Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 5834–5845
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Stability analysis for a delayed SIR model with a nonlinear incidence rate

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Communicated by X. Z. Liu

Abstract

We develop an SIR vector-bone epidemic model incorporating incubation time delay and the nonlinear incidence rate, where the growth of susceptibles is governed by the logistic equation. The threshold parameter R_0 is used to determine whether the disease persists in the population. The model always has the trivial equilibrium and the disease-free equilibrium whereas admits the endemic equilibrium if R_0 exceeds one. The disease-free equilibrium is globally asymptotically stable if R_0 is less than one, while the system is persistent if R_0 is greater than one. Furthermore, by applying the time delay as a bifurcation parameter, the local stability of the endemic equilibrium is discussed and it loses stability and Hopf bifurcation occurs as the length of the time delay increases past τ_0 under certain conditions. An example is carried out to illustrate the main results.

Keywords: Stability analysis, delayed SIR model, nonlinear incidence rate, Lyapunov function, Hopf bifurcation.

2010 MSC: 34D23, 92D30.

1. Introduction

In recent years, numerous dynamic models of infectious diseases have received more and more attention. To better understand the transmission pattern of infectious diseases, a myriad of excellent results have been developed (see [1–6, 9–12, 14–17, 19–29] and references therein). Recently, based on the classical SIR epidemic model, Takeuchi et al. [19] formulated a delayed SIR epidemic model with bilinear incidence rate in order to investigate the spread of the vector-bone diseases, and McCluskey [15] studied the global stability of equilibria. Later, Wang et al. [24] analyzed the following SIR vector-bone disease model with incubation time delay and logistic growth rate with carrying capacity K:

$$\begin{aligned}
\frac{dS(t)}{dt} &= rS(t)\left(1-\frac{S(t)}{K}\right) - \beta S(t)I(t-\tau),\\
\frac{dI(t)}{dt} &= \beta S(t)I(t-\tau) - (\mu_1+\delta)I(t),\\
\frac{dR(t)}{dt} &= \delta I(t) - \mu_2 R(t).
\end{aligned} \tag{1.1}$$

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doi:10.22436/jnsa.010.11.21

Received 2016-07-15

$S(t)$, $I(t)$, and $R(t)$ are the numbers of susceptible, infective, and recovered host individuals at time t, respectively. r denotes the intrinsic birth rate. β denotes the average number of contacts per infective per unit time. τ is the incubation time. μ_1 and μ_2 represent the death rate of infective and recovered, respectively. δ is the recovered rate of infective individuals. All the parameters are positive constants. The dynamic properties of system (1.1) was established. More precisely, the disease-free equilibrium is globally asymptotically stable if the basic reproduction number $R_0 < 1$; while the unique endemic equilibrium is absolutely stable if $1 < R_0 < 3$, and it is conditionally stable when $R_0 > 3$. Moreover, Hopf bifurcation occurs under some conditions.

Enatsu et al. [4] extended system (1.1) and proposed the following vector-bone disease model with nonlinear incidence rate:

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(t)\left(1-\frac{S(t)}{K}\right) - \beta S(t)G(I(t-\tau)), \\ \frac{dI(t)}{dt} &= \beta S(t)G(I(t-\tau)) - (\mu_1+\delta)I(t), \\ \frac{dR(t)}{dt} &= \delta I(t) - \mu_2 R(t), \end{aligned} \tag{1.2}$$

where the parameters have the same biological meaning as that defined in model (1.1). The stability of model (1.2) was investigated. If R_0 is less than one, the disease-free equilibrium is globally asymptotically stable; while the unique endemic equilibrium may be absolutely stable or conditionally stable depending on the relationship between R_0 and one. Furthermore, the model (1.2) exhibits bifurcation properties as the length of the delay increases past a critical value provided that $1 < \overline{R}_0 < R_0$.

Liu [14] also considered another extended epidemic model based on the consideration of some biological meaning:

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(t)\left(1-\frac{S(t)}{K}\right) - \beta F(S(t))I(t-\tau), \\ \frac{dI(t)}{dt} &= \beta F(S(t))I(t-\tau) - (\mu_1+\delta)I(t), \\ \frac{dR(t)}{dt} &= \delta I(t) - \mu_2 R(t). \end{aligned} \tag{1.3}$$

Liu [14] discussed the dynamic behaviors of system (1.3). The trivial equilibrium is always unstable, and the disease-free equilibrium is stable if the basic reproduction number is less than one. Furthermore, the endemic equilibrium may lose its stability under certain conditions if the basic reproduction number is greater than one, which admits periodic behavior.

Inspired by the works of Wang et al. [24], Enatsu et al. [4], Zhang et al. [28], and Liu [14], it is reasonable to construct a more realistic disease model with the general nonlinear incidence rate of the form $\frac{\beta S(t)I(t-\tau)}{1+\alpha_1 S(t)+\alpha_2 I(t-\tau)}$. Then the delayed SIR vector-bone disease model can be written as

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(t)\left(1-\frac{S(t)}{K}\right) - \frac{\beta S(t)I(t-\tau)}{1+\alpha_1 S(t)+\alpha_2 I(t-\tau)}, \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t-\tau)}{1+\alpha_1 S(t)+\alpha_2 I(t-\tau)} - (\mu_1+\delta)I(t), \\ \frac{dR(t)}{dt} &= \delta I(t) - \mu_2 R(t), \end{aligned} \tag{1.4}$$

where α_1 and α_2 are constants in order to illustrate the saturation effects.

