Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 5922–5931
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

The split feasibility problems in an infinite dimensional space

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Communicated by X. Qin

Abstract

The purpose of this article is to investigate the approximation of common solutions of fixed point and split feasibility problems. A viscosity iterative algorithm is introduced and studied for this approximation problem. Strong convergence theorems are established in an infinite dimensional real Hilbert space.

Keywords: Image reconstruction, approximation solution, viscosity method, split feasibility problem.
2010 MSC: 47J20, 65J15.

1. Introduction

Let A be an M by N matrix. Let $C \in R^N$ and $Q \in R^M$ be nonempty closed convex sets. Let $\text{Proj}_C^{R^N}$ and $\text{Proj}_Q^{R^M}$ be the orthogonal projections onto C and Q, respectively. Recall that the split feasibility problem by Censor and Elfving [7] is to find $x \in C$ with $Ax \in Q$, if such x exists. The split feasibility problem was first introduced in 1994 for modeling inverse problems that arise from phase retrievals and in medical image reconstruction. Many image reconstruction problems can be formulated as the split feasibility problem; see, for example, [6, 8] and the references therein.

Censor and Elfving [7] introduced and investigated the following CQ algorithm

$$x_0 \in R^N, x_{n+1} = \text{Proj}_C^{R^N}(x_n - \delta A^T(I - \text{Proj}_Q^{R^M})Ax_n), \quad n \geqslant 0,$$

where $\delta \in (0, \frac{2}{E})$, E is the largest eigenvalue of the matrix A^TA, and I is the identity matrix.

Recently, Byrne [5] developed the split feasibility problem in the setting of infinite dimensional Hilbert spaces.

Let C and Q be nonempty, closed, and convex subsets in Hilbert spaces H_1 and H_2, respectively. Then the split feasibility problem in the framework of infinite dimensional spaces is formulated as finding a point $x \in C$ with the property:

$$x \in C, \quad Ax \in Q, \tag{1.1}$$

where $A : C \subset H_1 \to H_2$ is a bounded linear operator. In view of the applications, the splitting feasibility problem has been studied by many authors in the framework of infinite dimensional Hilbert spaces; see [2, 9, 17, 20, 22] and the references therein.

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doi:10.22436/jnsa.010.11.29

Received 2017-07-01

We denote by $\mathrm{SFP}(A)$ the solution set of the split feasibility problem, that is,

$$\mathrm{SFP}(A) = \{x \in H_1 : x \in C, \ \ Ax \in Q\} = A^{-1}(Q) \cap C.$$

It is clear that $A^{-1}(Q)$ is a closed convex subset of H_1. Therefore, $\mathrm{SFP}(A)$ is also a closed convex subset of H_1. It is known that the split feasibility problem is very general. It, which includes convex feasibility problem which is to find a common element in the intersection of a family of nonempty closed and convex subsets of a Hilbert space, has been extensively investigated; see [9, 12, 15] and the references therein.

Let $\mathrm{Proj}_C^{H_1}$ and $\mathrm{Proj}_Q^{H_2}$ be metric projections onto sets C and Q, respectively. It is well-known that if $\mathrm{SFP}(A) \neq \emptyset$, then solving split feasibility problem (1.1) is equivalent to solving a fixed point equation

$$x = \mathrm{Proj}_C^{H_1}(x - \delta A^*(I - \mathrm{Proj}_Q^{H_2})Ax),$$

where $\delta > 0$ is a parameter and A^* is the adjoint operator of A. If we define a mapping U_δ by

$$U_\delta x = x - \delta A^*(I - \mathrm{Proj}_Q^{H_2})Ax,$$

then one has $x = \mathrm{Proj}_C^{H_1} U_\gamma x$.

Assume that split feasibility problem (1.1) is consistent, i.e., the problem has a solution. It is easy to see that $\mathrm{Fix}(U_\delta) = A^{-1}(Q)$ and hence $\mathrm{SFP}(A) = C \cap \mathrm{Fix}(U_\delta) = \mathrm{Fix}(\mathrm{Proj}_C^{H_1} U_\delta)$, where $\mathrm{Fix}(U_\delta)$ and $\mathrm{Fix}(\mathrm{Proj}_C^{H_1} U_\delta)$ denote the fixed point set of U_δ and $\mathrm{Proj}_C^{H_1} U_\delta$, respectively, for sufficiently small $\delta > 0$; see Wang, Zhou [20], Zhou [21] and Zhou, Wang [22] for the details.

