (1.1) *and* (1.2) *in* $(0, T)$ *with nonnegative initial data* $u_{1,0}(x)$, $v_{1,0}(x)$ *and* $u_{2,0}(x)$, $v_{2,0}(x)$, *respectively. If* $(u_{1,0}, v_{1,0}) \leqslant (u_{2,0}, v_{2,0})$ *a.e. in* $\mathbb{R}^n$, *then* $(u_1, v_1) \leqslant (u_2, v_2)$ *a.e. in* $\mathbb{R}^n \times (0, T)$.

3. Auxiliary lemmas

In this section, we illustrate some auxiliary lemmas which will be used to investigate the blow-up property of solutions to the problem (1.1)-(1.3). In this paper, we always assume that $p \geqslant q$ without loss of generality if there is no instruction.

Lemma 3.1. *Assume that* $0 \leqslant b \in C^1([0, +\infty))$ *satisfies* (1.4) *with* $0 \leqslant \kappa < +\infty$. *Let* (u, v) *be a solution to the problem* (1.1)-(1.3). *Then there exist three numbers* $R_0 > 0$, $\delta > 1$ *and* $M_0 > 0$ *depending only on* n *and* b, *such that for any* $R > R_0$,

$$\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\phi_R(|x|)dx \geqslant -M_0R^{-2}\int_{B_{\delta R}\setminus B_R} u(x,t)\phi_R(|x|)dx + \int_{\mathbb{R}^n} v^p(x,t)\phi_R(|x|)dx, \quad t > 0, \tag{3.1}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^n} v(x,t)\phi_R(|x|)dx \geqslant -M_0R^{-2}\int_{B_{\delta R}\setminus B_R} v(x,t)\phi_R(|x|)dx + \int_{\mathbb{R}^n} u^q(x,t)\phi_R(|x|)dx, \quad t > 0, \tag{3.2}$$

in the distribution sense, where

$$\phi_R(r) = \begin{cases} h(r), & 0 \leqslant r \leqslant R, \\ \frac{1}{2}h(r)\left(1 + \cos\frac{(r-R)\pi}{(\delta-1)R}\right), & R < r < \delta R, \\ 0, & r \geqslant \delta R, \end{cases}$$

with

$$h(r) = \exp\left\{\int_0^r sb(s)ds\right\}, \quad r \geqslant 0,$$

while B_r *denotes the open ball in* $\mathbb{R}^n$ *with radius* r *and centered at the origin.*

Proof. Set $\delta > 1$. It is obvious that $\phi_R \in C^1([0, +\infty)) \cap C^2((0, R) \cup (R, \delta R) \cup (\delta R, +\infty))$ with $\phi_R'(0) = 0$. Multiplying (1.1) by $\phi_R(|x|)$ and then integrating by parts, one has

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\phi_R(|x|)dx = &\int_{B_{\delta R}} u(x,t)\big(\Delta\phi_R(|x|) - \operatorname{div}(b(|x|)\phi_R(|x|)x)\big)dx \\ &+ \int_{\mathbb{R}^n} v^p(x,t)\phi_R(|x|)dx, \quad t > 0, \end{aligned} \tag{3.3}$$

in the distribution sense.

Let us compute $\Delta\phi_R(|x|) - \mathrm{div}(b(|x|)\phi_R(|x|)x)$ in $B_R \setminus \{0\}$ and $B_{\delta R} \setminus \overline{B}_R$, respectively. It is clear that for $0 < |x| < R$,

$$\begin{aligned}\Delta\phi_R(|x|) &= \phi_R''(|x|) + \frac{n-1}{|x|}\phi_R'(|x|)\\ &= \Big(|x|b'(|x|) + b(|x|) + |x|^2b^2(|x|) + (n-1)b(|x|)\Big)\phi_R(|x|),\\ \mathrm{div}(b(|x|)\phi_R(|x|)x) &= |x|b'(|x|)\phi_R(|x|) + |x|b(|x|)\phi_R'(|x|) + nb(|x|)\phi_R(|x|)\\ &= \Big(|x|b'(|x|) + |x|^2b^2(|x|) + nb(|x|)\Big)\phi_R(|x|),\end{aligned}$$

thus

$$\Delta\phi_R(|x|) - \mathrm{div}(b(|x|)\phi_R(|x|)x) = 0, \quad 0 < |x| < R. \tag{3.4}$$