Notice that if $\alpha_1 = \alpha_2 = 0$, system (1.4) becomes the model investigated by Wang et al. [24]. If $\alpha_2 = 0$, the incidence rate becomes the saturated one [25, 28]. If $\alpha_1 = 0$, system (1.4) reduces to an example given by Enatsu et al. [4].

We introduce the non-dimensional quantities by writing

$$\tilde{S}(\tilde{t}) = \frac{S(t)}{K},\ \tilde{I}(\tilde{t}) = \frac{I(t)}{K},\ \tilde{R}(\tilde{t}) = \frac{R(t)}{K},\ \tilde{t} = \beta K t,\ \tilde{\tau} = \beta K \tau,$$

and

$$\tilde{r} = \frac{r}{\beta K},\ \tilde{\mu}_1 = \frac{\mu_1}{\beta K},\ \tilde{\delta} = \frac{\delta}{\beta K},\ \tilde{\mu}_2 = \frac{\mu_2}{\beta K},\ \tilde{\alpha_1} = K\alpha_1,\ \tilde{\alpha_2} = K\alpha_2,$$

which on substituting into (1.4) becomes

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(t)(1-S(t)) - \frac{S(t)I(t-\tau)}{1+\alpha_1 S(t) + \alpha_2 I(t-\tau)}, \\ \frac{dI(t)}{dt} &= \frac{S(t)I(t-\tau)}{1+\alpha_1 S(t) + \alpha_2 I(t-\tau)} - (\mu_1+\delta)I(t), \\ \frac{dR(t)}{dt} &= \delta I(t) - \mu_2 R(t), \end{aligned} \tag{1.5}$$

where, for notational simplicity, we have omitted the ˜ on all variables and parameters.

The initial conditions of system (1.5) take the following form

$$S(\theta) = \phi_1(\theta),\ I(\theta) = \phi_2(\theta),\ R(\theta) = \phi_3(\theta),\ \phi_i(0) > 0,\ i = 1,2,3, \tag{1.6}$$

where $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], \mathbb{R}^3_+)$, here $\mathbb{R}^3_+ = \{(x_1, x_2, x_3), x_i \geqslant 0, i = 1,2,3\}$. The fundamental theory of functional differential equations [7] implies that system (1.5) has a unique nonnegative solution $(S(t), I(t), R(t))$ satisfying the initial conditions (1.6). We define the basic reproduction number by

$$R_0 = \frac{1}{(\mu_1+\delta)(1+\alpha_1)}.$$

The rest of the paper is structured as follows. In Section 2, the stabilities of the trivial equilibrium and the disease-free equilibrium are described. In Section 3, we consider the permanence for system (1.5) when $R_0 > 1$. Section 4 deals with the existence and stability of the endemic equilibrium. A numerical example is given in Section 5 followed by a brief conclusion in Section 6.

2. Stabilities of the trivial equilibrium and the disease-free equilibrium

In this section, we are only concerned with the stabilities of the trivial equilibrium and the disease-free equilibrium when $R_0 < 1$. It is straightforward to see that $E_0(0,0,0)$ is a trivial equilibrium and $E_1(1,0,0)$ is a disease-free equilibrium from (1.5).

Theorem 2.1. *The trivial equilibrium* $E_0(0,0,0)$ *of system* (1.5) *is always unstable.*

Proof. For the equilibrium E_0, the characteristic equation is of the form

$$(\lambda+\mu_2)(\lambda-r)(\lambda+\mu_1+\delta) = 0. \tag{2.1}$$

It is easy to see that one of the eigenvalues in equation (2.1) is $\lambda = r > 0$. This in turn implies the equilibrium E_0 is unstable. □

Theorem 2.2. *If* $R_0 < 1$, *the disease-free equilibrium* $E_1(1,0,0)$ *of system* (1.5) *is globally asymptotically stable; while if* $R_0 > 1$, *the disease-free equilibrium* E_1 *of system* (1.5) *is unstable.*

Proof. First we assume that $R_0 < 1$. The characteristic equation about the equilibrium E_1 becomes as follows

$$(\lambda+\mu_2)(\lambda+r)(\lambda+\mu_1+\delta-\frac{e^{-\lambda\tau}}{1+\alpha_1}) = 0. \tag{2.2}$$

It is clear that the equation (2.2) has solutions $\lambda = -\mu_2 < 0$, $\lambda = -r < 0$ and the solution of the transcendental equation

$$\lambda + \mu_1 + \delta - \frac{e^{-\lambda\tau}}{1+\alpha_1} = 0.$$

Let $G(\lambda) = \lambda + \mu_1 + \delta - \frac{e^{-\lambda\tau}}{1+\alpha_1}$. Suppose $Re(\lambda) \geqslant 0$, then $G(\lambda) = 0$ implies

$$\begin{aligned} Re(\lambda) &= -(\mu_1+\delta) + \frac{1}{1+\alpha_1} e^{-Re(\lambda)\tau} \cos Im(\lambda)\tau \\ &= (\mu_1+\delta)[R_0 e^{-Re(\lambda)\tau} \cos Im(\lambda)\tau - 1] \\ &\leqslant (\mu_1+\delta)(R_0 e^{-Re(\lambda)\tau} - 1) \\ &\leqslant (\mu_1+\delta)(R_0 - 1), \end{aligned}$$

which is a contradiction because of the fact that $R_0 < 1$. Then if follows that E_1 is locally asymptotically stable.

