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# Infinitely many periodic solutions for second-order discrete Hamiltonian systems

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## Abstract

Infinitely many periodic solutions are obtained for a second-order discrete Hamiltonian systems by using the minimax methods in critical point theory. Our results extend and improve previously known results. 

Keywords: Minimax methods, periodic solutions, sublinear, discrete Hamiltonian systems, critical point.
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## 1. Introduction

Consider the following second order discrete Hamiltonian system

$$\begin{cases} \triangle^2 u(t-1) + \nabla F(t,u(t)) = 0, & t \in \mathbb{Z}[1,T], \\ u(0) = u(T), \end{cases} \tag{1.1}$$

where $T \in \mathbb{Z}$, $\mathbb{Z}[1,T]$ denotes the discrete interval $\{1,2,\cdots,T\}$, $\triangle u(t) = u(t+1) - u(t)$, $\triangle^2 u(t) = \triangle(\triangle u(t))$ and $\nabla F(t,x)$ denotes the gradient of $F$ with respect to the second variable. $F$ satisfies the following assumption:

(A) $F(t,x) \in C^1(\mathbb{R}^N,\mathbb{R})$ for any $t \in \mathbb{Z}[0,T]$ and $F$ is T-periodic in the first variable.

Since Guo and Yu developed a new method to study the existence and multiplicity of periodic solutions of difference equations by using critical point theory (see [4–6, 18], the existence and multiplicity of periodic solutions for problem (1.1) have been extensively studied and lots of interesting results have been worked out, see [1–3, 7, 8, 10–17] and the references therein. In particular, when the nonlinearity $\nabla F(t,x)$ is bounded, that is, there exists $M > 0$ such that $|\nabla F(t,x)| \leqslant M$ for all $(t,x) \in \mathbb{Z}[0,T] \times \mathbb{R}^N$, and that

$$\sum_{t=0}^{T} F(t,x) \to +\infty \text{ as } |x| \to \infty.$$

Guo and Yu [6] obtained one periodic solution to problem (1.1).

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In [12, 13], Xue and Tang generalized these results to the sublinear case:

$$|\nabla F(t,x)| \leqslant M_1|x|^{\alpha} + M_2, \quad \forall\ (t,x) \in \mathbb{Z}[0,T] \times \mathbb{R}^N,$$

and

$$|x|^{-2\alpha} \sum_{t=0}^{T} F(t,x) \to \pm\infty \text{ as } |x| \to \infty,$$

where $M_1 > 0$, $M_2 > 0$ and $\alpha \in [0,1)$.

In [10], Tang and Zhang considered the nonlinearity $\nabla F(t,x)$ satisfies the following condition:

$$|\nabla F(t,x)| \leqslant f(t)|x|^{\alpha} + g(t), \quad \forall\ (t,x) \in \mathbb{Z}[0,T] \times \mathbb{R}^N, \tag{1.2}$$

or

$$|\nabla F(t,x)| \leqslant f(t)|x| + g(t), \quad \forall\ (t,x) \in \mathbb{Z}[0,T] \times \mathbb{R}^N, \tag{1.3}$$

where $f, g : \mathbb{Z}[0,T] \to \mathbb{R}^+$, $\alpha \in (0,1)$. Under these conditions, periodic solutions of problem (1.1) have been obtained, which completed and extended the results in [12, 13].

Recently, Che and Xue [1] obtained infinitely many periodic solutions for problem (1.1) when (1.2) holds, and

$$\limsup_{r\to+\infty} \inf_{x\in\mathbb{R}^N, |x|=r} \sum_{t=0}^{T} F(t,x) = +\infty, \tag{1.4}$$

and

$$\liminf_{R\to+\infty} \sup_{x\in\mathbb{R}^N, |x|=R} |x|^{-2\alpha} \sum_{t=0}^{T} F(t,x) = -\infty, \tag{1.5}$$

where $\alpha \in (0,1)$.

In this paper, motivated by the results mentioned above, we will further investigate infinitely many periodic solutions to the problem (1.1) under conditions (1.2) or (1.3).

