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A fixed point theorem for systems of operator equations and its application

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Abstract

A new fixed point theorem in product cones is established for systems of operator equations, where the components are expressed by partial ordering. In applications, this allows the nonlinear term of a differential system to have different behaviors in components.

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1. Introduction

Let E be a Banach space, $P \subset E$ be a cone. Then E becomes an ordered Banach space equipped with the partial ordering $'' \leqslant ''$ induced by P. P is called normal if there exists a constant $N > 0$ such that $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$. One can refer the concepts and the properties of the cone to [8, 10]. In this paper we investigate the existence of fixed point for systems of the operator equations

$$\begin{cases} u_1 = N_1(u_1, u_2), \\ u_2 = N_2(u_1, u_2), \end{cases} \tag{1.1}$$

where $N_i \in C[E \times E, E]$ $(i = 1, 2)$.

Recently, much work has been carried out on the existence of positive solutions of various type of nonlinear problems (see [1–7, 9, 11, 15–25]). One of the most common approaches is to set down an equivalent abstract operator for nonlinear problems and then applying the topological degree, the fixed point theorem or the fixed point index theory in cones to get the desired results. For example, in [16, 23], the authors applied the fixed point index theory to study the existence and multiplicity of positive solutions of boundary value problems for systems of second order ordinary differential equations. Precup in [18] considered the existence of positive periodic solutions of a first order differential system by establishing a new version of Krasnosel'skiĭ's fixed point theorem in cones for systems of operator equations, where the compression-expansion conditions are expressed on components. In [4], the authors discussed the

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multiplicity of positive solutions for a class of (p_1, p_2)-Laplacian systems based on the product formula of the fixed point index and Leray-Schauder degree theory.

Motivated by works mentioned above, we employ the e-positive operator or/and functional and the partial ordering to get some new fixed point theorem for operator equation (1.1). As application of our main result, the existence of positive solutions for system of second and fourth order ordinary differential equations is considered.

2. Main results

We first recall some concepts and conclusions on the fixed point index in [8, 10] for the proof of fixed point theorem. Let E be a Banach space and let $P \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E with boundary $\partial\Omega$, and let $A : P \cap \Omega \to P$ be a completely continuous operator. If $Au \neq u$ for $u \in P \cap \partial\Omega$, then the fixed point index $i(A, P \cap \Omega, P)$ can be defined. One important fact is that if $i(A, P \cap \Omega, P) \neq 0$, then A has a fixed point in $P \cap \Omega$.

The following lemmas is needed in our proofs.

Lemma 2.1 ([8, 10])**.** *Let $A : P \to P$ be completely continuous. We have*

(i) *if $Au \neq \mu u$ for all $u \in P \cap \partial\Omega$ and $\mu \geqslant 1$, then $i(A, P \cap \Omega, P) = 1$;*

(ii) *if there exists $u_0 \in P$, $u_0 \neq \theta$ such that $u - Au \neq \mu u_0$ for all $u \in P \cap \partial\Omega$ and $\mu \geqslant 0$, then $i(A, P \cap \Omega, P) = 0$.*

Lemma 2.2 ([4])**.** *Let E be a real Banach space, $P_i \subset E$ be a cone and W_i be a bounded open subset of E with boundary ∂W_i $(i = 1, 2)$. Suppose that $A_i : P_i \cap W_i \to P_i$ is a completely continuous operator and that $A_i u_i \neq u_i$, for all $u_i \in P_i \cap \partial W_i$. Then*

$$i(A, (P_1 \times P_2) \cap (W_1 \times W_2), P_1 \times P_2) = i(A_1, P_1 \cap W_1, P_1) \cdot i(A_2, P_2 \cap W_2, P_2),$$

where $A(u, v) = (A_1 u, A_2 v)$, $\forall\ (u, v) \in (P_1 \times P_2) \cap (W_1 \times W_2)$.

