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Strong convergence of Halpern method for firmly type nonexpansive mappings

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Abstract

In this paper, Halpern method is applied to find fixed points of a class of firmly type nonexpansive mappings. A strong convergence result is obtained under the control conditions (C1) and (C2). Our conclusion obtained in this paper gives the affirmative answer of the Halpern open problem for this class of mapping.

Keywords: Fixed point, Halpern method, firmly type nonexpansive mappings, strong convergence.
2010 MSC: 47H10, 47J25.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space equipped up its inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be a given set. Recall that a mapping $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ is said to be nonexpansive if the following inequality holds

$$\|\mathcal{T}u - \mathcal{T}u^\dagger\| \leqslant \|u - u^\dagger\|, \quad \forall u, u^\dagger \in \mathcal{C}.$$

A mapping $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ is called averaged, if and only if it can be written as the average of the identity operator and a nonexpansive operator, that is,

$$\mathcal{T} = \lambda I + (1-\lambda)\mathcal{S},$$

where λ is a number in $(0,1)$ and $\mathcal{S}\colon \mathcal{C} \to \mathcal{C}$ is nonexpansive.

A point $v^\dagger \in \mathcal{C}$ is a fixed point of $\mathcal{T}$ provided $\mathcal{T}v^\dagger = v^\dagger$. Denote by $\mathrm{Fix}(\mathcal{T})$ the set of fixed points of $\mathcal{T}$, that is, $\mathrm{Fix}(\mathcal{T}) = \{v^\dagger \in \mathcal{C} : \mathcal{T}v^\dagger = v^\dagger\}$. It is assumed throughout the paper that $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$.

Iterative computation of fixed points of nonlinear mappings is an interesting topic in a large number of applied areas, in particular in image recovery and signal processing. Constructed iteration approaches

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to find fixed points of nonexpansive mappings have received vast investigation, see, e.g., [1–3, 8–11, 17–19, 21, 22, 24].

The aim of the present paper focuses our attention on the parameter control of iterative methods for finding fixed points of nonexpansive mappings. We next briefly reproduce several historic approaches.

It is well-known that the Picard scheme $u_{n+1} = \mathcal{T}u_n = \cdots = \mathcal{T}^{n+1}u_0$ of the mapping $\mathcal{T}$ at an initial guess $u_0 \in \mathcal{C}$ may, in general, not behave well. This means that it may not converge even in the weak topology. One pathway to overcome this disadvantage is to apply Mann's ([14]) iteration algorithm that generates a sequence $\{u_n\}$ via the iterative manner:

$$u_{n+1} = \alpha_n u_n + (1-\alpha_n)\mathcal{T}u_n, \quad n \geqslant 0. \tag{1.1}$$

Though simple in form, the Mann iteration (1.1) is remarkably useful for finding fixed points of a nonexpansive mapping and provides a unified framework for many algorithms from various different fields. However, Mann iterative algorithm (1.1) for nonexpansive mappings has only weak convergence in the infinite dimensional spaces.

A natural question rises: could we acquire the strong convergence conclusion by using the normal Mann's method (1.1) for nonexpansive mappings? In this respect, in 1975, Genel and Lindenstrass [4] demonstrated a counterexample. Thus, some rectifications are necessary in order to guarantee the strong convergence of the modified method.

In 1967, Halpern [6] constructed the following iteration scheme for computing a fixed point of a nonexpansive mapping $\mathcal{T}$.

For fixed $u \in \mathcal{C}$ and an initial guess $u_0 \in \mathcal{C}$, let the sequence $\{u_n\}$ be generated iteratively by

$$u_{n+1} = \alpha_n u + (1-\alpha_n)\mathcal{T}u_n, \quad n \geqslant 0, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Algorithm (1.2) was referred to as the Halpern algorithm. Halpern pointed out that the control conditions:

(C1): $\lim_{n\to\infty} \alpha_n = 0$;

(C2): $\sum_{n=0}^{\infty} \alpha_n = \infty$ are necessary for the strong convergence of the iteration (1.2) to a fixed point of $\mathcal{T}$.

At the same time, he also put forth the following open problem.