When $R < |x| < \delta R$, a direct calculation gives

$$\begin{aligned}&\Delta\phi_R(|x|) - \mathrm{div}(b(|x|)\phi_R(|x|)x)\\ &\quad= \phi_R''(|x|) + \frac{n-1}{|x|}\phi_R'(|x|) - |x|b'(|x|)\phi_R(|x|) - |x|b(|x|)\phi_R'(|x|) - nb(|x|)\phi_R(|x|)\\ &\quad= -\frac{\pi}{2(\delta-1)R|x|}\big(|x|^2b(|x|) + n - 1\big)h(|x|)\sin\frac{(r-R)\pi}{(\delta-1)R} - \frac{\pi^2}{2(\delta-1)^2R^2}h(|x|)\cos\frac{(r-R)\pi}{(\delta-1)R}.\end{aligned} \tag{3.5}$$

It follows from (1.4) that there exists $R_0 > 0$ such that

$$0 < s^2b(s) + n - 1 < \kappa + n, \quad s > R_0. \tag{3.6}$$

Choose $\delta = \frac{\pi}{n+\kappa} + 1$. Then for any $R > R_0$, one gets from (3.5) and (3.6) that

$$\begin{aligned}\Delta\phi_R(|x|) - \mathrm{div}(b(|x|)\phi_R(|x|)x) &\geqslant -\frac{\pi^2}{2(\delta-1)^2R^2}h(|x|) - \frac{\pi^2}{2(\delta-1)^2R^2}h(|x|)\cos\frac{(r-R)\pi}{(\delta-1)R}\\ &= -M_0R^{-2}\phi_R(|x|), \quad R < |x| < \delta R,\end{aligned} \tag{3.7}$$

with $M_0 = (\delta-1)^{-2}\pi^2$. Substituting (3.4) and (3.7) into (3.3) leads to (3.1). Similarly, one can prove that (3.2) holds. □

Remark 3.2. Lemma 3.1 still holds if (1.4) is relaxed by

$$\overline{\lim_{s\to+\infty}}\, s^2b(s) = \kappa.$$

Remark 3.3. The proof of Lemma 3.1 is invalid if $\kappa = +\infty$. In this case, both (3.1) and (3.2) hold for any fixed $R > 0$, but $\delta > 1$ and $M_0 > 0$ depend also on R.

To show the existence of a nontrivial global solution to the problem (1.1)-(1.3), we study self-similar supersolutions to the system (1.1) and (1.2) of the form

$$u(x,t) = (t+\tau)^{-\alpha}U((t+\tau)^{-1/2}|x|), \quad v(x,t) = (t+\tau)^{-\beta}V((t+\tau)^{-1/2}|x|), \quad x \in \mathbb{R}^n,\ t \geqslant 0, \tag{3.8}$$

with

$$\alpha = \frac{p+1}{pq-1}, \quad \beta = \frac{q+1}{pq-1},$$

and $\tau > 0$ will be determined later. If $U, V \in C^1([0,+\infty))$ solve

$$U''(r) + \frac{n-1}{r}U'(r) + (t+\tau)rb((t+\tau)^{1/2}r)U'(r) + \frac{1}{2}rU'(r) + \alpha U(r) + V^p(r) \leqslant 0, \quad r > 0, \tag{3.9}$$

$$V''(r) + \frac{n-1}{r}V'(r) + (t+\tau)rb((t+\tau)^{1/2}r)V'(r) + \frac{1}{2}rV'(r) + \beta V(r) + U^q(r) \leqslant 0, \quad r > 0, \tag{3.10}$$

for any $r > 0$, respectively. Then (u, v) given by (3.8) is a supersolution to the system (1.1) and (1.2).