Let D be a nonempty closed and convex subset of a real Hilbert space H. Recall that a mapping $T : D \to D$ is a contractive mapping if and only if there exists a constant $\alpha \in [0, 1)$ such that

$$\|Tx - Ty\| \leqslant \alpha \|x - y\|, \quad \forall x, y \in D.$$

$T : D \to D$ is a nonexpansive mapping if and only if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in D.$$

From Browder [3], we know that the fixed point set of T is not empty provided that C is bounded, closed and convex. The theory of nonexpansive mappings has been recently applied to solve various convex optimization theories; see [1, 10, 11, 14, 16, 19] and the references therein. Recall that a mapping $S : D \to D$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e., $S := (1 - \alpha)I + \alpha S$ where $\alpha \in (0, 1)$, $S : D \to D$ is a nonexpansive mapping and I is the identity operator on D. We note that averaged mappings are nonexpansive. It is known that the composite of finitely many averaged mappings is still averaged. If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a nonempty common fixed point set, then $\bigcap_{i=1}^N \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1, T_2, \ldots, T_N)$.

It is well-known that if $\delta \in (0, 2/\|A\|^2)$, then U_δ is averaged and hence $\mathrm{Proj}_C^{H_1} U_\delta$ is also averaged, consequently, as a direct consequence of Reich's weak convergence theorem [18], the sequence $\{x_n\}$ is generated by the following procedure:

$$x_0 \in H_1, \ \ x_{n+1} = \mathrm{Proj}_C^{H_1}[(I - \delta A^*(I - \mathrm{Proj}_Q^{H_2})A)x_n], \ \ n \geqslant 0, \tag{1.2}$$

where I denotes the identity mapping on H_1 and H_2, converges weakly to a solution of the feasibility problem; see Byrne [5] for the details. (1.2) is referred to as the Byrne's CQ algorithm in the existing literature.

Recall that $T : D \to D$ is a strict pseudocontraction if and only if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\| + \kappa \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D.$$

The class of strict pseudocontractions was introduced and investigated by Browder and Petryshyn [4] in 1967. It is clear that the class of strict pseudocontractions includes the class of nonexpansions as special cases.

Recall that a mapping $F : D \to H$ is said to be monotone if and only if

$$\langle Fx - Fy, x - y\rangle \geqslant 0, \quad \forall x, y \in D.$$

$F : D \to H$ is said to be ν-inverse strongly monotone if and only if

$$\langle Fx - Fy, x - y\rangle \geqslant \nu\|Fx - Fy\|^2, \quad \forall x, y \in D.$$

$F : D \to H$ is said to be L-Lipschitzian if and only if $\|Fx - Fy\| \leqslant L\|x - y\|$ for all $x, y \in D$. We remark that if F is ν-inverse strongly monotone, then it is $\frac{1}{\nu}$-Lipschitzian and monotone. Let $T : D \to H$ be a nonexpansive mapping and define an operator $F : D \to H$ by $Fx = x - Tx$. Then, $F : D \to H$ is $\frac{1}{2}$-inverse strongly monotone.

Recently, many authors investigated the splitting feasibility problem in infinite dimensional Hilbert spaces via fixed point methods for the weak convergence of methods. In this paper, we consider a Halpern-like viscosity approximation method for the norm convergence of the method. The organization is as follows. In Section 2, some definitions and lemmas are provided. In Section 3, strong convergence theorems are established and some reduced results are also provided to support the main results.

2. Preliminaries

Let D be a nonempty closed and convex subset of a Hilbert space H. Let T be a mapping. From now on, the fixed point set of T will be denoted by $Fix(T)$. For every point $x \in H$, there exists a unique nearest point in D denoted by $Proj_D^H x$ such that $\| x - Proj_D^H x \| \leqslant \| x - y \|$ for all $y \in D$. $Proj_D^H$ is called the metric projection of H onto D. It is well-known that $Proj_D^H$ is nonexpansive mapping and satisfies $\langle x - y, Proj_D^H x - Proj_D^H y\rangle \geqslant \| Proj_D^H x - Proj_D^H y \|^2$ for all $x, y \in H$. Moreover, $Proj_D^H x$ is characterized by the fact $Proj_D^H x \in D$ and $\langle x - Proj_D^H x, y - Proj_D^H x\rangle \leqslant 0$, and $\| x - y \|^2 \geqslant \| x - Proj_D^H x \|^2 + \| y - Proj_D^H x \|^2$ for all $x \in H, y \in D$. In a real Hilbert space the following holds: $\| \lambda x + (1 - \lambda)y \|^2 = \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda(1 - \lambda) \| x - y \|^2$ for all $x, y \in H$ and $\lambda \in (0, 1)$. It is well-known that every nonexpansive operator $T : H \to H$ satisfies, for all $x, y \in H \times H$, the inequality $\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leqslant \frac{1}{2} \| (T(x) - x) - (T(y) - y) \|^2$, and therefore, we get, for all $(x, y) \in H \times Fix(T)$, $\langle x - T(x), y - T(y)\rangle \leqslant \frac{1}{2} \| T(x) - x \|^2$.