Let $H_T$ be a Hilbert space defined by

$$H_T = \{u : \mathbb{Z} \to \mathbb{R}^N |\ u(t) = u(t+T),\ \forall t \in \mathbb{Z}\},$$

with the inner product

$$\langle u, v\rangle = \sum_{t=0}^{T} (u(t), v(t)),$$

and the norm

$$\|u\| = \left(\sum_{t=0}^{T} |u(t)|^2\right)^{\frac{1}{2}}.$$

Let

$$\|u\|_\infty = \max_{t\in\mathbb{Z}[0,T]} |u(t)|.$$

Since $H_T$ is finite dimensional, one has that:

$$\frac{1}{\sqrt{T}}\|u\| \leqslant \|u\|_\infty \leqslant \|u\|.$$

Let

$$\Phi(u) = \frac{1}{2}\sum_{t=0}^{T} |\triangle u(t)|^2 - \sum_{t=0}^{T} F(t, u(t)), \quad \forall u \in H_T.$$

It is well-known that the solutions of problem (1.1) correspond to the critical points of $\Phi$ (see [9]).

**Lemma 1.1** ([14]). *As a subspace of* $H_T$, $N_k$ *is defined by*

$$N_k = \{u \in H_T | -\triangle^2 u(t-1) = \lambda_k u(t)\},$$

*where* $\lambda_k = 2 - 2\cos k\omega$, $\omega = \frac{2\pi}{T}$, $k \in \mathbb{Z}[0, [\frac{T}{2}]]$ *(where* $[c]$ *denotes the largest integer less than* $c$*). Then we have*

(1) $N_k \perp N_j$ *for* $k \neq j$ *and* $j, k \in \mathbb{Z}[0, [\frac{T}{2}]]$.

(2) $H_T = \oplus_{k=0}^{[\frac{T}{2}]} N_k$.

*Set* $H_1 = N_0$ *and* $H_2 = \oplus_{k=1}^{[\frac{T}{2}]} N_k$. *Then* $H_T = H_1 \oplus H_2$ *and*

$$\sum_{t=0}^{T} |\triangle u(t)|^2 \geqslant \lambda_1 \|u\|, \quad \forall\, u \in H_2.$$

The element $u$ of $H_1$ is just the eigenvector corresponding to $\lambda_0 = 0$ which satisfies $u(t) \equiv u(0)$ for $t \in \mathbb{Z}[0, T]$.

Our main results are the following theorems.

**Theorem 1.2.** *Suppose that* (A), (1.2) *and* (1.4) *hold, and*

$$\liminf_{r \to +\infty} \sup_{x \in \mathbb{R}^N, |x| = r} |x|^{-2\alpha} \sum_{t=0}^{T} F(t, x) < -\frac{\left(\sum_{t=0}^{T} f(t)\right)^2}{2\lambda_1}. \tag{1.6}$$

*Then*

(i) *the problem* (1.1) *has infinitely many periodic solutions* $\{u_n\}$ *such that* $\Phi(u_n) \to +\infty$ *as* $n \to \infty$*;*

(ii) *the problem* (1.1) *has infinitely many periodic solutions* $\{u_m^*\}$ *such that* $\Phi(u_m^*) \to -\infty$ *as* $m \to \infty$.

**Theorem 1.3.** *Suppose that* (A), (1.3) *with* $\sum_{t=0}^{T} f(t) < \frac{\lambda_1}{4}$ *and* (1.4) *hold, and*

$$\liminf_{r \to +\infty} \sup_{x \in \mathbb{R}^N, |x| = r} |x|^{-2} \sum_{t=0}^{T} F(t, x) < -\frac{\left(\sum_{t=0}^{T} f(t)\right)^2}{2\left(\lambda_1 - 2\sum_{t=0}^{T} f(t)\right)}. \tag{1.7}$$

*Then*

(i) *the problem* (1.1) *has infinitely many periodic solutions* $\{u_n\}$ *such that* $\Phi(u_n) \to +\infty$ *as* $n \to \infty$*;*

(ii) *the problem* (1.1) *has infinitely many periodic solutions* $\{u_m^*\}$ *such that* $\Phi(u_m^*) \to -\infty$ *as* $m \to \infty$.

*Remark* 1.4. Obviously, the condition (1.6) is different from condition (1.5) that of in [1]; Theorem 1.3 is completely new comparing with main result of [1] since we allow $\alpha = 1$ although the method using in this paper is same as that of in [1].

## 2. Proof of main results

Since the proof of Theorem 1.2 is similar to that of Theorem 1.3, we only prove Theorem 1.3.