Lemma 3.4. *Assume that* $b \in C^1([0,+\infty))$ *satisfies* (1.4) *with* $0 \leqslant \kappa \leqslant +\infty$. *Let* $pq > (pq)_c$ *and*

$$U(r) = V(r) = \varepsilon e^{-A(r)}, \quad r \geqslant 0, \tag{3.11}$$

where $\varepsilon > 0$ *will be determined, while* $A \in C^1([0,+\infty))$ *satisfies* $A(0) = 0$ *and*

$$A'(r) = \begin{cases} A_1 r, & 0 \leqslant r \leqslant l^2, \\ A_2 r + (A_1 - A_2)\dfrac{l^{2(n+\kappa_2)}}{r^{n+\kappa_2-1}}, & l^2 < r < l, \\ A_2 r + (A_1 - A_2) l^{n+\kappa_2} r, & r \geqslant l, \end{cases}$$

where $0 < l < 1$ *to be determined,*

$$A_i = \frac{2(p+1)}{(n+\kappa_i)(pq+(pq)_c-2)}, \quad i = 1, 2,$$

with $-n < \kappa_1 < \kappa_2 < \kappa$ *satisfying*

$$\inf\{s^2 b(s) : s > 0\} > \kappa_1, \quad (n+\kappa_2)(pq+(pq)_c-2) > 4(p+1).$$

Then there exist $0 < l < 1$, $\tau > 0$ *and* $\varepsilon > 0$ *such that* (u, v) *given by* (3.8) *and* (3.11) *is a supersolution to the system* (1.1) *and* (1.2).

Proof. For $0 < r < l^2$ and $t > 0$,

$$\begin{aligned} U''(r) &+ \frac{n-1}{r}U'(r) + (t+\tau) r b((t+\tau)^{1/2} r) U'(r) + \frac{1}{2} r U'(r) + \alpha U(r) \\ &\leqslant U(r)\left(-(n+\kappa_0)A_1 + \frac{p+1}{pq-1} + \left(A_1 - \frac{1}{2}\right) A_1 r^2\right) \\ &= U(r)\left(-\frac{2(\kappa_0-\kappa_1)(p+1)}{(n+\kappa_1)(pq+(pq)_c-2)} + \left(A_1 - \frac{1}{2}\right) A_1 r^2\right) - U(r)\frac{(pq-(pq)_c)(p+1)}{(pq+(pq)_c-2)(pq-1)}, \end{aligned}$$

where $\kappa_0 = \inf\{s^2 b(s) : s > 0\}$. Therefore, there exists $0 < l_1 < 1$ such that for any $0 < l < l_1$,

$$\begin{aligned} U''(r) &+ \frac{n-1}{r}U'(r) + (t+\tau) r b((t+\tau)^{1/2} r) U'(r) + \frac{1}{2} r U'(r) + \alpha U(r) \\ &\leqslant -\frac{(pq-(pq)_c)(p+1)}{2(pq+(pq)_c-2)(pq-1)} U(r), \quad 0 < r < l^2, \ t > 0. \end{aligned} \tag{3.12}$$

From the definition of A, one gets

$$\begin{aligned} U''(r) &+ \frac{n+\kappa_2-1}{r}U'(r) + \frac{1}{2} r U'(r) + \alpha U(r) \\ &= U(r)\left((A'(r))^2 - A''(r) - \frac{n+\kappa_2-1}{r}A'(r) - \frac{1}{2} r A'(r) + \frac{p+1}{pq-1}\right) \\ &= U(r)\left(\left(A_2 + (A_1-A_2)\frac{l^{2(n+\kappa_2)}}{r^{n+\kappa_2}}\right)\left(A_2 + (A_1-A_2)\frac{l^{2(n+\kappa_2)}}{r^{n+\kappa_2}} - \frac{1}{2}\right) r^2\right) - U(r)(n+\kappa_2)A_2 + \frac{p+1}{pq-1} \\ &\leqslant U(r)\left(\left(A_1 - \frac{1}{2}\right) A_1 r^2 - \frac{(pq-(pq)_c)(p+1)}{(pq+(pq)_c-2)(pq-1)}\right), \quad l^2 < r < l, \end{aligned}$$

which implies that there exists $0 < l_2 < l_1$ such that for any $0 < l < l_2$,

$$U''(r) + \frac{n+\kappa_2-1}{r}U'(r) + \frac{1}{2} r U'(r) + \alpha U(r) \leqslant -\frac{(pq-(pq)_c)(p+1)}{2(pq+(pq)_c-2)(pq-1)} U(r), \quad l^2 < r < l. \tag{3.13}$$

