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Refinements of Hermite-Hadamard inequality for operator convex function

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Abstract

In this paper, we present several operator versions of the Hermite-Hadamard inequality for the operator convex function, which are refinements of some operator convex inequalities presented by Dragomir [S. S. Dragomir, Appl. Math. Comput., **218** (2011), 766–772] and [S. S. Dragomir, RGMIA Research Report Collection, **2016** (2016), 15 pages].

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1. Introduction

The following classical inequality, which was first discovered by Hermite in 1883 in the journal Mathesis (see [10]) and independently proved in 1893 by Hadamard in [9], is well-known as the Hermite-Hadamard inequality in the literature:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a convex function on an interval $[a,b]$. Both inequalities hold in the reversed direction if f is concave.

For any convex function defined on a segment $[a,b]$, one can easily observe that (1.1) is equivalent to the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \int_0^1 f[(1-t)a+tb]dt \leqslant \frac{f(a)+f(b)}{2}.$$

We note that Hermite-Hadamard inequality, regarded as a refinement of the concept of convexity, has several applications in nonlinear analysis and the geometry of Banach spaces. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [2, 6, 12]).

A real-valued continuous function f on an interval I is said to be operator convex if

$$f((1-\lambda)A+\lambda B) \leqslant (1-\lambda)f(A)+\lambda f(B),$$

in the operator order, for all $\lambda \in [0,1]$ and for self-adjoint operators A and B on a Hilbert space H whose

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spectra are contained in I.

In 2011, Dragomir [4] established the following Hermite-Hadamard type inequality for operator convex function.

Theorem 1.1. *Let* $f: I \to \mathbb{R}$ *be an operator convex function on the interval* I. *Then for any self-adjoint operators* A *and* B *with spectra in* I *we have the inequality*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leqslant \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right] \\ &\leqslant \int_0^1 f((1-t)A+tB)dt \leqslant \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right] \leqslant \frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.2}$$

In 2016, Dragomir [5] gave a refinement of the above Hermite-Hadamard inequality for operator convex function (1.2) as follows.

Theorem 1.2. *Let* $f: I \to \mathbb{R}$ *be an operator convex function on the interval* I. *Then for any self-adjoint operators* A *and* B *with spectra in* I *and for any* $\lambda \in [0,1]$ *we have the inequality*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leqslant (1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \\ &\leqslant \int_0^1 f((1-t)A+tB)dt \leqslant \frac{1}{2}\left[f\left((1-\lambda)A+\lambda B\right)+(1-\lambda)f(B)+\lambda f(A)\right] \leqslant \frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.3}$$

For recent related results on Hermite-Hadamard type operator inequality, see [1, 8, 11].

In this paper, we present several operator versions of the Hermite-Hadamard inequality for the operator convex function, which are refinements of operator convex inequalities (1.2) and (1.3).

2. Main results

For the first and the second inequalities in the Hermite-Hadamard inequality (1.1), Feng and Burqan constructed the following refinements, respectively.

Lemma 2.1 ([7]). *Let* f *be a real-valued function which is convex on the interval* $[a,b]$. *Then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right] \leqslant \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Lemma 2.2 ([3]). *Let* f *be a real-valued function which is convex on the interval* $[a,b]$. *Then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}. \tag{2.2}$$

We present a refinement of the above Hermite-Hadamard inequality (2.1) and (2.2) as follows.

Lemma 2.3. *Let* f *be a real-valued function which is convex on the interval* $[a,b]$ *and let* n *be a positive integer. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{2}\left\{f\left[\frac{(2^n+1)a+(2^n-1)b}{2^{n+1}}\right]+f\left[\frac{(2^n-1)a+(2^n+1)b}{2^{n+1}}\right]\right\} \\ &\leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{1}{2(n+1)}\left[nf(a)+2f\left(\frac{a+b}{2}\right)+nf(b)\right] \leqslant \frac{f(a)+f(b)}{2}. \end{aligned}$$

Proof. Since f is convex on $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2}.$$

Thus

$$\frac{1}{2(n+1)}\left[nf(a)+2f\left(\frac{a+b}{2}\right)+nf(b)\right]\leqslant\frac{1}{2(n+1)}[nf(a)+f(a)+f(b)+nf(b)]=\frac{f(a)+f(b)}{2}.$$

This completes the proof of the fourth inequality.