In what follows, we shall consider two normal cones P_1, P_2 of a Banach space E, normal cone K_1 of a Banach space $(E_1, \|\cdot\|_1)$ with normal constant $\overline{N}_1$, normal cone K_2 of a Banach space $(E_2, \|\cdot\|_2)$ with normal constant $\overline{N}_2$, cone $P_1 \times P_2$ of a Banach space $E \times E$ with the norm $\|(u_1, u_2)\| = \max\{\|u_1\|, \|u_2\|\}$, and we shall use the same symbol $'' \leqslant''$ to denote the partial ordering relations induced by $P_1 \times P_2$ in $E \times E$, and by P_1, P_2, K_1, K_2 in E, E_1 and E_2.

Definition 2.3. Let P be a cone in a Banach space E, and K_1 be a cone in a Banach space E_1. Let $e \in K_1 \backslash \{\theta\}$. A linear operator $T : P \to K_1$ is called e-positive, if for every nonzero $x \in P$ there exist two positive number $c(x)$, $d(x)$ such that

$$c(x)e \leqslant Tx \leqslant d(x)e.$$

Remark 2.4. The above definition of e-positive operator is different from the previous definition in [12, 13]. The main difference is that $E \neq E_1$, or $E = E_1$ but $P \neq K_1$. When $E = E_1$ and $P \subset K_1$, Definition 2.3 is the same as [21, Definition 2.2].

We now state and prove the main abstract result of this paper.

Theorem 2.5. *Let $N = (N_1, N_2) : P_1 \times P_2 \to P_1 \times P_2$ be a completely continuous operator. Assume that for each $i \in \{1, 2\}$ one of the following conditions is satisfied:*

(a) *There exist $k_{i1} \in (1, +\infty)$, $k_{i2} \in [0, 1)$, $k_i, r_i, M_i \in (0, +\infty)$, $e_i \in K_i \backslash \{\theta\}$ and two linear e_i-positive operators $B_{i1}, B_{i2} : P_i \to K_i$, such that*

$$B_{i1} N_i(u) \geqslant k_{i1} B_{i1} u_i, \quad \|u_i\| \leqslant r_i, \quad u = (u_1, u_2) \in P_1 \times P_2, \quad B_{i2} N_i(u) \leqslant k_{i2} B_{i2} u_i + M_i e_i,$$

and

$$\|B_{i2} u_i\|_i \geqslant k_i \|u_i\|, \quad u_i \in P_i.$$

(b) *There exist $k_{i1} \in [0, 1)$, $k_{i2} \in (1, +\infty)$, $k_i, r_i, M_i \in (0, +\infty)$, $e_i \in K_i \backslash \{\theta\}$ and two e_i-positive operators*

$B_{i1}, B_{i2} : P_i \to K_i$ *such that*

$$B_{i1}N_i(u) \leqslant k_{i1}B_{i1}u_i, \quad \|u_i\| \leqslant r_i, \quad u = (u_1, u_2) \in P_1 \times P_2, \quad B_{i2}N_i(u) \geqslant k_{i2}B_{i2}u_i - M_i e_i,$$

and

$$\|B_{i2}u_i\|_i \geqslant k_i\|u_i\|, \ \ u_i \in P_i.$$

Then N *has a fixed point* u *in* $P_1 \times P_2$ *with* $\|u_i\| > 0$ $(i = 1, 2)$.

Proof. We prove Theorem 2.5 for the following case:

$$B_{11}N_1(u) \geqslant k_{11}B_{11}u_1, \quad \|u_1\| \leqslant r_1, \tag{2.1}$$
$$B_{12}N_1(u) \leqslant k_{12}B_{12}u_1 + M_1 e_1, \quad u \in P_1 \times P_2, \tag{2.2}$$
$$B_{21}N_2(u) \leqslant k_{21}B_{21}u_2, \quad \|u_2\| \leqslant r_2, \tag{2.3}$$
$$B_{22}N_2(u) \geqslant k_{22}B_{22}u_2 - M_2 e_2, \quad u \in P_1 \times P_2. \tag{2.4}$$

Since the proof of other cases are analogous to the above case, we omit it.