Lemma 2.1 ([13])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1 - t_n)a_n + b_n + c_n, \quad \forall n \geqslant 0,$$

where $\{c_n\}$ is a sequence of nonnegative real numbers, $\{t_n\} \subset (0, 1)$, and $\{b_n\}$ is a sequence of real numbers. Assume that

(a) $\limsup_{n\to\infty} \frac{b_n}{t_n} \leqslant 0$, $\sum_{n=0}^{\infty} t_n = \infty$;
(b) $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([3])**.** *Let H be a Hilbert space and let D be a nonempty closed and convex subset of H. Let S be a strict pseudocontraction on D with fixed points. If $x_n \rightharpoonup x^*$, where $\rightharpoonup$ denotes the weak convergence, and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then x^* is a fixed point of T, that is, $p = Tp$. In addition, $Fix(T)$ is closed and convex.*

The following two lemmas are known and not hard to derive.

Lemma 2.3. *Let $Proj_D^H : H \to D$ be the metric projection from H on a nonempty, closed, and convex subset D. Then the following conclusions hold true*

(a) $\langle (I - Proj_D^H)x - (I - Proj_D^H)y, x - y\rangle \geqslant \|(I - Proj_D^H)x - (I - Proj_D^H)y\|^2, \forall x, y \in H$.
(b) $\|x - y\|^2 - \|(I - Proj_D^H)x - (I - Proj_D^H)y\|^2 \geqslant \|Proj_D^H x - Proj_D^H y\|^2, \forall x, y \in H$

Lemma 2.4. *Let H be a Hilbert space. Then the following inequality holds*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

3. Main results

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $\mathrm{Proj}_C^{H_1}$ be the metric projection from H_1 onto C and let $\mathrm{Proj}_Q^{H_2}$ be the metric projection from H_2 onto Q. Let $f : C \to C$ be a contractive mapping with constant $0 \leqslant \alpha < 1$ and let $T : C \to C$ be a strict pseudocontraction with constant $0 \leqslant \kappa < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator such that split feasibility problem (1.1) is consistent. Assume that $\mathrm{Sol(SFP)} \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative algorithm*

$$\begin{cases} x_1 \in C, \\ y_n = \gamma_n x_n + (1-\gamma_n)Tx_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)\mathrm{Proj}_C^{H_1}\Big((1-\beta_n)(x_n - \delta_n A^*(I - \mathrm{Proj}_Q^{H_2})Ax_n) + \beta_n f(x_n)\Big), \quad n \geqslant 1, \end{cases}$$

where $\{\delta_n\}$ is a positive real sequence such that $\sum_{n=1}^{\infty} |\delta_n - \delta_{n+1}| < \infty$, $0 < \delta \leqslant \delta_n \leqslant \delta' < \frac{2}{\|A\|^2}$, where δ and δ' are two real numbers, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences in $(0,1)$ such that $0 < \alpha \leqslant \alpha_n \leqslant \alpha' < 1$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty$, $0 < \gamma \leqslant \gamma_n \leqslant \kappa < 1$, where γ is a constant in $(0,1)$, $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$. If $\mathrm{Fix}(T) \cap \mathrm{SFP}(A)$ is not empty, then $\{x_n\}$ converges strongly to a point $x^* \in \mathrm{Fix}(T) \cap \mathrm{SFP}(A)$ and x^* is the unique solution to the variational inequality

$$\langle f(x^*) - x^*, x - x^* \rangle \leqslant 0, \quad \forall x \in \mathrm{Fix}(T) \cap \mathrm{SFP}(A).$$

Proof. Note that the common solution set is not empty. Fixing $p \in \mathrm{Fix}(T) \cap \mathrm{SFP}(A)$, we find from Lemma 2.3 that

$$\begin{aligned} \|y_n - p\|^2 &= \|\gamma_n(x_n - p) + (1-\gamma_n)(Tx_n - p)\|^2 \\ &= \gamma_n\|x_n - p\|^2 - \gamma_n(1-\gamma_n)\|x_n - Tx_n\|^2 + (1-\gamma_n)\|Tx_n - p\|^2 \\ &\leqslant \gamma_n\|x_n - p\|^2 - \gamma_n(1-\gamma_n)\|x_n - Tx_n\|^2 + (1-\gamma_n)(\|x_n - p\|^2 + \kappa\|x_n - Tx_n\|^2) \\ &\leqslant \|x_n - p\|^2 - (1-\gamma_n)(\gamma_n - \kappa)\|x_n - Tx_n\|^2. \end{aligned}$$