An easy way to see the global attractivity of equilibrium E_1 when $R_0 < 1$ is to consider the following Lyapunov functional,

$$V(t) = \frac{1}{1+\alpha_1}(S(t) - 1 - \ln(S(t))) + (\mu_1+\delta)\int_{-\tau}^{0} I(t+\theta)d\theta + I(t).$$

Calculating the time derivative of $V(t)$ along with the solution of system (1.5), we obtain

$$\dot{V}(t)|_{(1.5)} = \frac{1}{1+\alpha_1}(1 - \frac{1}{S(t)})\dot{S}(t) + (\mu_1+\delta)I(t) - (\mu_1+\delta)I(t-\tau) + \dot{I}(t).$$

Using the first two equations of system (1.5) gives

$$\begin{aligned} \dot{V}(t)|_{(1.5)} &= \frac{-r(S(t)-1)^2}{1+\alpha_1} + \frac{I(t-\tau)(1+\alpha_1 S(t))}{(1+\alpha_1)(1+\alpha_1 S(t) + \alpha_2 I(t-\tau))} - (\mu_1+\delta)I(t-\tau) \\ &\leqslant \frac{-r(S(t)-1)^2}{1+\alpha_1} + \frac{I(t-\tau)}{(1+\alpha_1)} - (\mu_1+\delta)I(t-\tau) \\ &= \frac{-r(S(t)-1)^2}{1+\alpha_1} + (\mu_1+\delta)I(t-\tau)(R_0-1) \leqslant 0. \end{aligned}$$

Lyapunov-LaSalle asymptotic stability theorem implies $\lim_{t\to\infty} S(t) = 1$ if $R_0 < 1$. By the second and third equations of system (1.5), it follows from $\lim_{t\to\infty} S(t) = 1$ that $\lim_{t\to\infty} I(t) = 0$ and $\lim_{t\to\infty} R(t) = 0$. Thus equilibrium E_1 is globally asymptotically stable if $R_0 < 1$.

Next we focus on the instability of equilibrium E_1 when $R_0 > 1$. If $R_0 > 1$, then $G(0) = (\mu_1+\delta)(1 - R_0) < 0$. When λ tends to infinity, $G(\lambda)$ approaches infinity as well. Then $G(\lambda) = 0$ has at least one positive solution. Hence E_1 is unstable. □

3. Permanence

In this section, the permanence for system (1.5) is obtained. Before the main results are established, we give the following lemmas first.

Lemma 3.1 ([13]). *Consider the following equation*

$$u'(t) = au(t-\tau) - bu(t),$$

where $a, b, \tau > 0$ *and* $u(t) > 0$ *for* $-\tau \leqslant t \leqslant 0$. *We have*

(i) *if* $a < b$, *then* $\lim_{t\to\infty} u(t) = 0$;

(ii) *if* $a > b$, *then* $\lim_{t\to\infty} u(t) = +\infty$.

Lemma 3.2. *All the solutions of system* (1.5) *satisfying conditions* (1.6) *are always nonnegative and ultimately bounded.*

Proof. By the first equation of the system (1.5), we have $\dot{S}(t)\leqslant r(1-S(t))S(t)$, which implies

$$\limsup_{t\to\infty} S(t)\leqslant 1.$$

Then for sufficiently large t, adding the equations for system (1.5) yields

$$\begin{aligned}\frac{d(S(t)+I(t)+R(t))}{dt} &= rS(t)(1-S(t)) - \mu_1 I(t) - \mu_2 R(t)\\ &\leqslant rS(t) - \mu_1 I(t) - \mu_2 R(t)\\ &= (r+1)S(t) - S(t) - \mu_1 I(t) - \mu_2 R(t)\\ &\leqslant (r+1)S(t) - \mu_m(S(t)+I(t)+R(t))\\ &\leqslant (r+1) - \mu_m(S(t)+I(t)+R(t)).\end{aligned}$$

Then we have $\limsup_{t\to\infty}(S(t)+I(t)+R(t))\leqslant\frac{r+1}{\mu_m}$. Then all the solutions of system (1.5) are ultimately bounded, which completes the proof. □

By the third equation of system (1.5), it is easy to get Lemma 3.3.

Lemma 3.3. *Permanence of* $S(t)$, $I(t)$ *in system* (1.5) *implies that of* $R(t)$.

Before introducing the main theorem, some notations and a lemma are presented firstly from [8]. Let $X := C^+([-\tau, 0], \mathbb{R}^2_+)$ be the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^2_+$. Define

$$\begin{aligned}X_1 &= \{(\phi_1, \phi_2)\in X : \phi_1(\theta) = 0, \theta\in[-\tau, 0]\},\\ X_2 &= \{(\phi_1, \phi_2)\in X : \phi_1(\theta) > 0, \phi_2(\theta) = 0, \theta\in[-\tau, 0]\}.\end{aligned}$$

Denote $X_0 = X_1\cup X_2$, $X^0 = X/X_0$. Denote $T(t)$ for $t \geqslant 0$ as the solution operators corresponding to system (1.5). The ω-limit set is defined as $\omega(x):=\{y\in X :$ there is a sequence $t_n\to\infty$ as $n\to\infty$ such that $T(t_n)x\to y$ as $n\to\infty\}$. Then we have the following lemma.