Let

$$R_1 = \max\{\frac{M_1\overline{N}_1\|e_1\|_1}{k_1(1-k_{12})} + 1, r_1 + 1\}, \quad R_2 = \max\{\frac{M_2\overline{N}_2\|e_2\|_2}{k_2(k_{22}-1)} + 1, r_2 + 1\},$$
$$U = \left(B_{R_1}\backslash\overline{B_{\frac{r_1}{2}}}\right) \times \left(B_{R_2}\backslash\overline{B_{\frac{r_2}{2}}}\right) = \{(u_1, u_2) \in P_1 \times P_2 : \frac{r_i}{2} < \|u_i\| < R_i, \ \ i = 1, 2\}.$$

Clearly, U is an open set of $P_1 \times P_2$. For $t \in [0, 1]$, we consider the completely continuous homotopy $H : [0, 1] \times \overline{U} \to P_1 \times P_2$ defined by

$$H(t, u) = (N_1(u_1, tu_2), N_2(tu_1, u_2)), \quad u = (u_1, u_2) \in P_1 \times P_2.$$

We will prove that $\{H(t, u)\}_{t\in[0,1]}$ satisfy $H(t, u) \neq u$ for each $t \in [0, 1]$ and $u \in \partial U = \{(u_1, u_2) \in P_1 \times P_2 : \|u_1\| = \frac{r_1}{2}$ or $\|u_1\| = R_1$ or $\|u_2\| = \frac{r_2}{2}$ or $\|u_2\| = R_2\}$, the sufficient conditions for the homotopy invariance of fixed point index on ∂U. Next, we separate the proof of Theorem 2.5 into four steps.

Step 1. We claim that

$$u_1 - N_1(u_1, tu_2) \neq \mu\widetilde{e_1}, \quad \mu \geqslant 0, \quad t \in [0, 1] \quad \text{and} \quad (u_1, u_2) \in \partial B_{\frac{r_1}{2}} \times P_2, \tag{2.5}$$

where $\widetilde{e_1}$ is a fixed element in $P_1\backslash\{\theta\}$. In fact, if there exist $\mu \geqslant 0$, $t \in [0, 1]$ and $(u_1, u_2) \in \partial B_{\frac{r_1}{2}} \times P_2$ such that $u_1 - N_1(u_1, tu_2) = \mu\widetilde{e_1}$, then from (2.1), one deduces that

$$B_{11}u_1 = B_{11}N_1(u_1, tu_2) + \mu B_{11}\widetilde{e_1} \geqslant B_{11}N_1(u_1, tu_2) \geqslant k_{11}B_{11}u_1,$$

that is, $(k_{11} - 1)B_{11}u_1 \leqslant \theta$. Notice that $k_{11} > 1$, by the definition of e_i-positive operator, we obtain that $u_1 = \theta$, which is a contradiction with $u \in \partial B_{\frac{r_1}{2}}$.

Step 2. We claim that

$$N_1(u_1, tu_2) \neq \mu u_1, \quad \mu \geqslant 1, \quad t \in [0, 1] \quad \text{and} \quad (u_1, u_2) \in \partial B_{R_1} \times P_2. \tag{2.6}$$

Suppose that there exist $\mu \geqslant 1$, $t \in [0, 1]$ and $(u_1, u_2) \in \partial B_{R_1} \times P_2$ with $\mu u_1 = N_1(u_1, tu_2)$. So, by (2.2), we have

$$B_{12}u_1 \leqslant \mu B_{12}u_1 = B_{12}N_1(u_1, tu_2) \leqslant k_{12}B_{12}u_1 + M_1 e_1.$$

As a result,

$$(1 - k_{12})B_{12}u_1 \leqslant M_1 e_1.$$

Considering the normality of cone, we get

$$k_1(1 - k_{12})\|u_1\| \leqslant \|(1 - k_{12})B_{12}u_1\|_1 \leqslant \overline{N}_1\|M_1 e_1\|_1,$$

which implies that $\|u_1\| \leqslant \frac{\overline{N}_1 M_1\|e_1\|_1}{k_1(1-k_{12})} < R_1$. This contradicts $u_1 \in \partial B_{R_1}$.