Using Hermite-Hadamard inequality, we have

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&=f\left[\frac{1}{2}\cdot\frac{(2^n+1)a+(2^n-1)b}{2^{n+1}}+\frac{1}{2}\cdot\frac{(2^n-1)a+(2^n+1)b}{2^{n+1}}\right]\\&\leqslant\frac{1}{2}\left\{f\left[\frac{(2^n+1)a+(2^n-1)b}{2^{n+1}}\right]+f\left[\frac{(2^n-1)a+(2^n+1)b}{2^{n+1}}\right]\right\}.\end{aligned}$$

This completes the proof of the first inequality.

To prove the second and the third inequalities, it is only needed to prove the following inequalities by Lemmas 2.1 and 2.2:

$$f\left[\frac{(2^n+1)a+(2^n-1)b}{2^{n+1}}\right]+f\left[\frac{(2^n-1)a+(2^n+1)b}{2^{n+1}}\right]\leqslant f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right) \tag{2.3}$$

and

$$\frac{1}{2}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\leqslant\frac{1}{n+1}\left[nf(a)+2f\left(\frac{a+b}{2}\right)+nf(b)\right]. \tag{2.4}$$

Next, we prove inequality (2.3) by induction. By Lemma 2.2, we have

$$f(a)+f(b)\geqslant f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right). \tag{2.5}$$

So (2.3) holds trivially for the case $n=1$. Now suppose the assertion (2.3) holds for the case $n=k$. By the induction hypothesis, we have

$$\begin{aligned}&f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\\&\geqslant f\left[\frac{(2^k+1)a+(2^k-1)b}{2^{k+1}}\right]+f\left[\frac{(2^k-1)a+(2^k+1)b}{2^{k+1}}\right]\\&\geqslant f\left[\frac{3\cdot\frac{(2^k+1)a+(2^k-1)b}{2^{k+1}}+\frac{(2^k-1)a+(2^k+1)b}{2^{k+1}}}{4}\right]+f\left[\frac{\frac{(2^k+1)a+(2^k-1)b}{2^{k+1}}+3\cdot\frac{(2^k-1)a+(2^k+1)b}{2^{k+1}}}{4}\right]\quad(\text{by }(2.5))\\&=f\left[\frac{(2^{k+1}+1)a+(2^{k+1}-1)b}{2^{(k+1)+1}}\right]+f\left[\frac{(2^{k+1}-1)a+(2^{k+1}+1)b}{2^{(k+1)+1}}\right].\end{aligned}$$

Hence, (2.3) holds for the case $n=k+1$. Similarly, the inequality (2.4) holds by induction. Given all that, the proof of Lemma 2.3 is complete. □

Remark 2.4. When $n=1$, it is easy to see that Lemma 2.1 and Lemma 2.2 are special cases of Lemma 2.3.

By Lemma 2.3, we obtain our first refinement of Hermite-Hadamard inequality for operator convex function.

Theorem 2.5. *Let* $f: I\to\mathbb{R}$ *be an operator convex function on the interval* I *and let* n *be a positive integer. Then for any self-adjoint operators* A *and* B *on a Hilbert space* H *with spectra in* I *we have the inequality*

$$\begin{aligned}f\left(\frac{A+B}{2}\right)&\leqslant\frac{1}{2}\left\{f\left[\frac{(2^n+1)A+(2^n-1)B}{2^{n+1}}\right]+f\left[\frac{(2^n-1)A+(2^n+1)B}{2^{n+1}}\right]\right\}\\&\leqslant\int_0^1 f((1-t)A+tB)dt\leqslant\frac{1}{2(n+1)}\left[nf(A)+2f\left(\frac{A+B}{2}\right)+nf(B)\right]\leqslant\frac{f(A)+f(B)}{2}.\end{aligned}\tag{2.6}$$

Proof. Let $x \in H$ be a unit vector and two self-adjoint operators A and B with spectra in I. Define the real-valued function $\rho(t) = \langle f((1-t)A + tB)x, x\rangle$ on the interval $[0,1]$. Since f is operator convex, then for any $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$ we have

$$\begin{aligned}\rho(\alpha t_1 + \beta t_2) &= \langle f((1-(\alpha t_1 + \beta t_2))A + (\alpha t_1 + \beta t_2)B)x, x\rangle \\ &= \langle f(\alpha[(1-t_1)A + t_1 B] + \beta[(1-t_2)A + t_2 B])x, x\rangle \\ &\leqslant \alpha\langle f((1-t_1)A + t_1 B)x, x\rangle + \beta\langle f((1-t_2)A + t_2 B)x, x\rangle = \alpha\rho(t_1) + \beta\rho(t_2).\end{aligned}$$