Problem 1.1. Are the control conditions (C1) and (C2) sufficient for the convergence of the Halpern iteration (1.2) to a fixed point of $\mathcal{T}$?

Subsequently, many researchers carefully considered this problem, for instance, [7, 15]. However, this problem was still not solved until the following important conclusion was presented by Suzuki [13] in 2005.

Conclusion 1.2 ([13])**.** Let $\mathcal{X}$ be a Banach space. Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

$$0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $\{u_n\} \subset \mathcal{X}$ and $\{v_n\} \subset \mathcal{X}$ are two bounded sequences satisfying the conditions

(i) $u_{n+1} = (1-\beta_n)u_n + \beta_n v_n$, for all $n \geqslant 0$;
(ii) $\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|u_{n+1} - u_n\|) \leqslant 0$.

Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

By applying Conclusion 1.2, one can prove that the sequence $\{u_n\}$ generated by Halpern method (1.2) converges strongly to the fixed point of $\mathcal{T}$ under the control conditions (C1) and (C2) provided $\mathcal{T}$ is averaged.

Remark 1.3. The Halpern open problem is solved for the class of averaged mappings. In this case, we can rewrite (1.2) as

$$u_{n+1} = \alpha_n u + (1-\alpha_n)(\lambda u_n + (1-\lambda)\mathcal{S}u_n), \quad n \geqslant 0,$$

or its general form

$$u_{n+1} = \alpha_n u + \beta_n u_n + \gamma_n \mathcal{S}u_n, \quad n \geqslant 0, \tag{1.3}$$

where $\mathcal{S}$ is a nonexpansive.

Note that the above iterative algorithm (1.3) has been applied to find the fixed points of a large number of nonlinear mappings, see for instance [20, 23] and the references therein.

Recently, Song and Cai [12] introduced a class of firmly type nonexpansive mappings and proved the strong convergence of Halpern iteration (1.2) for a firmly type nonexpansive mapping under conditions (C1) and (C2). However, there is a gap in the proof of [12, Theorem 3.1].

The main purpose of this paper is to demonstrate the strong convergence of Halpern iteration (1.2) for a firmly type nonexpansive mapping under conditions (C1) and (C2) by using a new technique. Our conclusion obtained in this paper gives the affirmative answer of the Halpern open problem for this class of firmly type nonexpansive mappings.

2. Preliminaries

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$.

Definition 2.1. A mapping $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ is said to be firmly type nonexpansive if there exists a positive constant $k \in (0, \infty)$ such that

$$\|\mathcal{T}u - \mathcal{T}u^\dagger\|^2 \leqslant \|u - u^\dagger\|^2 - k\|(I-\mathcal{T})u - (I-\mathcal{T})u^\dagger\|^2 \tag{2.1}$$

for all $u, u^\dagger \in \mathcal{C}$.

For every point $z \in \mathcal{H}$, there exists a unique nearest point in $\mathcal{C}$, denoted by $\mathrm{proj}_{\mathcal{C}} z$ such that

$$\|z - \mathrm{proj}_{\mathcal{C}} z\| \leqslant \|z - u\|, \quad \forall z \in \mathcal{C}.$$

The mapping $\mathrm{proj}_{\mathcal{C}}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{C}$. It is well-known that $\mathrm{proj}_{\mathcal{C}}$ is a nonexpansive mapping and is characterized by the following property:

$$\langle z - \mathrm{proj}_{\mathcal{C}} z, u - \mathrm{proj}_{\mathcal{C}} z\rangle \leqslant 0, \quad \forall z \in \mathcal{H}, \ \ u \in \mathcal{C}. \tag{2.2}$$

Lemma 2.2 ([5]). *Let $\mathcal{C}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$, and let $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ be a nonexpansive mapping with $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Assume that $\{u_n\} \subset \mathcal{C}$ is a sequence such that $u_n \rightharpoonup x^\dagger$ and $(I-\mathcal{T})u_n \to 0$. Then $x^\dagger \in \mathrm{Fix}(\mathcal{T})$.*

Lemma 2.3 ([16]). *Assume that $\{\delta_n\}$ is a sequence of nonnegative real numbers such that*

$$\delta_{n+1} \leqslant (1-\alpha_n)\delta_n + \alpha_n \sigma_n, \quad n \geqslant 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n \sigma_n| < \infty$.