Since $\gamma_n \leqslant \kappa$, we find that

$$\|y_n - p\| \leqslant \|x_n - p\|.$$

Define a mapping $W : H_1 \to H_1$ by

$$Wx = A^*(Ax - \mathrm{Proj}_Q^{H_2}Ax), \quad \forall x \in H_1.$$

Since $I - \mathrm{Proj}_Q^{H_2}$ is inverse-strongly monotone, we find that

$$\begin{aligned} \langle x - y, Wx - Wy \rangle &= \langle x - y, A^*(Ax - \mathrm{Proj}_Q^{H_2}Ax) - A^*(Ay - \mathrm{Proj}_Q^{H_2}Ay) \rangle \\ &= \langle Ax - Ay, (Ax - \mathrm{Proj}_Q^{H_2}Ax) - (Ay - \mathrm{Proj}_Q^{H_2}Ay) \rangle \\ &\geqslant \|(Ax - \mathrm{Proj}_Q^{H_2}Ax) - (Ay - \mathrm{Proj}_Q^{H_2}Ay)\|^2 \\ &\geqslant \frac{1}{\|A\|^2}\|Wx - Wy\|^2, \quad \forall x, y \in H_1. \end{aligned} \tag{3.1}$$

This shows that W is $\frac{1}{\|A\|^2}$-inverse-strongly monotone. It follows that

$$\begin{aligned} \|(I - \delta_n W)x - (I - \delta_n W)y\|^2 &= \delta_n^2\|Wx - Wy\|^2 - 2\delta_n\langle Wx - Wy, x - y \rangle + \|x - y\|^2 \\ &\leqslant \delta_n^2\|Wx - Wy\|^2 - \frac{2\delta_n}{\|A\|^2}\|Wx - Wy\|^2 + \|x - y\|^2 \\ &= \delta_n(\delta_n - \frac{2}{\|A\|^2})\|Wx - Wy\|^2 + \|x - y\|^2. \end{aligned}$$

Since $\delta_n \leqslant \frac{2}{\|A\|^2}$, we find that $(I-\mu_n W)$ is a nonexpansive mapping with

$$\mathrm{Fix}(I-\mu_n W) = W^{-1}(0).$$

On the other hand, one has $W^{-1}(0) = A^{-1}(Q)$. Indeed, letting $x \in A^{-1}(Q)$, we find from the definition of W that $x \in W^{-1}(0)$. This proves $A^{-1}(Q) \subset W^{-1}(0)$. Let $x \in W^{-1}(0)$, that is, $Wx = 0$. Since $\mathrm{Sol}(SFP) \cap \mathrm{Fix}(T) \neq \emptyset$, we can take a point $y \in \mathrm{Fix}(T) \cap SFP(A)$. This implies

$$\mathrm{Proj}_Q^{H_2} Ay = Ay \text{ and } y = Ty.$$

Hence, $Wy = 0$. Using (3.1), we have

$$0 = \langle x-y, Wx - Wy\rangle \geqslant \|(Ax - \mathrm{Proj}_Q^{H_2} Ax) - (Ay - \mathrm{Proj}_Q^{H_2} Ay)\|^2 = \|(Ax - \mathrm{Proj}_Q^{H_2} Ax)\|^2,$$

which implies that

$$(I - \mathrm{Proj}_Q^{H_2})Ax = 0,$$

that is, $x \in A^{-1}(Q)$. This shows that $W^{-1}(0) \subset A^{-1}(Q)$. Hence, one has $W^{-1}(0) = A^{-1}(Q)$. Since C, Q are closed and convex, we see that $SFP(A)$ is also closed and convex. Since T is strictly pseudocontractive, we find that $\mathrm{Fix}(T)$ is closed and convex. Since $\mathrm{Proj}^{H_1}_{\mathrm{Fix}(T)\cap SFP(A)} f$ is α-contractive, we see that $\mathrm{Proj}^{H_1}_{\mathrm{Fix}(T)\cap SFP(A)} f$ has a unique fixed point. Next, we use x^* to denote the unique fixed point, that is, $x^* = \mathrm{Proj}^{H_1}_{\mathrm{Fix}(T)\cap SFP(A)} f(x^*)$. Putting

$$z_n = \mathrm{Proj}_C^{H_1}\Big((1-\beta_n)(x_n - \delta_n A^*(I - \mathrm{Proj}_Q^{H_2})Ax_n) + \beta_n f(x_n)\Big),$$