For $r > l$, it holds that

$$\begin{aligned}
U''(r)+&\frac{n+\kappa_2-1}{r}U'(r)+\frac{1}{2}rU'(r)+\alpha U(r)\\
&=U(r)\left((A'(r))^2-A''(r)-\frac{n+\kappa_2-1}{r}A'(r)-\frac{1}{2}rA'(r)+\frac{p+1}{pq-1}\right)\\
&=U(r)\left(\left(A_2+(A_1-A_2)l^{n+\kappa_2}\right)\left(A_2+(A_1-A_2)l^{n+\kappa_2}-\frac{1}{2}\right)r^2\right.\\
&\quad\left.-(n+\kappa_2)(A_1-A_2)l^{n+\kappa_2}-\frac{(pq-(pq)_c)(p+1)}{(pq+(pq)_c-2)(pq-1)}\right)\\
&\leqslant U(r)\left(A_2+(A_1-A_2)l^{n+\kappa_2}\right)\left(A_2+(A_1-A_2)l^{n+\kappa_2}-\frac{1}{2}\right)r^2-U(r)\frac{(pq-(pq)_c)(p+1)}{(pq+(pq)_c-2)(pq-1)}.
\end{aligned}$$

The choice of κ_1, κ_2 leads to

$$\lim_{l\to 0^+}\left(A_2+(A_1-A_2)l^{n+\kappa_2}\right)=A_2<\frac{1}{2},$$

which yields that there exists $0 < l_3 < l_2$ such that for any $0 < l < l_3$,

$$A_2+(A_1-A_2)l^{n+\kappa_2}<\frac{1}{2},$$

and thus

$$U''(r)+\frac{n+\kappa_2-1}{r}U'(r)+\frac{1}{2}rU'(r)+\alpha U(r)\leqslant-\frac{(pq-(pq)_c)(p+1)}{2(pq+(pq)_c-2)(pq-1)}U(r),\quad r>l. \tag{3.14}$$

Fixing $0 < l < l_3$, (1.4) ensures that one can choose $\tau > 0$ sufficiently large such that

$$(t+\tau)rb((t+\tau)^{1/2}r)\geqslant\frac{\kappa_2}{r},\quad r>l^2,\ t>0. \tag{3.15}$$

It follows from (3.12), (3.13), (3.14), (3.15) that for $t > 0$,

$$\begin{aligned}
U''(r)+&\frac{n-1}{r}U'(r)+(t+\tau)rb((t+\tau)^{1/2}r)U'(r)+\frac{1}{2}rU'(r)+\alpha U(r)\\
&\leqslant U''(r)+\frac{n+\kappa_2-1}{r}U'(r)+\frac{1}{2}rU'(r)+\alpha U(r)\\
&\leqslant-\frac{(pq-(pq)_c)(p+1)}{2(pq+(pq)_c-2)(pq-1)}U(r),\quad r\in(0,l^2)\cup(l^2,l)\cup(l,+\infty).
\end{aligned}\tag{3.16}$$

The same as what we calculate above, we can obtain that for $t > 0$,

$$\begin{aligned}
V''(r)+&\frac{n-1}{r}V'(r)+(t+\tau)rb((t+\tau)^{1/2}r)V'(r)+\frac{1}{2}rV'(r)+\beta V(r)\\
&\leqslant-\left(\frac{(pq-(pq)_c)(p+1)}{2(pq+(pq)_c-2)(pq-1)}+\frac{p-q}{pq-1}\right)V(r)\\
&\leqslant-\frac{(pq-(pq)_c)(p+1)}{2(pq+(pq)_c-2)(pq-1)}V(r),\quad r\in(0,l^2)\cup(l^2,l)\cup(l,+\infty).
\end{aligned}\tag{3.17}$$