Lemma 3.4. *Suppose we have the following:*

(i) *the solution operators* $T(t)$ *satisfy* $T(t) : X^0\to X^0$; $T(t) : X_0\to X_0$;

(ii) X^0 *is open and dense in* X *with* $X = X^0\cup X_0$ *and* $X = X^0\cap X_0 = \emptyset$;

(iii) $T(t)$ *is asymptotically smooth;*

(iv) $T(t)$ *is point dissipative in* X;

(v) $\gamma^+(U)$ *is bounded in* X *if* U *is bounded in* X, *where* $\gamma^+(x)$ *is the positive orbit through* x;

(vi) $\Omega = \bigcup_{x\in Y}\omega(x)$ *is isolated and has an acyclic covering* M, *where* Y *is the global attractor of* $T(t)$ *restricted to* X_0 *and* $M = \cup_{i=1}^k M_i$;

(vii) *for each* $M_i\in M$, $W^s(M_i)\cap X^0 = \emptyset$, *where* W^s *refers to the stable set.*

Then $T(t)$ *is uniformly persistent, i.e., there is an* $\eta > 0$ *such that for any* $x\in X^0$, $\liminf_{t\to\infty} d(T(t)x, X_0)\geqslant\eta$.

Theorem 3.5. *If* $R_0 > 1$, *there exists an* $\eta > 0$ *such that any solution* $(S(t), I(t))$ *of system* (3.1) *with initial value* $(S_0, I_0) \in X^0$ *satisfies* $\liminf_{t\to\infty} S(t) > \eta$, *and* $\liminf_{t\to\infty} I(t) > \eta$.

Proof. Obviously, it is sufficient to consider the following subsystem of system (1.5) and the persistence of $(S(t), I(t))$ for system (3.1) when $R_0 > 1$:

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(t)(1-S(t)) - \frac{S(t)I(t-\tau)}{1+\alpha_1 S(t)+\alpha_2 I(t-\tau)}, \\ \frac{dI(t)}{dt} &= \frac{S(t)I(t-\tau)}{1+\alpha_1 S(t)+\alpha_2 I(t-\tau)} - (\mu_1+\delta)I(t), \end{aligned} \tag{3.1}$$

where $S(\theta)$, $I(\theta) \geqslant 0$ are continuous on the interval $[-\tau, 0]$, and $S(0)$, $I(0) > 0$.

By Lemma 3.2, it is straightforward to see that (i)-(v) of Lemma 3.4 always hold. Thus, we only need to verify the conditions (vi) and (vii). To do this, set

$$M_\partial := \{\phi \in X_0 : \ T(t)\phi \text{ satisfies system (3.1) and } T(t)\phi \in X_0, \forall t \geqslant 0\}.$$

We first show that

$$M_\partial = \{\tilde{E_0}, \tilde{E_1}\}.$$

System (3.1) has two equilibria in X_0: $\tilde{E_0} \in X_1$, $\tilde{E_1} \in X_2$ with

$$\begin{aligned} \tilde{E_0} &= \{(\phi_1, \phi_2) \in X : \phi_1(\theta) \equiv \phi_2(\theta) \equiv 0, \theta \in [-\tau, 0]\}, \\ \tilde{E_1} &= \{(\phi_1, \phi_2) \in X : \phi_1(\theta) \equiv 1, \phi_2(\theta) \equiv 0, \theta \in [-\tau, 0]\}, \end{aligned}$$

and we get $\dot{S}(t)|_{(\phi_1,\phi_2)\in X_1} \equiv 0$, then we obtain $S(t)|_{(\phi_1,\phi_2)\in X_1} \equiv 0$ for any $t \geqslant 0$. The second equation of system (3.1) implies $\dot{I}(t)|_{(\phi_1,\phi_2)\in X_1} = -(\mu_1+\delta)I(t) \leqslant 0$, hence all the points in X_1 tend to $\tilde{E_0}$, that is to say, $X_1 = W^s(\tilde{E_0})$. Similarly, we have all the points in X_2 approach $\tilde{E_1}$, that is to say, $X_2 = W^s(\tilde{E_1})$. Those show that the invariant sets $\tilde{E_0}$ and $\tilde{E_1}$ are isolated, then $\{\tilde{E_0}, \tilde{E_1}\}$ is isolated and is an acyclic covering. Let $\Omega = \bigcup_{x\in Y} \omega(x)$, where Y is the global attractor of $T(t)$ restricted to X_0. It is easy to see that $\Omega = \{\tilde{E_0}, \tilde{E_1}\}$. Therefore, the condition (vi) of Lemma 3.4 is satisfied.

Next, we show that $W^s(\tilde{E_i}) \cap X^0 = \emptyset$, $i = 0, 1$. It is sufficient to prove that $W^s(\tilde{E_1}) \cap X^0 = \emptyset$ because the proof for $W^s(\tilde{E_0}) \cap X^0 = \emptyset$ is analogous.

Suppose that on the contrary, i.e., $W^s(\tilde{E_1}) \cap X^0 \neq \emptyset$, then there is a positive solution $(S(t), I(t)) \in X^0$ of system (3.1) with $\lim_{t\to\infty} (S(t), I(t)) = (1, 0)$. Therefore there exists a sufficiently small $\varepsilon > 0$ and a positive constant $T = T(\varepsilon)$ such that

$$S(t) > 1-\varepsilon > 0, \ 0 < I(t) < \varepsilon, \ \forall t \geqslant T, \quad \mu_1+\delta < \frac{1-\varepsilon}{1+\alpha_1(1-\varepsilon)+\alpha_2\varepsilon}$$

because of $R_0 = \frac{1}{(\mu_1+\delta)(1+\alpha_1)} > 1$.