one has

$$\begin{aligned}
\|z_n - x^*\| &\leqslant \|\Big((1-\beta_n)(x_n - \delta_n W x_n) + \beta_n f(x_n)\Big) - x^*\| \\
&\leqslant \|(1-\beta_n)((x_n - \delta_n W x_n) - (x^* - \delta_n W x^*)) + \beta_n (f(x_n) - x^*)\| \\
&\leqslant (1-\beta_n)\|(x_n - \delta_n W x_n) - (x^* - \delta_n W x^*)\| + \beta_n \|f(x_n) - f(x^*)\| + \beta_n \|f(x^*) - x^*\| \\
&\leqslant \beta_n \|f(x^*) - x^*\| + (1 - \beta_n(1-\alpha))\|x_n - x^*\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leqslant \alpha_n \|y_n - x^*\| + (1-\alpha_n)\|z_n - x^*\| \\
&\leqslant \big(1 - \beta_n(1-\alpha_n)(1-\alpha)\big)\|x_n - x^*\| + \beta_n(1-\alpha_n)(1-\alpha)\frac{\|f(x^*) - x^*\|}{1-\alpha}.
\end{aligned}$$

By mathematical induction, we find that

$$\|x_{n+1} - x^*\| \leqslant \max\{\frac{\|f(x^*) - x^*\|}{1-\alpha}, \|x_1 - x^*\|\}.$$

This shows that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Putting

$$S_n = \gamma_n I + (1-\gamma_n)T,$$

we find that $\mathrm{Fix}(S_n) = \mathrm{Fix}(T)$ for each n and

$$\begin{aligned}
\|S_n x_n - S_n x_{n-1}\|^2 &= \gamma_n \|x_n - x_{n-1}\|^2 - \gamma_n(1-\gamma_n)\|(x_n - x_{n-1}) - (Tx_n - Tx_{n-1})\|^2 \\
&\quad + (1-\gamma_n)\|Tx_n - Tx_{n-1}\|^2 \\
&\leqslant \gamma_n(\gamma_n - 1)\|(x_n - x_{n-1}) - (Tx_n - Tx_{n-1})\|^2 + \gamma_n \|x_n - x_{n-1}\|^2 \\
&\quad + (1-\gamma_n)(\|x_n - x_{n-1}\|^2 + \kappa\|(x_n - x_{n-1}) - (Tx_n - Tx_{n-1})\|^2) \\
&\leqslant \|x_n - x_{n-1}\|^2 - (1-\gamma_n)(\gamma_n - \kappa)\|(x_n - x_{n-1}) - (Tx_n - Tx_{n-1})\|^2 \\
&\leqslant \|x_n - x_{n-1}\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}\|y_n - y_{n-1}\| &\leqslant \|S_n x_n - S_n x_{n-1}\| + \|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\ &\leqslant \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - Tx_{n-1}\|.\end{aligned} \tag{3.2}$$

Since $(I - \delta_n W)$ is nonexpansive, we find that

$$\begin{aligned}\|(I-\delta_n W)x_n - (I-\delta_{n-1}W)x_{n-1}\| &\leqslant \|(I-\delta_n W)x_n - (I-\delta_n W)x_{n-1}\| \\ &\quad + \|(I-\delta_n W)x_{n-1} - (I-\delta_{n-1}W)x_{n-1}\| \\ &\leqslant \|(I-\delta_n W)x_{n-1} - (I-\delta_{n-1}W)x_{n-1}\| + \|x_n - x_{n-1}\| \\ &\leqslant |\delta_n - \delta_{n-1}|\|Wx_{n-1}\| + \|x_{n-1} - x_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|z_n - z_{n-1}\| &\leqslant \|\Big((1-\beta_n)(x_n - \delta_n W x_n) + \beta_n f(x_n)\Big) \\ &\quad - \Big((1-\beta_{n-1})(x_{n-1} - \delta_{n-1}Wx_{n-1}) + \beta_{n-1}f(x_{n-1})\Big)\| \\ &\leqslant (1-\beta_n)\|(x_n - \delta_n W x_n) - (x_{n-1} - \delta_{n-1}Wx_{n-1})\| + \beta_n\|f(x_n) - f(x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|W_{n-1}x_{n-1} - f(x_n)\| \\ &\leqslant (1-\beta_n)\|x_{n-1} - x_n\| + (1-\beta_n)|\delta_n - \delta_{n-1}|\|Wx_{n-1}\| + \beta_n\|f(x_n) - f(x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|W_{n-1}x_{n-1} - f(x_n)\| \\ &\leqslant (1-\beta_n(1-\alpha))\|x_{n-1} - x_n\| + |\delta_n - \delta_{n-1}|\|Wx_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|W_{n-1}x_{n-1} - f(x_n)\|.\end{aligned} \tag{3.3}$$