Then $\lim_{n\to\infty} \delta_n = 0$.

3. Main results

Let $\mathcal{C}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ be a firmly type nonexpansive mapping with $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$.

Algorithm 3.1 (Initialization)**.** For fixed $u \in \mathcal{C}$ and given an initial guess $x_0 \in \mathcal{C}$ arbitrarily.

For the constructed $\{x_n\}$, x_{n+1} is generated iteratively by the manner

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\mathcal{T}x_n, \quad n \geqslant 0, \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Theorem 3.2. *Assume $\{\alpha_n\}$ satisfies the following conditions*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $\mathrm{proj}_{\mathrm{Fix}(\mathcal{T})}(u)$.

Proof. Set $z = \mathrm{proj}_{\mathrm{Fix}(\mathcal{T})}(u)$. Firstly, we show that the sequence $\{x_n\}$ is bounded. From (3.1), we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(u-z) + (1-\alpha_n)(\mathcal{T}x_n - z)\| \\ &\leqslant \alpha_n\|u-z\| + (1-\alpha_n\|\mathcal{T}x_n - z\| \\ &\leqslant \alpha_n\|u-z\| + (1-\alpha_n)\|x_n - z\| \\ &\leqslant \max\{\|u-z\|, \|x_n - z\|\}. \end{aligned}$$

By induction, we get

$$\|x_{n+1} - z\| \leqslant \max\{\|u-z\|, \|x_0 - z\|\}.$$

Hence, the sequence $\{x_n\}$ is bounded and so is $\{\mathcal{T}x_n\}$.

By virtue of (2.1) and (3.1), we deduce

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(u-z) + (1-\alpha_n)(\mathcal{T}x_n - z)\|^2 \\ &\leqslant (1-\alpha_n)\|\mathcal{T}x_n - z\|^2 + 2\alpha_n\langle u-z, x_{n+1} - z\rangle \\ &\leqslant (1-\alpha_n)[\|x_n - z\|^2 - k\|x_n - \mathcal{T}x_n\|^2] + 2\alpha_n\langle u-z, x_{n+1} - z\rangle \\ &= \alpha_n\left[2\langle u-z, x_{n+1} - z\rangle - \frac{(1-\alpha_n)k\|x_n - \mathcal{T}x_n\|^2}{\alpha_n}\right] \\ &\quad + (1-\alpha_n)\|x_n - z\|^2. \end{aligned} \tag{3.2}$$

Set $\delta_n = \|x_n - z\|^2$ and

$$\sigma_n = 2\langle u-z, x_{n+1} - z\rangle - \frac{(1-\alpha_n)k\|x_n - \mathcal{T}x_n\|^2}{\alpha_n} \tag{3.3}$$

for all $n \geqslant 0$.

According to (3.2) and (3.3), we obtain

$$\delta_{n+1} \leqslant (1-\alpha_n)\delta_n + \alpha_n\sigma_n, \quad n \geqslant 0. \tag{3.4}$$

Next, we show that $\limsup_{n\to\infty} \sigma_n$ is finite. From (3.3), we get

$$\sigma_n \leqslant 2\langle u-z, x_{n+1} - z\rangle \leqslant 2\|u-z\|\|x_{n+1} - z\|.$$

Since $\{x_n\}$ is bounded, it follows that

$$\limsup_{n\to\infty} \sigma_n < +\infty.$$

Next we prove

$$\limsup_{n\to\infty} \sigma_n \geqslant -1,$$

by contradiction.

If we assume on the contrary $\limsup_{n\to\infty} \sigma_n < -1$, then there exists m_0 such that $\sigma_n \leqslant -1$ for all $n \geqslant m_0$. It then follows from (3.4) that

$$\begin{aligned}\delta_{n+1} &\leqslant (1-\alpha_n)\delta_n - \alpha_n\\ &\leqslant \delta_n - \alpha_n\end{aligned}$$

for all $n \geqslant m_0$.