By the second equation of (3.1), we have

$$\dot{I}(t) \geqslant \frac{1-\varepsilon}{1+\alpha_1(1-\varepsilon)+\alpha_2\varepsilon} I(t-\tau) - (\mu_1+\delta)I(t), \ \forall t \geqslant T+\tau. \tag{3.2}$$

Now consider the following comparison system

$$\begin{aligned} \dot{z}(t) &= \frac{1-\varepsilon}{1+\alpha_1(1-\varepsilon)+\alpha_2\varepsilon} z(t-\tau) - (\mu_1+\delta)z(t), \ \forall t \geqslant T+\tau, \\ z(t) &= I(t) > 0 \ \text{ for } T \leqslant t \leqslant T+\tau. \end{aligned} \tag{3.3}$$

By applying Lemma 3.1 and the second equation of (3.3), we get $\lim_{t\to\infty} z(t) = +\infty$. System (3.2) together with the comparison principle [18] implies $I(t) \geqslant z(t)$ for all $t \geqslant T$. Hence, $\lim_{t\to\infty} I(t) = +\infty$, which contradicts $I(t) < \varepsilon$ for all $t \geqslant T+\tau$. Then $W^s(\tilde{E_1}) \cap X^0 = \emptyset$ holds. By applying Lemma 3.4 we obtain that for some constant $\eta > 0$

$$\liminf_{t\to\infty} S(t) > \eta, \text{ and } \liminf_{t\to\infty} I(t) > \eta,$$

which completes the proof of Theorem 3.5. □

4. The uniqueness and stability of the endemic equilibrium

In this section, we are about to concentrate particularly on the uniqueness and stability of the endemic equilibrium, and Hopf bifurcation as well when $R_0 > 1$.

Theorem 4.1. *If* $R_0 > 1$, *system* (1.5) *admits exactly one endemic equilibrium* $E_*(S^*, I^*, R^*)$, *where*

$$S^* = \frac{(\mu_1+\delta)\alpha_1 + \alpha_2 r - 1 + \sqrt{[(\mu_1+\delta)\alpha_1+\alpha_2 r - 1]^2 + 4\alpha_2 r(\mu_1+\delta)}}{2\alpha_2 r},\ 0 < S^* < 1,$$
$$I^* = \frac{rS^*(1-S^*)}{\mu_1+\delta},\ R^* = \frac{\delta}{\mu_2}I^*.$$

Proof. At the endemic equilibrium E_*, it follows from the first equation of the system (1.5) that

$$r(1-S^*) = \frac{I^*}{1+\alpha_1 S^* + \alpha_2 I^*}. \tag{4.1}$$

The second equation of the system (1.5) gives

$$\mu_1 + \delta = \frac{S^*}{1+\alpha_1 S^* + \alpha_2 I^*}. \tag{4.2}$$

Equation (4.1) divided by equation (4.2) yields

$$I^* = \frac{rS^*(1-S^*)}{\mu_1+\delta}, \tag{4.3}$$

which on substituting into the equation (4.2) gives S^* as solutions of the quadratic equation

$$\alpha_2 r(S^*)^2 + (1-\alpha_1(\mu_1+\delta)-\alpha_2 r)S^* - (\mu_1+\delta) = 0. \tag{4.4}$$

Let $H(S) = \alpha_2 rS^2 + (1-\alpha_1(\mu_1+\delta)-\alpha_2 r)S - (\mu_1+\delta)$. It is a straightforward matter to calculate $H(0) = -(\mu_1+\delta) < 0$, then equation $H(S) = 0$ has one negative real root and one positive real root. Observe that $H(1) = (\alpha_1+1)(\mu_1+\delta)(R_0-1) > 0$ since $R_0 > 1$. Therefore $H(S) = 0$ has exactly one positive solution $S^* \in (0,1)$. To be specific, S^* satisfies

$$S^* = \frac{(\mu_1+\delta)\alpha_1 + \alpha_2 r - 1 + \sqrt{[(\mu_1+\delta)\alpha_1+\alpha_2 r - 1]^2 + 4\alpha_2 r(\mu_1+\delta)}}{2\alpha_2 r}.$$

It is not difficult to compute the expression R^* from system (1.5) at the endemic equilibrium E_*. □

The characteristic equation at endemic equilibrium $E_*(S^*, I^*, R^*)$ can be turned into

$$(\lambda+\mu_2)[\lambda^2 + a\lambda + b - e^{-\lambda\tau}(c\lambda+d)] = 0,$$

where

$$\begin{aligned}
a &= \mu_1 + \delta - r(1-2S^*) + \frac{I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2},\\
b &= (\mu_1+\delta)[\frac{I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2} - r(1-2S^*)],\\
c &= \frac{S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2},\quad d = -r(1-2S^*)\frac{S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}.
\end{aligned} \tag{4.5}$$

Then the eigenvalues at E_* are $-\mu_2$ and the solutions of the following equation:

$$\lambda^2 + a\lambda + b - e^{-\lambda\tau}(c\lambda+d) = 0. \tag{4.6}$$

Proposition 4.2. *Assume* $R_0 > 1$ *and* $b + d \geqslant 0$*, then all the solutions of equation* (4.6) *have negative real part for* $\tau = 0$.