In view of (3.2) and (3.3), we find that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leqslant \alpha_n\|y_n - y_{n-1}\| + (1-\alpha_n)\|z_n - z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|y_{n-1} - z_{n-1}\| \\ &\leqslant \big(1 - (1-\alpha_n)\beta_n(1-\alpha)\big)\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - Tx_{n-1}\| \\ &\quad + |\delta_n - \delta_{n-1}|\|Wx_{n-1}\| + |\beta_n - \beta_{n-1}|\|W_{n-1}x_{n-1} - f(x_n)\| + |\alpha_n - \alpha_{n-1}|\|y_{n-1} - z_{n-1}\|.\end{aligned}$$

Using Lemma 2.1 and the conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$, we find that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Since W is $\frac{1}{\|A\|^2}$-inverse-strongly monotone, we find that

$$\begin{aligned}\|(I-\delta_n W)x_n - x^*\|^2 &= \delta_n^2\|Wx_n - Wx^*\|^2 + \|x_n - x^*\|^2 - 2\delta_n\langle Wx_n - Wx^*, x_n - x^*\rangle \\ &\leqslant \delta_n^2\|Wx_n - Wx^*\|^2 + \|x_n - x^*\|^2 - \frac{2\delta_n}{\|A\|^2}\|Wx_n - Wx^*\|^2 \\ &= (\delta_n^2 - \frac{2\delta_n}{\|A\|^2})\|Wx_n - Wx^*\|^2 + \|x_n - x^*\|^2 \\ &= \delta_n(\delta_n - \frac{2}{\|A\|^2})\|Wx_n\|^2 + \|x_n - x^*\|^2.\end{aligned} \tag{3.5}$$

Since $\|\cdot\|^2$ is convex, we find from (3.5) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leqslant \alpha_n\|S_n x_n - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 \\ &\leqslant \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|(1-\beta_n)(x_n - \delta_n W x_n) + \beta_n f(x_n) - x^*\|^2 \\ &\leqslant \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)(1-\beta_n)\|(x_n - \delta_n W x_n) - x^*\|^2 + \beta_n(1-\alpha_n)\|f(x_n) - x^*\|^2 \\ &\leqslant (1-\alpha_n)(1-\beta_n)\delta_n(\delta_n - \frac{2}{\|A\|^2})\|Wx_n\|^2 + \beta_n\|f(x_n) - x^*\|^2 + \big(1 - \beta_n(1-\alpha_n)\big)\|x_n - x^*\|^2.\end{aligned}$$

Hence, we have

$$(1-\alpha_n)(1-\beta_n)\delta_n(\frac{2}{\|A\|^2}-\delta_n)\|Wx_n\|^2 \leqslant \beta_n\|f(x_n)-x^*\|^2+\big(1-\beta_n(1-\alpha_n)\big)\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2$$
$$\leqslant \alpha_n\|f(x_n)-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.$$

Using the conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$, we find from (3.4) that

$$\lim_{n\to\infty}\|Wx_n\|=0. \tag{3.6}$$

Note that

$$\begin{aligned} x_n-S_nx_n &= x_n-x_{n+1}+(1-\alpha_n)(z_n-S_nx_n)\\ &= x_n-x_{n+1}+(1-\alpha_n)(z_n-x_n)+(1-\alpha_n)(x_n-S_nx_n). \end{aligned}$$

It follows that

$$\begin{aligned} \|S_nx_n-x_n\| &\leqslant \frac{\|x_{n+1}-x_n\|}{\alpha_n}+\frac{1-\alpha_n}{\alpha_n}\|x_n-\mathrm{Proj}_C^{H_1}\big((1-\beta_n)(x_n-\mu_nWx_n)+\beta_nf(x_n)\big)\|\\ &\leqslant \frac{\|x_{n+1}-x_n\|}{\alpha_n}+\frac{1-\alpha_n}{\alpha_n}\|x_n-\big((1-\beta_n)(x_n-\mu_nWx_n)+\beta_nf(x_n)\big)\|\\ &\leqslant \frac{\|x_{n+1}-x_n\|}{\alpha_n}+\frac{1-\alpha_n}{\alpha_n}(\beta_n\|x_n-f(x_n)\|+(1-\beta_n)\|x_n-(x_n-\mu_nWx_n)\|)\\ &\leqslant \frac{\|x_{n+1}-x_n\|}{\alpha_n}+\frac{(1-\alpha_n)\beta_n}{\alpha_n}\|x_n-f(x_n)\|+\frac{(1-\alpha_n)(1-\beta_n)\mu_n}{\alpha_n}\|Wx_n\|. \end{aligned}$$

In view of (3.4) and (3.6), we arrive at

$$\lim_{n\to\infty}\|S_nx_n-x_n\|=0. \tag{3.7}$$

On the other hand, we have

$$\|Tx_n-x_n\|\leqslant\|x_n-S_nx_n\|+\|Tx_n-S_nx_n\|\leqslant\|x_n-S_nx_n\|+\gamma_n\|Tx_n-x_n\|.$$