For the sake of convenience, we denote

$$\gamma = \sum_{t=0}^{T} f(t), \quad \beta = \sum_{t=0}^{T} g(t).$$

**Lemma 2.1.** *Suppose that* (1.3) *with* $\sum_{t=0}^{T} f(t) < \frac{\lambda_1}{4}$ *holds, then*

$$\Phi(u) \to +\infty \text{ as } \|u\| \to \infty \text{ in } H_2.$$

*Proof.* From (1.3), for all $u$ in $H_2$ we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\sum_{t=0}^{T} |\triangle u(t)|^2 - \sum_{t=0}^{T} F(t,u(t))\\ &\geqslant \frac{\lambda_1}{2}\|u\|^2 - \sum_{t=0}^{T} f(t)|u(t)|^2 - \sum_{t=0}^{T} g(t)|u(t)|\\ &\geqslant \frac{\lambda_1}{2}\|u\|^2 - \|u\|_\infty^2 \sum_{t=0}^{T} f(t) - \|u\|_\infty \sum_{t=0}^{T} g(t)\\ &\geqslant \frac{\lambda_1}{2}\|u\|^2 - \|u\|^2 \sum_{t=0}^{T} f(t) - \|u\| \sum_{t=0}^{T} g(t)\\ &= \left(\frac{\lambda_1}{2} - \gamma\right)\|u\|^2 - \beta\|u\|.\end{aligned}$$

So, $\Phi(u) \to +\infty$ as $\|u\| \to \infty$ in $H_2$. □

**Lemma 2.2.** *Suppose that* (1.4) *holds. Then there exists positive real sequence* $\{a_n\}$ *such that*

$$\lim_{n\to\infty} a_n = +\infty,$$
$$\lim_{n\to\infty} \sup_{u\in H_2, \|u\|=a_n} \Phi(u) = -\infty.$$

*Proof.* By (1.4), it is easy to obtain this result, so we omit the detail here. □

**Lemma 2.3.** *Suppose that* (1.3) *with* $\sum_{t=0}^{T} f(t) < \frac{\lambda_1}{4}$ *and* (1.7) *hold. Then there exists positive real sequence* $\{b_m\}$ *such that*

$$\lim_{m\to\infty} b_m = +\infty,$$
$$\lim_{m\to\infty} \inf_{u\in H_{b_m}} \Phi(u) = +\infty,$$

*where* $H_{b_m} = \{u \in H_1 : \|u\| = b_m\} \bigoplus H_2$.

*Proof.* By (1.7), let $a > \frac{1}{\lambda_1 - 2\gamma}$ such that

$$\liminf_{r\to+\infty} \sup_{x\in\mathbb{R}^N, |x|=r} |x|^{-2} \sum_{t=0}^{T} F(t,x) < -\frac{a}{2}\gamma^2.$$

Let $u \in H_{b_m}$, $u = \overline{u} + \widetilde{u}$, where $\overline{u} \in H_1$, $\widetilde{u} \in H_2$. So, we have

$$\begin{aligned}\left|\sum_{t=0}^{T} F(t,u(t)) - \sum_{t=0}^{T} F(t,\overline{u})\right| &= \left|\sum_{t=0}^{T}\int_0^1 \nabla F(t,\overline{u}(0) + s\widetilde{u}(t), \widetilde{u}(t))ds\right|\\ &\leqslant \sum_{t=0}^{T}\int_0^1 f(t)|\overline{u}(0) + s\widetilde{u}(t)||\widetilde{u}(t)|ds + \sum_{t=0}^{T}\int_0^1 g(t)|\widetilde{u}(t)|ds\\ &\leqslant \sum_{t=0}^{T} f(t)\left(|\overline{u}(0)| + |\widetilde{u}(t)|\right)|\widetilde{u}(t)| + \sum_{t=0}^{T} g(t)|\widetilde{u}(t)|\end{aligned}$$

$$
\begin{aligned}
&\leqslant \gamma|\overline{u}(0)|\|\widetilde{u}\|_\infty + \gamma\|\widetilde{u}\|_\infty^2 + \beta\|\widetilde{u}\|_\infty \\
&\leqslant \frac{1}{2a}\|\widetilde{u}\|_\infty^2 + \frac{a}{2}\gamma^2|\overline{u}(0)|^2 + \gamma\|\widetilde{u}\|_\infty^2 + \beta\|\widetilde{u}\|_\infty \\
&\leqslant \left(\frac{1}{2a} + \gamma\right)\|\widetilde{u}\|^2 + \frac{a}{2}\gamma^2\|\overline{u}\|^2 + \beta\|\widetilde{u}\|
\end{aligned}
$$