Due to p, $q > 1$ and the definition of A,

$$0<K_0=\sup_{r>0}e^{-(p-1)A(r)}<+\infty.$$

Choose $\varepsilon > 0$ sufficiently small such that

$$\max\{\varepsilon^{p-1},\varepsilon^{q-1}\}\leqslant\frac{(pq-(pq)_c)(p+1)}{2K_0(pq+(pq)_c-2)(pq-1)}.$$

Then (3.16) and (3.17) yield that (3.9) and (3.10) for $r \in (0, l^2) \cup (l^2, l) \cup (l, +\infty), t > 0$. Since $U, V \in C^{1,1}([0, +\infty))$, (u, v) given by (3.8) and (3.11) is a supersolution to the system (1.1) and (1.2). □

Remark 3.5. Lemma 3.4 still holds if $b \in C^1([0,+\infty))$ and (1.4) are relaxed by that b is Lipschitz continuous in $[0,+\infty)$ and

$$\overline{\lim_{s\to+\infty}} s^2 b(s) = \kappa,$$

respectively.

Also, we need the following elementary inequality.

Lemma 3.6. *Assume that* $p \geqslant q > 1$ *and* $a, b \geqslant 0$, *then*

$$a^p + b^q \geqslant \begin{cases} 2^{-p}(a+b)^p, & \textit{if } 0 \leqslant a, b \leqslant 1, \\ 2^{-p}(a+b)^q, & \textit{otherwise}. \end{cases}$$

Proof. For $0 \leqslant a, b \leqslant 1$, the convexity of the function $x \mapsto x^p$ with $p > 1$ yields

$$(a+b)^p \leqslant 2^{p-1}(a^p + b^p) \leqslant 2^p(a^p + b^q),$$

that is,

$$a^p + b^q \geqslant 2^{-p}(a+b)^p, \quad 0 \leqslant a, b \leqslant 1.$$

Now, we divide the remaining range of (a, b) into the following two parts

$$D_1 = \{(a, b);\ a > 1,\ b \geqslant 0\}, \quad D_2 = \{(a, b);\ 0 \leqslant a \leqslant 1,\ b > 1\}.$$

If $(a, b) \in D_1$, then the convexity of the function $x \mapsto x^q$ with $q > 1$ ensures

$$(a+b)^q \leqslant 2^{q-1}(a^q + b^q) \leqslant 2^p(a^p + b^q),$$

i.e.,

$$a^p + b^q \geqslant 2^{-p}(a+b)^q, \quad (a, b) \in D_1.$$

Finally, if $(a, b) \in D_2$, then

$$\frac{a^p + b^q}{(a+b)^q} \geqslant \left(\frac{b}{a+b}\right)^q \geqslant \left(\frac{b}{1+b}\right)^q \geqslant \left(\frac{b}{1+b}\right)^p. \tag{3.18}$$

It follows from $b > 1$ that $\frac{b}{1+b} \geqslant \frac{1}{2}$, which together with (3.18), shows

$$\frac{a^p + b^q}{(a+b)^q} \geqslant 2^{-p},$$

i.e.,

$$a^p + b^q \geqslant 2^{-p}(a+b)^q, \quad (a, b) \in D_2.$$

The lemma is proven. □

4. Blow-up theorems of Fujita type

In this section, we establish the blow-up theorems of Fujita type for the problem (1.1)-(1.3) by using auxiliary lemmas in Section 3. In the future proofs, let ϕ_R, h, R_0, δ and M_0 are given in Lemma 3.1.

Firstly, we consider the case $pq < (pq)_c$ with $0 \leqslant \kappa < +\infty$.