From the restriction imposed on $\{\gamma_n\}$ and (3.7), we find that

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0.$$

Next, we show that

$$\limsup_{n\to\infty}\langle z_n-x^*,f(x^*)-x^*\rangle\leqslant 0.$$

We take a subsequence $\{z_{n_m}\}$ of $\{z_n\}$ such that

$$\lim_{m\to\infty}\langle f(x^*)-x^*,z_{n_m}-x^*\rangle=\limsup_{n\to\infty}\langle f(x^*)-x^*,z_n-x^*\rangle. \tag{3.8}$$

Note that

$$\begin{aligned} \|x_n-z_n\| &\leqslant \|x_n-(1-\beta_n)(x_n-\delta_nA^*(I-\mathrm{Proj}_Q^{H_2})Ax_n)-\beta_nf(x_n)\|\\ &\leqslant \beta_n\|f(x_n)-x_n\|+(1-\beta_n)\|(x_n-\delta_nWx_n)-x_n\|\\ &\leqslant \beta_n\|f(x_n)-x_n\|+\delta_n\|Wx_n\|. \end{aligned}$$

In view of $\lim_{n\to\infty}\beta_n=0$, we find from (3.6) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.9}$$

This proves that $\{z_n\}$ is bounded. This shows that $\{z_{n_m}\}$ is also bounded. We may assume that $\{z_{n_m}\}$

converges weakly to $z \in H_1$. Since C is weakly closed, we see that $z \in C$. Since

$$\|Tz_n - z_n\| \leqslant \|z_n - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tz_n\|,$$

we find from (3.9) and the Lipschitz continuity of T that

$$\lim_{n\to\infty} \|Tz_n - z_n\| = 0.$$

From (3.6), we also have $\lim_{n\to\infty} \|Wz_n\| = 0$. Using Lemma 2.2, we find that z is a fixed point of T. Since W is inverse-strongly monotone, we have

$$\frac{1}{\|A\|^2}\|Wz_{n_m} - Wz\|^2 \leqslant \langle Wz_{n_m} - Wz, z_{n_m} - z\rangle. \tag{3.10}$$

Letting $m \to \infty$ in (3.10), we find that $z \in W^{-1}(0)$. This proves that

$$z \in C \cap W^{-1}(0) \cap \text{Fix}(T) = SFP(A) \cap \text{Fix}(T).$$

Using (3.8), one obtains that

$$\limsup_{n\to\infty} \langle z_n - x^*, f(x^*) - x^*\rangle \leqslant 0.$$

Finally, we prove that $x_n \to x^*$ in norm as $n \to \infty$. Using Lemma 2.4, we find that

$$\begin{aligned}
\|z_n - x^*\|^2 &\leqslant \|(1-\beta_n)(x_n - \delta_n A^*(I - \text{Proj}_Q^{H_2})Ax_n) + \beta_n f(x_n) - x^*\|^2 \\
&\leqslant (1-\beta_n)^2\|(x_n - \delta_n A^*(I - \text{Proj}_Q^{H_2})Ax_n) - x^*\| + 2\beta_n\langle f(x_n) - x^*, z_n - x^*\rangle \\
&\leqslant (1-\beta_n)^2\|x_n - x^*\|^2 + 2\beta_n\alpha\|x_n - x^*\|\|z_n - x^*\| + 2\beta_n\langle f(x^*) - x^*, z_n - x^*\rangle \\
&\leqslant (1 - 2\beta_n + \beta_n^2 + \beta_n\alpha)\|x_n - x^*\|^2 + \beta_n\alpha\|z_n - x^*\|^2 + 2\beta_n\langle f(x^*) - x^*, z_n - x^*\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_n - x^*\|^2 &\leqslant \Big(1 - \frac{2\beta_n(1-\alpha)}{1-\beta_n\alpha}\Big)\|x_n - x^*\|^2 \\
&\quad + \frac{2\beta_n(1-\alpha)}{1-\beta_n\alpha}\Big(\frac{1}{1-\alpha}\langle f(x^*) - x^*, z_n - x^*\rangle + \frac{\beta_n}{2(1-\alpha)}\|x_n - x^*\|^2\Big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leqslant \alpha_n\|y_n - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 \\
&\leqslant \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)(1-\overline{\alpha}_n)\|x_n - x^*\|^2 + (1-\alpha_n)\overline{\alpha}_n\lambda_n \\
&\leqslant \big(1 - \overline{\alpha}_n(1-\alpha_n)\big)\|x_n - x^*\|^2 + \overline{\alpha}_n(1-\alpha_n)\lambda_n,
\end{aligned}$$

where

$$\lambda_n = \frac{\beta_n}{2(1-\alpha)}\|x_n - x^*\|^2 + \frac{1}{1-\alpha}\langle f(x^*) - x^*, z_n - x^*\rangle.$$