for all $u \in H_{b_m}$. Therefore, one has that

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}\sum_{t=0}^{T} |\triangle u(t)|^2 - \sum_{t=0}^{T} F(t, u(t)) \\
&= \frac{1}{2}\sum_{t=0}^{T} |\triangle \widetilde{u}(t)|^2 - \left(\sum_{t=0}^{T} F(t, u(t)) - \sum_{t=0}^{T} F(t, \overline{u}(t))\right) - \sum_{t=0}^{T} F(t, \overline{u}(t)) \\
&\geqslant \left(\frac{\lambda_1}{2} - \frac{1}{2a} - \gamma\right)\|\widetilde{u}\|^2 - \beta\|\widetilde{u}\| \\
&\quad - \|\overline{u}\|^2\left(\|\overline{u}\|^{-2}\sum_{t=0}^{T} F(t, \overline{u}(t)) + \frac{a}{2}\gamma^2\right)
\end{aligned}
$$

for all $u \in H_{b_m}$. From condition (1.7) and the above inequality the proof is finished. □

Now we give the proof of Theorem 1.3.

*The proof of Theorem 1.3.* Let $B_{a_n}$ be a ball in $H_1$ with radius $a_n$. Set

$$
\Gamma_n = \{\gamma \in C(B_{a_n}, H_T), \gamma\,|_{\partial B_{a_n}} = \mathrm{Id}\,|_{\partial B_{a_n}}\},
$$

and

$$
c_n = \inf_{\gamma \in \Gamma_n} \max_{x \in B_{a_n}} \Phi(\gamma(x)).
$$

It is easy to obtain that $\Phi$ is coercive on $H_2$ from Lemma 2.1. So, there is a constant $M$ such that

$$
\max_{x \in B_{a_n}} \Phi(\gamma(x)) \geqslant \inf_{u \in H_2} \Phi(u) \geqslant M.
$$

On the other hand, it is easy to see that $\gamma(Ba_n)\bigcap H_2 \neq \emptyset$ for any $\gamma \in \Gamma_n$. Therefore

$$
c_n \geqslant \inf_{u \in H_2} \Phi(u) \geqslant M.
$$

By Lemma 2.2, for any large value of $n$, one has that

$$
c_n > \max_{u \in \partial B_{a_n}} \Phi(u).
$$

For such $n$, there exists a sequence $\{\gamma_k\}$ in $\Gamma_n$ such that

$$
\max_{x \in B_{a_n}} \Phi(\gamma_k(x)) \to c_n, \quad k \to \infty.
$$

Applying [9, Theorem 4.3 and Corollary 4.3], there exists a sequence $\{v_k\}$ in $H_T$ satisfying

$$
\Phi(v_k) \to c_n, \quad \mathrm{dist}(v_k, \gamma_k(B_{a_n})) \to 0, \quad \Phi'(v_k) \to 0,
$$

as $k \to \infty$. So, for any large enough $k$, one has that

$$
c_n \leqslant \max_{x \in B_{a_n}} \Phi(\gamma_k(x)) \leqslant c_n + 1,
$$

and there exists $w_k \in \gamma_k(B_{a_n})$ such that

$$\|v_k - w_k\| \leqslant 1.$$

For fix $n$, by Lemma 2.3, let $m$ be large enough such that

$$b_m > a_n, \quad \text{and} \quad \inf_{u \in H_{b_m}} \Phi(u) > c_n + 1.$$

This implies that $\gamma(B_{a_n})$ cannot intersect the hyperplane $H_{b_m}$ for each $k$.

Let $w_k = \overline{w}_k + \widetilde{w}_k$, where $\overline{w}_k \in H_1$ and $\widetilde{w}_k \in H_2$. Then we have $|\overline{w}_k| < b_m$ for each $k$.

From (1.3), we have that

$$\begin{aligned}
c_n + 1 \geqslant \Phi(w_k) &= \frac{1}{2}\sum_{t=0}^{T} |\triangle w_k(t)|^2 - \sum_{t=0}^{T} F(t, w_k(t)) \\
&\geqslant \frac{\lambda_1}{2}\|\widetilde{w}_k\|^2 - \sum_{t=0}^{T} f(t)|w_k(t)|^2 - \sum_{t=0}^{T} g(t)|w_k(t)| \\
&\geqslant \frac{\lambda_1}{2}\|\widetilde{w}_k\|^2 - 2\sum_{t=0}^{T} f(t)[|\overline{w}_k(0)|^2 + |\widetilde{w}_k(t)|^2] - \sum_{t=0}^{T} g(t)(|\overline{w}_k(0)| + |\widetilde{w}_k(t)|) \\
&\geqslant \left(\frac{\lambda_1}{2} - 2\gamma\right)\|\widetilde{w}_k\|^2 - 2b_m^2\gamma - \|\widetilde{w}_k\|\beta - b_m\beta.
\end{aligned}$$