Step 3. We claim that

$$\mu u_2 \neq N_2(tu_1, u_2), \quad \mu \geqslant 1, \quad t \in [0,1] \quad \text{and} \quad (u_1, u_2) \in P_1 \times \partial B_{\frac{r_2}{2}}. \tag{2.7}$$

If this is false, then there exist $\mu \geqslant 1$, $t \in [0,1]$ and $(u_1, u_2) \in P_1 \times \partial B_{\frac{r_2}{2}}$ such that $\mu u_2 = N_2(tu_1, u_2)$. This together with (2.3) yields

$$B_{21}u_2 \leqslant \mu B_{21}u_2 = B_{21}N_2(tu_1, u_2) \leqslant k_{21}B_{21}u_2 < B_{21}u_2,$$

a contradiction follows from the definition of e_i-positive operator and $u_2 \in P_2 \backslash \{\theta\}$. So, (2.7) holds.

Step 4. We claim that

$$u_2 - N_2(tu_1, u_2) \neq \mu \widetilde{e_2}, \quad \mu \geqslant 0, \quad t \in [0,1] \quad \text{and} \quad (u_1, u_2) \in P_1 \times \partial B_{R_2}, \tag{2.8}$$

where $\widetilde{e_2}$ is a fixed element in $P_2 \backslash \{\theta\}$. Suppose that there exist $\mu \geqslant 0$, $t \in [0,1]$ and $(u_1, u_2) \in P_1 \times \partial B_{R_2}$ with $u_2 - N_2(tu_1, u_2) = \mu \widetilde{e_2}$. Therefore by (2.4), we have

$$B_{22}u_2 = B_{22}N_2(tu_1, u_2) + \mu B_{22}\widetilde{e_2} \geqslant B_{22}N_2(tu_1, u_2) \geqslant k_{22}B_{22}u_2 - M_2 e_2,$$

which implies

$$M_2 e_2 \geqslant (k_{22} - 1)B_{22}u_2.$$

In view of the normality of cone, we get

$$\overline{N}_2 M_2 \|e_2\|_2 \geqslant (k_{22} - 1)\|B_{22}u_2\|_2 \geqslant k_2(k_{22} - 1)\|u_2\|.$$

This guarantees

$$\|u_2\| \leqslant \frac{\overline{N}_2 M_2 \|e_2\|_2}{k_2(k_{22} - 1)} < R_2,$$

which is a contradiction. Hence (2.8) is true.

Based on the expressions (2.5), (2.6), (2.7) and (2.8), it is not difficult to verify that $\{H(t, u)\}_{t \in [0,1]}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂U. Moreover, by Lemma 2.1, we have

$$i(N_1(\cdot, \theta), B_{R_1}, P_1) = 1, \;\; i(N_1(\cdot, \theta), B_{\frac{r_1}{2}}, P_1) = 0,$$
$$i(N_2(\theta, \cdot), B_{R_2}, P_2) = 0, \;\; i(N_2(\theta, \cdot), B_{\frac{r_2}{2}}, P_2) = 1.$$