Since $\beta_n \to 0$ as $n \to \infty$ and $\{\|x_n - x^*\|\}$ is bounded, we find that $\limsup_{n\to\infty} \lambda_n \leqslant 0$. Using Lemma 2.1, we find that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

From Theorem 3.1, we immediately obtain the following result.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $\text{Proj}_C^{H_1}$ be the metric projection from H_1 onto C and let $\text{Proj}_Q^{H_2}$ be the metric projection from H_2 onto Q. Let $f : C \to C$ be a contractive mapping with constant $0 \leqslant \alpha < 1$ and let $T : C \to C$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator such that split feasibility problem (1.1) is consistent. Assume that $\text{Sol(SFP)} \cap \text{Fix}(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the* *following iterative algorithm*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n T x_n + (1-\alpha_n)\mathrm{Proj}_C^{H_1}\Big((1-\beta_n)(x_n - \delta_n A^*(I - \mathrm{Proj}_Q^{H_2})Ax_n) + \beta_n f(x_n)\Big), \quad n \geqslant 1, \end{cases}$$

where $\{\delta_n\}$ *is a positive real sequence such that* $\sum_{n=1}^{\infty}|\delta_n - \delta_{n+1}| < \infty$, $0 < \delta \leqslant \delta_n \leqslant \delta' < \frac{2}{\|A\|^2}$, *where* δ *and* δ' *are two real numbers,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two real sequences in* $(0,1)$ *such that* $0 < \alpha \leqslant \alpha_n \leqslant \alpha' < 1$, $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$, $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, $\sum_{n=1}^{\infty}|\beta_n - \beta_{n+1}| < \infty$. *If* $\mathrm{Fix}(T) \cap \mathrm{SFP}(A)$ *is not empty, then* $\{x_n\}$ *converges strongly to a point* $x^* \in \mathrm{Fix}(T) \cap \mathrm{SFP}(A)$ *and* x^* *is the unique solution to the variational inequality*

$$\langle f(x^*) - x^*, x - x^*\rangle \leqslant 0, \quad \forall x \in \mathrm{Fix}(T) \cap \mathrm{SFP}(A).$$

From Theorem 3.1, we also have the following results on splitting feasibility problem (1.1).

Corollary 3.3. *Let* H_1 *and* H_2 *be two real Hilbert spaces. Let* C *be a nonempty closed and convex subset of* H_1 *and let* Q *be a nonempty closed and convex subset of* H_2. *Let* $\mathrm{Proj}_C^{H_1}$ *be the metric projection from* H_1 *onto* C *and let* $\mathrm{Proj}_Q^{H_2}$ *be the metric projection from* H_2 *onto* Q. *Let* $f : C \to C$ *be a contractive mapping with constant* $0 \leqslant \alpha < 1$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator such that split feasibility problem* (1.1) *is consistent. Let* $\{x_n\}$ *be a sequence generated in the following iterative algorithm*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\mathrm{Proj}_C^{H_1}\Big((1-\beta_n)(x_n - \delta_n A^*(I - \mathrm{Proj}_Q^{H_2})Ax_n) + \beta_n f(x_n)\Big), \quad n \geqslant 1, \end{cases}$$

where $\{\delta_n\}$ *is a positive real sequence such that* $\sum_{n=1}^{\infty}|\delta_n - \delta_{n+1}| < \infty$, $0 < \delta \leqslant \delta_n \leqslant \delta' < \frac{2}{\|A\|^2}$, *where* δ *and* δ' *are two real numbers,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two real sequences in* $(0,1)$ *such that* $0 < \alpha \leqslant \alpha_n \leqslant \alpha' < 1$, $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$, $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, $\sum_{n=1}^{\infty}|\beta_n - \beta_{n+1}| < \infty$. *If* $\mathrm{SFP}(A)$ *is not empty, then* $\{x_n\}$ *converges strongly to a point* $x^* \in \mathrm{SFP}(A)$ *and* x^* *is the unique solution to the variational inequality*

$$\langle f(x^*) - x^*, x - x^*\rangle \leqslant 0, \quad \forall x \in \mathrm{SFP}(A).$$

Remark 3.4. The CQ algorithm heavily depends on metric projection Proj_C and Proj_Q. In the framework of Hilbert spaces, the projections are nonexpansive. Indeed, they are firmly nonexpansive. However, they may lose the good properties in the framework of Banach spaces. It is of interest to extend the results presented in this article to a Banach space.

Acknowledgment

The authors are grateful to the editor and the referees for useful suggestions which improved the contents of this article.

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