Therefore $\widetilde{w}_k(t)$ is bounded. Hence, $w_k$ is bounded since $\|w_k\| \leqslant C(\|\widetilde{w}_k\| + \|\overline{w}_k\|)$. Also, $\{v_k\}$ is bounded in $H_T$.

From the fact that $H_T$ is finite dimensional, we know there is a subsequence, which is still be denoted by $\{v_k\}$ such that $\{v_k\}$ converges to some point $u_n$. Therefore, in view of the continuity of $\Phi$ and $\Phi'$, it is easy to see that accumulation point $u_n$ of $\{v_k\}$ is a critical point and $c_n$ is a critical value of $\Phi$.

Let $n$ large enough such that $a_n > b_m$, then $\gamma(B_{a_n})$ intersects the hyperplane $H_{b_m}$ for any $\gamma \in \Gamma_n$. It follows that

$$\max_{x \in B_{a_n}} \Phi(\gamma(x)) \geqslant \inf_{u \in H_{b_m}} \Phi(u).$$

In view of above inequality and Lemma 2.3, we get $\lim_{n\to\infty} c_n = +\infty$. So, the proof of first result of Theorem 1.3 is finished.

Next, we prove (ii) of Theorem 1.3.

For fixed $m$, let

$$P_m = \{u \in H_T : u = \overline{u} + \widetilde{u}, |\overline{u}| \leqslant b_m, \widetilde{u} \in H_2\}.$$

For $u \in P_m$, one has that

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\sum_{t=0}^{T} |\triangle u(t)|^2 - \sum_{t=0}^{T} F(t, u(t)) \\
&\geqslant \frac{\lambda_1}{2}\|\widetilde{u}\|^2 - \sum_{t=0}^{T} f(t)|u(t)|^2 - \sum_{t=0}^{T} g(t)|u(t)| \\
&\geqslant \frac{\lambda_1}{2}\|\widetilde{u}\|^2 - 2\sum_{t=0}^{T} f(t)[|\overline{u}(0)|^2 + |\widetilde{u}(t)|^2] - \sum_{t=0}^{T} g(t)(|\overline{u}(0)| + |\widetilde{u}(t)|) \\
&\geqslant \left(\frac{\lambda_1}{2} - 2\gamma\right)\|\widetilde{u}\|^2 - 2b_m^2\gamma - \|\widetilde{u}\|\beta - b_m\beta.
\end{aligned} \tag{2.1}$$

So, $\Phi$ is bounded below on $P_m$. Let

$$\mu_m = \inf_{u \in P_m} \Phi(u),$$

and choose a minimizing sequence $\{u_k\}$ in $P_m$, that is

$$\Phi(u_k) \to \mu_m \quad \text{as} \quad k \to \infty.$$

According to (2.1), $\{u_k\}$ is bounded in $H_T$. Then there exists a subsequence, which is still be denoted by $\{u_k\}$ such that

$$u_k \rightharpoonup u_m^* \text{ weakly in } H_T.$$

Since $P_m$ is a convex closed subset of $H_T$ and $\Phi$ is weakly lower semicontinuous, $u_m^* \in P_m$ and

$$\mu_m = \lim_{k\to\infty} \Phi(u_k) \geqslant \Phi(u_m^*).$$

By $u_m^* \in P_m$,

$$\mu_m = \Phi(u_m^*).$$

Let $u_m^* = \overline{u}_m^* + \widetilde{u}_m^*$. In view of Lemma 2.2 and Lemma 2.3, $|\overline{u}_m^*| \neq b_m$ for large $m$, i.e., $u_m^*$ is in the interior of $P_m$. Then $u_m^*$ is a local minimum of functional. So, we have

$$\Phi(u_m^*) = \inf_{u\in P_m} \Phi(u) \leqslant \sup_{|u|=b_m} \Phi(u).$$

Then from Lemma 2.2 we see that $\Phi(u_m^*) \to -\infty$ as $m \to \infty$. Therefore, the proof is finished. □

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