By induction, we have

$$\delta_{n+1} \leqslant \delta_{m_0} - \sum_{i=m_0}^{n} \alpha_i. \tag{3.5}$$

By taking lim sup as $n \to \infty$ in (3.5), we have

$$\limsup_{n\to\infty} \delta_n \leqslant \delta_{m_0} - \lim_{n\to\infty} \sum_{i=m_0}^{n} \alpha_i = -\infty,$$

which induces a contradiction. So,

$$-1 \leqslant \limsup_{n\to\infty} \sigma_n < +\infty.$$

Hence, $\limsup_{n\to\infty} \sigma_n$ exists. Thus, we can take a subsequence $\{n_k\}$ such that

$$\begin{aligned}\limsup_{n\to\infty} \sigma_n &= \lim_{k\to\infty} \sigma_{n_k}\\ &= \lim_{k\to\infty} \left[2\langle u-z, x_{n_k+1}-z\rangle - \frac{(1-\alpha_{n_k})k\|x_{n_k} - \mathcal{T}x_{n_k}\|^2}{\alpha_{n_k}}\right].\end{aligned} \tag{3.6}$$

Since $\langle u-z, x_{n_k+1}-z\rangle$ is a bounded real sequence, without loss of generality, we may assume that the limit $\lim_{k\to\infty}\langle u-z, x_{n_k+1}-z\rangle$ exists. Consequently, from (3.6), the following limit also exists

$$\lim_{k\to\infty} \frac{(1-\alpha_{n_k})k\|x_{n_k} - \mathcal{T}x_{n_k}\|^2}{\alpha_{n_k}}. \tag{3.7}$$

Note that $\lim_{n\to\infty} \alpha_n = 0$. It follows from (3.7) that

$$\lim_{k\to\infty} \|x_{n_k} - \mathcal{T}x_{n_k}\| = 0.$$

It follows that any weak cluster point of $\{x_{n_k}\}$ belongs to $\mathrm{Fix}(\mathcal{T})$ by Lemma 2.2.

Note that

$$\|x_{n+1} - x_n\| \leqslant \alpha_n\|x_n - u\| + (1-\alpha_n)\|\mathcal{T}x_n - x_n\|.$$

This together with condition (C1) implies that

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0.$$

This implies that any weak cluster point of $\{x_{n_k+1}\}$ also belongs to $\mathrm{Fix}(\mathcal{T})$. Without loss of generality, we assume that $\{x_{n_k+1}\}$ converges weakly to $\bar{x}$. Therefore,

$$\limsup_{n\to\infty} \sigma_n \leqslant \lim_{k\to\infty} 2\langle u-z, x_{n_k+1}-z\rangle = 2\langle u-z, \bar{x}-z\rangle \leqslant 0,$$

due to the fact that $z = \mathrm{proj}_{\mathrm{Fix}(\mathcal{T})}(u)$ and (2.2).

Finally, applying Lemma 2.3 to (3.4), we get $x_n \to \mathrm{proj}_{\mathrm{Fix}(\mathcal{T})}(u)$. The proof is completed. □

Remark 3.3. Note that Suzuki's conclusion 1.2 can not be used to the class of firmly type nonexpansive mappings. Theorem 3.2 gives a positive answer to the Halpern open problem for the class of firmly nonexpansive mappings.

Recall that a mapping $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ is said to be firmly nonexpansive if

$$\|\mathcal{T}u - \mathcal{T}u^{\dagger}\|^2 \leqslant \|u - u^{\dagger}\|^2 - \|(I - \mathcal{T})u - (I - \mathcal{T})u^{\dagger}\|^2$$

for all $u, u^{\dagger} \in \mathcal{C}$.

Remark 3.4. It is obvious that the class of firmly type nonexpansive mappings includes the class of firmly nonexpansive mappings as a special case.

As an application of Theorem 3.2, we get the following corollary.

Corollary 3.5. *Let $\mathcal{C}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ be a firmly nonexpansive mapping with* $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. *Assume $\{\alpha_n\}$ satisfies the following conditions*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence $\{x_n\}$ generated by (3.1) *converges strongly to* $\mathrm{proj}_{\mathrm{Fix}(\mathcal{T})}(u)$.

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