Theorem 4.1. *Assume that* $0 \leqslant b \in C^1([0,+\infty))$ *satisfies* (1.4) *with* $0 \leqslant \kappa < +\infty$. *Let* $pq < (pq)_c$. *Then for any nontrivial* $0 \leqslant u_0, v_0 \in L^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *the solution to the problem* (1.1)-(1.3) *must blow up in a finite time.*

Proof. Owing to $0 \leqslant \kappa < +\infty$ and $pq < (pq)_c$,

$$\kappa < \frac{2(p+1)}{pq-1} - n.$$

Fix $\tilde{\kappa}$ to satisfy

$$\kappa < \tilde{\kappa} < \frac{2(p+1)}{pq-1} - n. \tag{4.1}$$

By (1.4) and (4.1), there exists $R_1 > 0$ such that

$$0 \leqslant s^2 b(s) < \tilde{\kappa}, \quad s > R_1.$$

For any $R > R_1$, one gets

$$\begin{aligned} 0 \leqslant \phi_R(|x|) \leqslant h(|x|)\chi_{[0,\delta R]}(|x|) &\leqslant \exp\left\{\int_0^{\delta R} sb(s)ds\right\}\chi_{[0,\delta R]}(|x|) \\ &\leqslant \exp\left\{\int_0^{R_1} sb(s)ds\right\}\exp\left\{\tilde{\kappa}\int_{R_1}^{\delta R}\frac{1}{s}ds\right\}\chi_{[0,\delta R]}(|x|) = MR^{\tilde{\kappa}}\chi_{[0,\delta R]}(|x|), \quad |x| \geqslant 0, \end{aligned} \tag{4.2}$$

where $\chi_{[0,\delta R]}$ is the characteristic function of the interval $[0, \delta R]$, while $M > 0$ depends only on n, b, R_1, δ and $\tilde{\kappa}$. Let (u, v) be the solution to the problem (1.1)-(1.3), and denote

$$w_R(t) = \int_{\mathbb{R}^n} \left(u(x,t) + R^\theta v(x,t)\right)\phi_R(|x|)dx, \quad t \geqslant 0,$$

with

$$\theta = -\frac{(p-q)(n+\tilde{\kappa})}{p+1}.$$

For any $R > \max\{R_0, R_1\}$, Lemma 3.1 shows

$$\frac{d}{dt}w_R(t) \geqslant -M_0R^{-2}w_R(t) + R^\theta\int_{\mathbb{R}^n} u^q(x,t)\phi_R(|x|)dx + \int_{\mathbb{R}^n} v^p(x,t)\phi_R(|x|)dx, \quad t > 0. \tag{4.3}$$

The Hölder inequality and (4.2) yield

$$\begin{aligned} \int_{\mathbb{R}^n} u(x,t)\phi_R(|x|)dx &\leqslant \left(\int_{\mathbb{R}^n}\phi_R(|x|)dx\right)^{(q-1)/q}\left(\int_{\mathbb{R}^n} u^q(x,t)\phi_R(|x|)dx\right)^{1/q} \\ &\leqslant \left(MR^{\tilde{\kappa}}\int_{B_{\delta R}} dx\right)^{(q-1)/q}\left(\int_{\mathbb{R}^n} u^q(x,t)\phi_R(|x|)dx\right)^{1/q} \\ &= M_1^{(q-1)/q}R^{(n+\tilde{\kappa})(q-1)/q}\left(\int_{\mathbb{R}^n} u^q(x,t)\phi_R(|x|)dx\right)^{1/q}, \quad t > 0, \end{aligned}$$

with $M_1 = M\omega_n\delta^n > 0$ and ω_n is the volume of the unit ball in $\mathbb{R}^n$, which implies

$$\int_{\mathbb{R}^n} u^q(x,t)\phi_R(|x|)dx \geqslant M_1^{1-q}R^{(n+\tilde{\kappa})(1-q)}\left(\int_{\mathbb{R}^n} u(x,t)\phi_R(|x|)dx\right)^q, \quad t > 0. \tag{4.4}$$