Proof. If incubation time delay $\tau = 0$, equation (4.6) yields

$$\lambda^2 + (a - c)\lambda + (b - d) = 0.$$

It follows from equations (4.2) and (4.3) that

$$\begin{aligned} b - d &= (\mu_1 + \delta)[\frac{I^*(1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} - r(1 - 2S^*)] + r(1 - 2S^*)\frac{S^*(1 + \alpha_1 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} \\ &= \frac{rS^*(1 - S^* + \alpha_2 S^* I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} > 0, \\ a - c &= (\mu_1 + \delta) - r(1 - 2S^*) + \frac{I^*(1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} - \frac{S^*(1 + \alpha_1 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} \\ &= \frac{I^*(1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} - [r(1 - 2S^*) - \frac{(\mu_1 + \delta)\alpha_2 I^*}{1 + \alpha_1 S^* + \alpha_2 I^*}], \\ b + d &= (\mu_1 + \delta)[\frac{I^*(1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} - r(1 - 2S^*)] - r(1 - 2S^*)\frac{S^*(1 + \alpha_1 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} \\ &= (\mu_1 + \delta)[\frac{I^*(1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} - r(1 - 2S^*)\frac{(2 + 2\alpha_1 S^* + \alpha_2 I^*)}{1 + \alpha_1 S^* + \alpha_2 I^*}]. \end{aligned}$$

Next we pay attention to the following scenarios: (I) $S^* \geqslant 0.5$ and (II) $S^* < 0.5$. In case (I), from the above-mentioned equations, we clearly have $b + d > 0$ and $a - c > 0$. In case (II), if $b + d \geqslant 0$, then we have

$$\frac{I^*(1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} \geqslant r(1 - 2S^*)\frac{(2 + 2\alpha_1 S^* + \alpha_2 I^*)}{1 + \alpha_1 S^* + \alpha_2 I^*}.$$

It is easy to see that

$$r(1 - 2S^*)\frac{(2 + 2\alpha_1 S^* + \alpha_2 I^*)}{1 + \alpha_1 S^* + \alpha_2 I^*} > r(1 - 2S^*) > r(1 - 2S^*) - \frac{(\mu_1 + \delta)\alpha_2 I^*}{1 + \alpha_1 S^* + \alpha_2 I^*}.$$

Therefore, we get

$$\frac{I^*(1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2} > r(1 - 2S^*) - \frac{(\mu_1 + \delta)\alpha_2 I^*}{1 + \alpha_1 S^* + \alpha_2 I^*},$$

which proves the theorem. □

Proposition 4.3. *Assume* $R_0 > 1$*, then the following statements hold.*

(i) *If* $(1 + 3\alpha_1(\mu_1 + \delta) - \alpha_2 r)S^* + (\mu_1 + \delta)(2 - \alpha_1) + \alpha_2 r - 1 \geqslant 0$*, then all the solutions of equation* (4.6) *have negative real part for* $\tau > 0$.

(ii) *If* $(1 + 3\alpha_1(\mu_1 + \delta) - \alpha_2 r)S^* + (\mu_1 + \delta)(2 - \alpha_1) + \alpha_2 r - 1 < 0$*, then there exists a monotone increasing sequence* $\{\tau_n\}_{n=0}^{\infty}$ *with* $\tau_0 > 0$ *such that equation* (4.6) *has a pair of imaginary roots for* $\tau = \tau_n (n = 0, 1, 2, \cdots)$.

Proof. Suppose that $\lambda = i\omega$ $(\omega > 0)$ is a solution of equation (4.6). We substitute $\lambda = i\omega$ into equation (4.6) to get

$$-\omega^2 + ia\omega + b - (\cos\omega\tau - i\sin\omega\tau)(ic\omega + d) = 0.$$

Equating real and imaginary parts gives

$$-\omega^2 + b = d\cos\omega\tau + c\omega\sin\omega\tau, \quad a\omega = c\omega\cos\omega\tau - d\sin\omega\tau. \tag{4.7}$$

Squaring and adding both equations in equation (4.7), we obtain

$$\omega^4 + (a^2 - 2b - c^2)\omega^2 + b^2 - d^2 = 0. \tag{4.8}$$

By equation (4.5), we get

$$\begin{aligned} a^2 - 2b - c^2 =& \frac{(\mu_1+\delta)\alpha_2 I^*}{1+\alpha_1 S^* + \alpha_2 I^*}((\mu_1+\delta) + \frac{S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2}) \\ &+ (r(1-2S^*) - \frac{I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2})^2 > 0, \end{aligned}$$

and

$$\begin{aligned} b - d &= \frac{rS^*(1-S^*+\alpha_2 S^* I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2} > 0, \\ b + d &= (\mu_1+\delta)[\frac{I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^*+\alpha_2 I^*)^2} - r(1-2S^*)\frac{(2+2\alpha_1 S^*+\alpha_2 I^*)}{1+\alpha_1 S^*+\alpha_2 I^*}]. \end{aligned}$$

Substitution of equations (4.1), (4.2), and (4.4) into above-mentioned equation gives

$$b + d = \frac{rS^*[(1+3\alpha_1(\mu_1+\delta) - r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1) + r\alpha_2 - 1]}{1+\alpha_1 S^*+\alpha_2 I^*}.$$

Firstly assume that $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1) + r\alpha_2 - 1 \geqslant 0$. Then we arrive at $a^2 - 2b - c^2 > 0$ and $b + d \geqslant 0$. That is to say, equation (4.8) has no positive real root ω, which is a contradiction. Therefore, all the roots of equation (4.6) have negative real part for $\tau > 0$. The first part of the proof is completed.