Then from the homotopy invariance property of fixed point index and Lemma 2.2, we have

$$\begin{aligned}
i(H(1, \cdot), &\left(B_{R_1} \backslash \overline{B_{\frac{r_1}{2}}}\right) \times \left(B_{R_2} \backslash \overline{B_{\frac{r_2}{2}}}\right), P_1 \times P_2) \\
&= i(H(0, \cdot), \left(B_{R_1} \backslash \overline{B_{\frac{r_1}{2}}}\right) \times \left(B_{R_2} \backslash \overline{B_{\frac{r_2}{2}}}\right), P_1 \times P_2) \\
&= i(N_1(\cdot, \theta), B_{R_1} \backslash \overline{B_{\frac{r_1}{2}}}, P_1) \cdot i(N_2(\theta, \cdot), B_{R_2} \backslash \overline{B_{\frac{r_2}{2}}}, P_2) \\
&= (i(N_1(\cdot, \theta), B_{R_1}, P_1) - i(N_1(\cdot, \theta), B_{\frac{r_1}{2}}, P_1) \cdot (i(N_2(\theta, \cdot), B_{R_2}, P_2) - i(N_2(\theta, \cdot), B_{\frac{r_2}{2}}, P_2)) \\
&= -1.
\end{aligned}$$

Thus we are able to apply the solution property of the fixed point index to show that N has a fixed point u in $P_1 \times P_2$ with $\|u_i\| > 0$ $(i = 1, 2)$.

□

Remark 2.6. In Theorem 2.5 other three cases are possible for $u = (u_1, u_2) \in P_1 \times P_2$:

(C1) $B_{11}N_1(u) \geqslant k_{11}B_{11}u_1,\ \|u_1\| \leqslant r_1; B_{12}N_1(u) \leqslant k_{12}B_{12}u_1 + M_1e_1,\ u \in P_1 \times P_2,$
$B_{21}N_2(u) \geqslant k_{21}B_{21}u_2,\ \|u_2\| \leqslant r_2; B_{22}N_2(u) \leqslant k_{22}B_{22}u_2 + M_2e_2,\ u \in P_1 \times P_2.$

(C2) $B_{11}N_1(u) \leqslant k_{11}B_{11}u_1,\ \|u_1\| \leqslant r_1; B_{12}N_1(u) \geqslant k_{12}B_{12}u_1 - M_1e_1,\ u \in P_1 \times P_2,$
$B_{21}N_2(u) \geqslant k_{21}B_{21}u_2,\ \|u_2\| \leqslant r_2; B_{22}N_2(u) \leqslant k_{22}B_{22}u_2 + M_2e_2,\ u \in P_1 \times P_2.$

(C3) $B_{11}N_1(u) \leqslant k_{11}B_{11}u_1,\ \|u_1\| \leqslant r_1; B_{12}N_1(u) \geqslant k_{12}B_{12}u_1 - M_1e_1,\ u \in P_1 \times P_2,$
$B_{21}N_2(u) \leqslant k_{21}B_{21}u_2,\ \|u_2\| \leqslant r_2; B_{22}N_2(u) \geqslant k_{22}B_{22}u_2 - M_2e_2,\ u \in P_1 \times P_2.$

Theorem 2.7. *The conclusion of Theorem 2.5 also holds if condition* (b) *is replaced with*

(b′) *There exist* $k_{i1} \in [0, 1)$, $k_{i2} \in (1, +\infty)$, $r_1, r_2 \in (0, +\infty)$ *with* $r_1 < r_2$, $e_i \in K_i$ *and two* e_i*-positive operators* B_{i1}, B_{i2} *such that*

$$B_{i1}N_i(u) \leqslant k_{i1}B_{i1}u_i, \quad \|u_i\| \leqslant r_1,$$

and

$$B_{i2}N_i(u) \geqslant k_{i2}B_{i2}u_i, \quad \|u_i\| \geqslant r_2.$$

3. Application

In this section, we shall apply Theorem 2.5 to the existence of positive solution for system of second and fourth order ordinary differential equations

$$\begin{cases} u_1^{(4)} = g_1(t)f_1(t, u_1, u_2), \quad t \in I = [0, 1], \\ u_2'' = -g_2(t)f_2(t, u_1, u_2), \\ u_1(0) = u_1(1) = u_1''(0) = u_1''(1) = 0, \\ u_2(0) = u_2(1) = 0. \end{cases} \tag{3.1}$$

To establish the existence of positive solutions, we make the following assumptions:

(H_1) $g_i \in C[(0, 1), R^+]$, $0 < \int_0^1 g_i(t)dt < +\infty$, $i = 1, 2$;

(H_2) $f_1, f_2 \in C[I \times R^+ \times R^+, R^+]$.