Similarly, one can prove that

$$\int_{\mathbb{R}^n} v^p(x,t)\phi_R(|x|)dx \geqslant M_1^{1-p}R^{(n+\tilde{\kappa})(1-p)}\left(\int_{\mathbb{R}^n} v(x,t)\phi_R(|x|)dx\right)^p, \quad t > 0. \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3) shows

$$\begin{aligned}\frac{d}{dt}w_R(t) \geqslant -M_0R^{-2}w_R(t) + M_1^{1-q}R^{(n+\tilde{\kappa})(1-q)+\theta}\Big(\int_{\mathbb{R}^n} u(x,t)\phi_R(|x|)dx\Big)^q \\ + M_1^{1-p}R^{(n+\tilde{\kappa})(1-p)-p\theta}\Big(\int_{\mathbb{R}^n} R^\theta v(x,t)\phi_R(|x|)dx\Big)^p, \quad t>0.\end{aligned} \tag{4.6}$$

It follows from

$$(n+\tilde{\kappa})(1-q)+\theta = (n+\tilde{\kappa})(1-p)-p\theta = -\frac{(pq-1)(n+\tilde{\kappa})}{p+1},$$

and Lemma 3.6 that (4.6) leads to

$$\begin{aligned}\frac{d}{dt}w_R(t) &\geqslant -M_0R^{-2}w_R(t) + M_2R^{-(pq-1)(n+\tilde{\kappa})/(p+1)} \\ &\quad\cdot\Big\{\Big(\int_{\mathbb{R}^n} u(x,t)\phi_R(|x|)dx\Big)^q + \Big(\int_{\mathbb{R}^n} R^\theta v(x,t)\phi_R(|x|)dx\Big)^p\Big\} \\ &\geqslant -M_0R^{-2}w_R(t) + 2^{-p}M_2R^{-(pq-1)(n+\tilde{\kappa})/(p+1)}\cdot\min\{w_R^p(t),\, w_R^q(t)\} \\ &= w_R(t)\Big(-M_0R^{-2} + 2^{-p}M_2R^{-(pq-1)(n+\tilde{\kappa})/(p+1)}\cdot\min\{w_R^{p-1}(t),\, w_R^{q-1}(t)\}\Big), \quad t>0,\end{aligned} \tag{4.7}$$

with $M_2 = \min\{M_1^{1-p},\, M_1^{1-q}\}$. Note that (4.1) implies

$$-\frac{(pq-1)(n+\tilde{\kappa})}{p+1} > -2,$$

while $w_R(0)$ is nondecreasing with respect to $R \in (0,+\infty)$ and

$$\sup\{w_R(0) : R>0\} > 0.$$

Therefore, there exists $R_2>0$ such that for any $R>R_2$,

$$M_0R^{-2} \leqslant 2^{-(p+1)}M_2R^{-(pq-1)(n+\tilde{\kappa})/(p+1)}\cdot\min\{w_R^{p-1}(0),\, w_R^{q-1}(0)\}. \tag{4.8}$$

Fix $R > \max\{R_0, R_1, R_2\}$. Then (4.7) and (4.8) yield

$$\frac{d}{dt}w_R(t) \geqslant 2^{-(p+1)}M_2R^{-(pq-1)(n+\tilde{\kappa})/(p+1)}\cdot\min\{w_R^p(t),\, w_R^q(t)\}, \quad t>0.$$

Since $p,\ q>1$, there exists $T_*>0$ such that

$$w_R(t) = \int_{\mathbb{R}^n}\big(u(x,t)+R^\theta v(x,t)\big)\,\phi_R(|x|)dx \to +\infty \quad \text{as} \quad t\to T_*^-.$$

From $\operatorname{supp}\phi_R(|x|) = \overline{B}_{\delta R}$, one gets

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} + \|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to +\infty \quad \text{as} \quad t\to T_*^-.$$

That is to say, (u,v) blows up in a finite time. □

Turn to the case $pq > (pq)_c$ with $0 \leqslant \kappa \leqslant +\infty$.