Secondly suppose $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1) + r\alpha_2 - 1 < 0$, which indicates $b + d < 0$. Therefore, there exists a unique positive real root ω_0 satisfying (4.8), where

$$\omega_0 = \sqrt{\frac{\sqrt{(a^2-2b-c^2)^2 - 4(b-d)(b+d)} - (a^2-2b-c^2)}{2}}.$$

It should be noted that $\lambda = -i\omega_0$ is also a root of equation (4.6). Then equation (4.6) has a single pair of purely imaginary roots $\pm i\omega_0$. Then using equation (4.7), we obtain

$$(ac-d)\omega_0^2 + bd = (c^2\omega_0^2 + d^2)\cos\omega_0\tau,$$

and it follows that

$$\tau_n = \frac{1}{\omega_0}\arccos\frac{(ac-d)\omega_0^2+bd}{c^2\omega_0^2+d^2} + \frac{2n\pi}{\omega_0}, n = 0, 1, 2, \cdots. \tag{4.9}$$

This completes the proof of the proposition. □

We give the following proposition without any proof since the proof is similar to that of paper [4].

Proposition 4.4. *If* $R_0 > 1$ *and* $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1) + r\alpha_2 - 1 < 0$, *then the transversality condition*

$$\frac{dRe(\lambda(\tau))}{d\tau}|_{\lambda=i\omega_0} > 0.$$

Summarizing the above propositions, we obtain the following theorem.

Theorem 4.5. *Assume* $R_0 > 1$, *then the following statements hold.*

(i) *If* $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1) + r\alpha_2 - 1 \geqslant 0$, *then the endemic equilibrium of system* (1.5) *is locally asymptotically stable for* $\tau \geqslant 0$.

(ii) *If* $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^*+(\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1<0$, *then the endemic equilibrium of system* (1.5) *is locally asymptotically stable for* $0 \leqslant \tau < \tau_0$ *and unstable for* $\tau > \tau_0$.

Remark 4.6. If both $R_0 > 1$ and $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^*+(\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1<0$ hold true, system (1.5) undergoes Hopf bifurcation at the endemic equilibrium E_* when τ crosses $\tau_n (n = 0, 1, \cdots)$.

5. An example

In this section, we consider the numerical results of system (1.5). In system (1.5), we set $r = \mu_1 = \mu_2 = \alpha_1 = \alpha_2 = 0.1$. If we choose $\delta = 0.1$, the endemic equilibrium of system (1.5) is $E_*(0.2057,\ 0.0817,\ 0.0817)$, $R_0 = 4.54555$, and $\tau_0 = 2.2224$ by applying (4.9). It should also be noted that $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^*+(\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1 = -0.3940 < 0$, which implies the endemic equilibrium E_* is conditionally stable. Furthermore, We can see that the endemic equilibrium E_* is asymptotically stable if time delay $\tau = 0.5 < \tau_0 = 2.2224$ (see Figure 1), while the endemic equilibrium E_* loses its stability, Hopf bifurcation occurs, and system (1.5) exhibits a stable period solution if $\tau = 2.5 > \tau_0$ (see Figure 2).

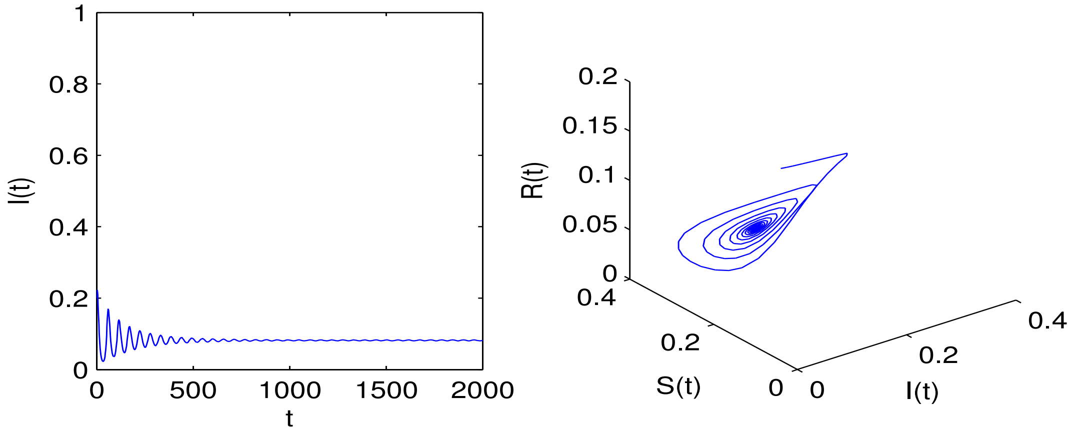

Figure 1: Temporal behavior of the infective individuals and corresponding three-dimensional phase for system (1.5) with $R_0 = 4.5455$ and $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^*+(\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1 = -0.3940 < 0$. The initial conditions are set to be $\phi_1(\theta) = 0.3$, $\phi_2(\theta) = 0.2$, $\phi_3(\theta) = 0.1$, $\theta\in[-\tau, 0]$, and $\tau = 0.5 < \tau_0 = 2.2224$.