We consider BVP (3.1) in $C[0, 1] \times C[0, 1]$. Evidently, $(C[0, 1], \|\cdot\|)$ is a Banach space with norm $\|\varphi\| = \max_{0 \leqslant t \leqslant 1} |\varphi(t)|$. We construct cones K, P by

$$K = \{\varphi \in C[0, 1] \mid \varphi(t) \geqslant 0,\ t \in [0, 1]\},$$

$$P = \{\varphi \in C[0, 1] \mid \varphi(t) \geqslant t(1 - t)\|\varphi\|,\ t \in [0, 1]\}.$$

Throughout the rest of paper, the partial ordering is always given by K.

We start by some preliminaries and a lemma. Obviously, $(u_1, u_2) \in C^4(0, 1) \times C^2[0, 1]$ is a solution of BVP (3.1) if and only if $(u_1, u_2) \in C[0, 1] \times C[0, 1]$ is a solution of the following nonlinear integral system:

$$\begin{cases} u_1(t) = \displaystyle\int_0^1 G_1(t, s)g_1(s)f_1(s, u_1(s), u_2(s))ds, \\ u_2(t) = \displaystyle\int_0^1 G_2(t, s)g_2(s)f_2(s, u_1(s), u_2(s))ds, \end{cases} \tag{3.2}$$

where

$$G_1(t, s) = \int_0^1 G_2(t, \tau)G_2(\tau, s)d\tau, \quad G_2(t, s) = \begin{cases} t(1 - s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1 - t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

It is an elementary fact that the function $G_i(t, s)$ has the following property:

$$\frac{1}{30}t(1 - t)s(1 - s) \leqslant G_i(t, s) = G_i(s, t) \leqslant t(1 - t)(\text{or } s(1 - s)), \quad t, s \in [0, 1]. \tag{3.3}$$

In order to solve (3.2), we consider the operator $N : K \times K \to K \times K$ defined by

$$Nu = (N_1u, N_2u),$$

where $N_i = T_iF_i,\ i = 1, 2,$

$$(T_i\varphi)(t) = \int_0^1 G_i(t,s)g_i(s)\varphi(s)ds, \quad \varphi \in K,$$

$$F_i(u_1,u_2)(t) = f_i(t,u_1(t),u_2(t)), \quad u = (u_1,u_2) \in K \times K.$$

Thus solving (3.2) is equivalent to finding fixed point of operator

$$N(u_1,u_2) = (T_1F_1u, T_2F_2u), \quad u \in K \times K.$$

Condition (H_1) and (3.3) imply that $T_i : P \to P$ $(i = 1, 2)$ are two completely continuous linear e-positive operators with $e(t) = t(1-t)$. Thus by the famous Krein-Rutman theorem (see [14]), we assert that $r(T_i) > 0$ and there exist $\varphi_i^* \in P$ $(i = 1, 2)$ with $\|\varphi_i^*\| = 1$ such that

$$r(T_i)\varphi_i^* = T_i\varphi_i^*,$$

which can be rewritten in the form

$$r(T_i)\varphi_i^*(t) = \int_0^1 G_i(t,s)g_i(s)\varphi_i^*(s)ds$$

for all $t \in [0, 1]$. Define operators $B_i : K \to K$ by

$$(B_i\varphi)(t) = \int_0^1 \varphi_i^*(s)g_i(s)\varphi(s)ds, \ \ \varphi \in K.$$