Theorem 4.2. *Assume that $0 \leqslant b \in C^1([0,+\infty))$ satisfies (1.4) with $0 \leqslant \kappa \leqslant +\infty$. Let $pq > (pq)_c$. Then there exist both nontrivial global and blow-up solutions to the problem (1.1)-(1.3).*

Proof. The comparison principle and Lemma 3.4 yield that the problem (1.1)-(1.3) with small initial data admits a nontrivial global solution . Let us show the existence of blow-up solution to the problem (1.1)-(1.3) with large initial data.

Fix $R > R_0$ with R_0 given in Lemma 3.1 and let (u, v) be the solution to the problem (1.1)-(1.3). Denote

$$\tilde{w}_R(t) = \int_{\mathbb{R}^n} (u(x,t) + v(x,t))\, \phi_R(|x|)dx, \quad t \geqslant 0.$$

It follows from Lemma 3.1 (the case $0 \leqslant \kappa < +\infty$), Remark 3.3 (the case $\kappa = +\infty$), the Hölder inequality and Lemma 3.6 that

$$\begin{aligned}
\frac{d}{dt}\tilde{w}_R(t) \geqslant& -M_0R^{-2}\tilde{w}_R(t) + \int_{\mathbb{R}^n} u^q(x,t)\phi_R(|x|)dx + \int_{\mathbb{R}^n} v^p(x,t)\phi_R(|x|)dx \\
\geqslant& -M_0R^{-2}\tilde{w}_R(t) + \Big(\int_{\mathbb{R}^n} \phi_R(|x|)dx\Big)^{1-q}\Big(\int_{\mathbb{R}^n} u(x,t)\phi_R(|x|)dx\Big)^q \\
&+ \Big(\int_{\mathbb{R}^n} \phi_R(|x|)dx\Big)^{1-p}\Big(\int_{\mathbb{R}^n} v(x,t)\phi_R(|x|)dx\Big)^p \\
\geqslant& -M_0R^{-2}\tilde{w}_R(t) + M_3\Big(\int_{\mathbb{R}^n} u(x,t)\phi_R(|x|)dx\Big)^q + M_3\Big(\int_{\mathbb{R}^n} v(x,t)\phi_R(|x|)dx\Big)^p \\
\geqslant& -M_0R^{-2}\tilde{w}_R(t) + 2^{-p}M_3\cdot\min\{\tilde{w}_R^p(t),\, \tilde{w}_R^q(t)\} \\
=&\tilde{w}_R(t)\left(-M_0R^{-2} + 2^{-p}M_3\cdot\min\{\tilde{w}_R^{p-1}(t),\, \tilde{w}_R^{q-1}(t)\}\right), \quad t > 0,
\end{aligned} \tag{4.9}$$

where ϕ_R and δ are mentioned in Lemma 3.1, and

$$M_3 = \min\left\{\Big(\int_{\mathbb{R}^n} \phi_R(|x|)dx\Big)^{1-q}, \Big(\int_{\mathbb{R}^n} \phi_R(|x|)dx\Big)^{1-p}\right\},$$

is a positive constant depended only on n, δ, p, q and R. If (u_0, v_0) is so large that

$$2^{-(p+1)}M_3\cdot\min\{\tilde{w}_R^{p-1}(0),\, \tilde{w}_R^{q-1}(0)\} \geqslant M_0R^{-2},$$

then (4.9) leads to

$$\frac{d}{dt}\tilde{w}_R(t) \geqslant 2^{-(p+1)}M_3\cdot\min\{\tilde{w}_R^p(t),\, \tilde{w}_R^q(t)\}, \quad t > 0.$$

By the same argument as in the end of the proof of Theorem 4.1, one can show that (u, v) must blow up in a finite time. □

Remark 4.3. Theorems 4.1 and 4.2 still hold if $b \in C^1([0, +\infty))$ is relaxed by that b is Lipschitz continuous in $[0, +\infty)$.

Remark 4.4. According to Remarks 3.2 and 3.5, Theorem 4.1 still holds if (1.4) is relaxed by

$$\overline{\lim_{s\to+\infty}}\, s^2b(s) = \kappa,$$

while Theorem 4.2 still holds if (1.4) is relaxed by

$$\underline{\lim_{s\to+\infty}}\, s^2b(s) = \kappa.$$

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