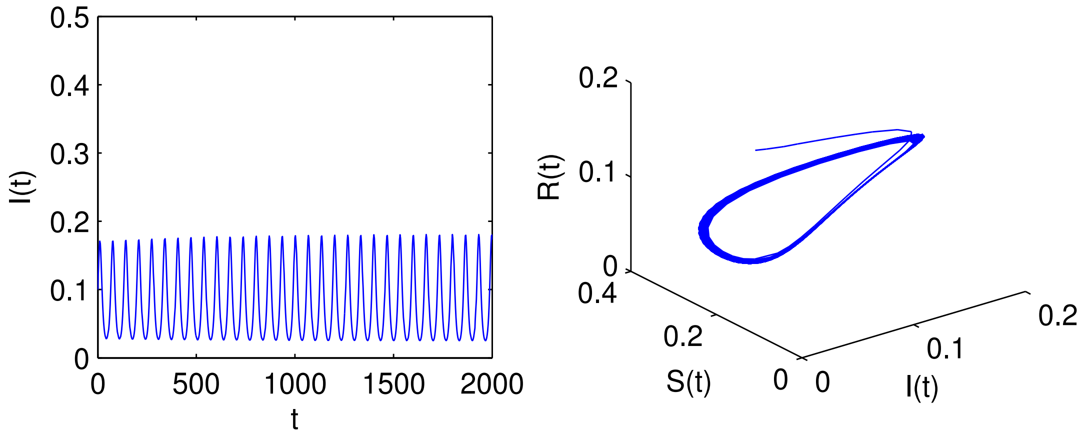

Figure 2: Temporal behavior of the infective individuals and corresponding three-dimensional phase for system (1.5) with $R_0 = 4.5455$ and $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^*+(\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1 = -0.3940 < 0$. The initial conditions are set to be $\phi_1(\theta) = 0.37$, $\phi_2(\theta) = 0.1$, $\phi_3(\theta) = 0.1$, $\theta\in[-\tau, 0]$, and $\tau = 2.5 > \tau_0 = 2.2224$.

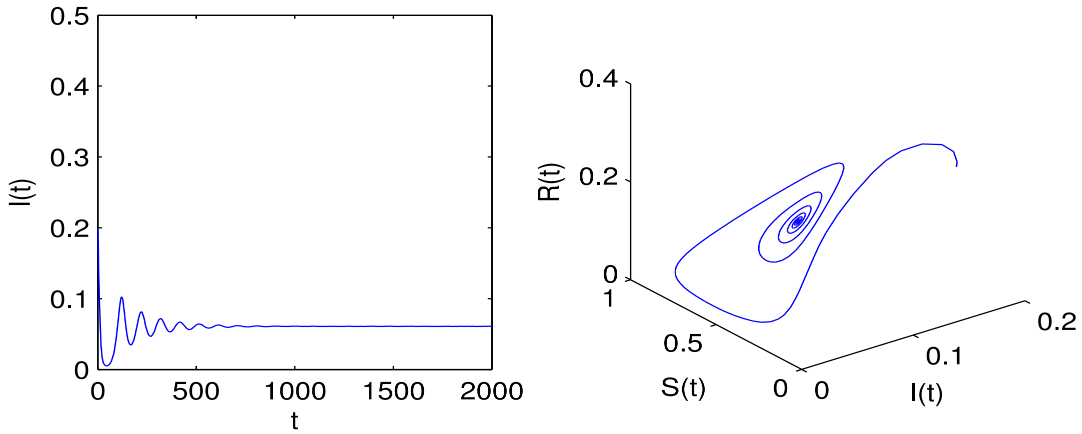

Figure 3: Temporal behavior of the infective individuals and corresponding three-dimensional phase for system (1.5) with $R_0 = 2.2727$ and $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1 = 0.2353 > 0$. The initial conditions are set to be $\phi_1(\theta) = 0.4$, $\phi_2(\theta) = 0.2$, $\phi_3(\theta) = 0.2$, $\theta\in[-\tau, 0]$, and $\tau = 10 > \tau_0 = 2.2224$. The values of parameters are as those in Figure 1 but $\delta = 0.3$.

If δ is chosen as 0.3 and other parameters are set as those in Figure 1, then the endemic equilibrium is $E_*(0.4192,\ 0.0609,\ 0.1826)$, $R_0 = 2.2727$, and $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1 = 0.2353 > 0$, which imply the condition (i) of Theorem 4.5 is satisfied. Moreover, from Figure 3, we can see the endemic equilibrium E_* is globally asymptotically stable although $\tau = 10 > \tau_0$.

6. Conclusion

In this paper a delayed SIR vector-bone disease model with incubation time delay is established, in which the growth of population follows the logistic model in the absence of disease and the more general form of the nonlinear incidence rate is considered. The stability of the equilibria has been discussed by analyzing the roots of characteristic equations and constructing the suitable Lyapunov functional. It is shown that the trivial equilibrium is always unstable. The stability of the disease-free equilibrium is completely determined by the threshold parameter R_0: the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$ while it is unstable if $R_0 > 1$. Moreover, if $R_0 > 1$, there exists a unique endemic equilibrium E_*. It is found that $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1 = 0$ is the condition which determines the absolute stability or conditional stability of the endemic equilibrium. To be specific, the endemic equilibrium is absolutely stable if $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1\geqslant 0$ holds true while it is conditionally stable if the above-mentioned formula is violated. That is to say, there is a certain threshold time value τ_0 such that the endemic equilibrium is locally asymptotically stable when $0 < \tau < \tau_0$ whereas it is unstable when $\tau > \tau_0$. Furthermore, it is worth noting that, if $R_0 > 1$ and $(1+3\alpha_1(\mu_1+\delta)-r\alpha_2)S^* + (\mu_1+\delta)(2-\alpha_1)+r\alpha_2-1 < 0$, the system exhibits Hoph bifurcation when time delay τ crosses $\tau_n (n = 0, 1, \cdots)$.

Acknowledgment

This work is supported in part by the National Nature Science Foundation of China (NSFC11101127 and 11301543), the Foundation for Leaders of Disciplines in Science of Haust (13490002).

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