Since $\varphi_i^* \in P$ is the positive eigenfunction of T_i, it follows from (3.3) that

$$\varphi_i^*(t) \geqslant \frac{1}{30r(T_i)}t(1-t)\int_0^1 s(1-s)g_i(s)\varphi_i^*(s)ds, \tag{3.4}$$

and

$$\varphi^*(t) \leqslant \frac{1}{r(T_i)}\int_0^1 s(1-s)g_i(s)\varphi_i^*(s)ds,$$

therefore $\int_0^1 s(1-s)g_i(s)\varphi_i^*(s)ds > 0$. Set $\delta_i = \frac{1}{30r(T_i)}\int_0^1 s(1-s)g_i(s)\varphi_i^*(s)ds > 0$. By the help of (3.4), we have

$$\delta_iG_i(t,s) \leqslant \varphi_i^*(s). \tag{3.5}$$

Lemma 3.1. *Let*

$$P_i = \{\varphi \in K \mid (B_i\varphi)(t) \geqslant r(T_i)\delta_i\|\varphi\|\}.$$

Then $T_i(K) \subset P_i$.

Proof. It follows from (3.5) that for every $\varphi \in K$

$$\begin{aligned}(B_iT_i\varphi)(t) &= \int_0^1 \varphi_i^*(s)g_i(s)(T_i\varphi)(s)ds\\ &= \int_0^1 \varphi_i^*(s)g_i(s)\int_0^1 G_i(s,\tau)g_i(\tau)\varphi(\tau)d\tau ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 g_i(\tau)\varphi(\tau)\int_0^1 G_i(\tau,s)g_i(s)\varphi_i^*(s)dsd\tau \\
&= r(T_i)\int_0^1 \varphi_i^*(\tau)g_i(\tau)\varphi(\tau)d\tau \\
&\geqslant r(T_i)\delta_i\int_0^1 G_i(t,\tau)g_i(\tau)\varphi(\tau)d\tau = r(T_i)\delta_i(T_i\varphi)(t).
\end{aligned}
$$

Then $(B_iT_i\varphi)(t) \geqslant r(T_i)\delta_i\|T_i\varphi\|$, i.e., $T_i(K) \subset P_i$.

□

Theorem 3.2. *Suppose that the conditions* (H_1)-(H_2) *are satisfied and that for each* $i \in \{1,2\}$ *one of the following conditions is satisfied:*

(H_3) *There exist* $k_{i1} \in (1,+\infty)$, $k_{i2} \in [0,1)$, $r_i, M_i \in (0,+\infty)$ *such that*

$$
f_i(t,u_1,u_2) \geqslant \frac{k_{i1}}{r(T_i)}u_i, \quad u_i \leqslant r_i,
$$

$$
f_i(t,u_1,u_2) \leqslant \frac{k_{i2}}{r(T_i)}u_i + M_i;
$$

(H_4) *There exist* $k_{i1} \in [0,1)$, $k_{i2} \in (1,+\infty)$, $r_i, M_i \in (0,+\infty)$ *such that*

$$
f_i(t,u_1,u_2) \leqslant \frac{k_{i1}}{r(T_i)}u_i, \quad u_i \leqslant r_i,
$$

$$
f_i(t,u_1,u_2) \geqslant \frac{k_{i2}}{r(T_i)}u_i - M_i.
$$

Then (3.1) *has a positive solution.*

Proof. Notice that

$$
\begin{aligned}
(B_iN_iu)(t) &= \int_0^1 \varphi_i^*(s)g_i(s)(T_iF_iu)(s)ds \\
&= \int_0^1 \varphi_i^*(s)g_i(s)\int_0^1 G_i(s,\tau)g_i(\tau)F_iu(\tau)d\tau ds \\
&= \int_0^1 g_i(\tau)F_iu(\tau)\int_0^1 G_i(\tau,s)g_i(s)\varphi_i^*(s)dsd\tau \\
&= r(T_i)\int_0^1 \varphi_i^*(\tau)g_i(\tau)F_iu(\tau)d\tau \\
&= r(T_i)(B_iF_iu)(t).
\end{aligned}
$$

Then Theorem 3.2 follows from Theorem 2.5 and Lemma 3.1 with $e_i = 1$ and $B_{i1} = B_{i2} = B_i$.

□

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