So $\rho(t)$ is a convex function on $[0,1]$. Applying Lemma 2.3 to the convex function $\rho(t)$ on $[0,1]$, we have

$$\begin{aligned}\rho\left(\frac{1}{2}\right) &\leqslant \frac{1}{2}\left[\rho\left(\frac{2^n-1}{2^{n+1}}\right) + \rho\left(\frac{2^n+1}{2^{n+1}}\right)\right] \\ &\leqslant \int_0^1 \rho(t)dt \leqslant \frac{1}{2(n+1)}\left[n\rho(0) + 2\rho\left(\frac{1}{2}\right) + n\rho(1)\right] \leqslant \frac{\rho(0)+\rho(1)}{2}.\end{aligned}$$

Hence

$$\begin{aligned}\left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle &\leqslant \frac{1}{2}\left\langle\left\{f\left[\frac{(2^n+1)A + (2^n-1)B}{2^{n+1}}\right] + f\left[\frac{(2^n-1)A + (2^n+1)B}{2^{n+1}}\right]\right\}x, x\right\rangle \\ &\leqslant \int_0^1 \langle f((1-t)A + tB)x, x\rangle dt \\ &\leqslant \frac{1}{2(n+1)}\left\langle\left[nf(A) + 2f\left(\frac{A+B}{2}\right) + nf(B)\right]x, x\right\rangle \leqslant \frac{1}{2}\langle[f(A) + f(B)]x, x\rangle.\end{aligned} \tag{2.7}$$

Finally, since by the continuity of the function f we have

$$\int_0^1 \langle f((1-t)A + tB)x, x\rangle dt = \left\langle \int_0^1 f((1-t)A + tB)dtx, x\right\rangle \tag{2.8}$$

for any $x \in H$, $\|x\| = 1$ and any self-adjoint operators A and B with spectra in I. Now (2.7) and (2.8) yield the whole inequalities (2.6) as desired. □

Remark 2.6. Theorem 1.1 is a special case of Theorem 2.5 when $n = 1$.

Corollary 2.7. *Under the assumptions of Theorem 2.5, if* $n = 2$, *then*

$$\begin{aligned}f\left(\frac{A+B}{2}\right) &\leqslant \frac{1}{2}\left[f\left(\frac{5A+3B}{8}\right) + f\left(\frac{3A+5B}{8}\right)\right] \\ &\leqslant \int_0^1 f((1-t)A + tB)dt \leqslant \frac{1}{3}\left[f(A) + f\left(\frac{A+B}{2}\right) + f(B)\right] \leqslant \frac{f(A)+f(B)}{2}.\end{aligned}$$

The following representation result is useful in the sequel.

Lemma 2.8 ([5]). *Let* $f : I \to \mathbb{C}$ *be a continuous function on the interval* I *and two self-adjoint operators* A *and* B *on a Hilbert space* H *with spectra in* I. *Then for any* $\lambda \in [0,1]$ *we have the representation*

$$\begin{aligned}\int_0^1 f((1-t)A + tB)dt &= (1-\lambda)\int_0^1 f[(1-t)((1-\lambda)A + \lambda B) + tB]dt \\ &\quad + \lambda\int_0^1 f[(1-t)A + t((1-\lambda)A + \lambda B)]dt.\end{aligned} \tag{2.9}$$

Theorem 2.9. *Let* $f: I \to \mathbb{R}$ *be an operator convex function on the interval* I *and let* n *be a positive integer. Then for any self-adjoint operators* A *and* B *on a Hilbert space* H *with spectra in* I *we have the inequality*

$$\begin{aligned}
f\left(\frac{A+B}{2}\right) &\leqslant (1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\\
&\leqslant \frac{1-\lambda}{2}\left\{f\left[\frac{(2^n+1)(1-\lambda)A+[(2^n+1)\lambda+(2^n-1)]B}{2^{n+1}}\right]\right.\\
&\quad\left.+f\left[\frac{(2^n-1)(1-\lambda)A+[(2^n-1)\lambda+(2^n+1)]B}{2^{n+1}}\right]\right\}\\
&\quad+\frac{\lambda}{2}\left\{f\left[\frac{[(2^n-1)(1-\lambda)+(2^n+1)]A+(2^n-1)\lambda B}{2^{n+1}}\right]\right.\\
&\quad\left.+f\left[\frac{[(2^n+1)(1-\lambda)+(2^n-1)]A+(2^n+1)\lambda B}{2^{n+1}}\right]\right\}\\
&\leqslant \int_0^1 f((1-t)A+tB)dt\\
&\leqslant \frac{1}{2(n+1)}\left\{\lambda n f(A)+(1-\lambda)n f(B)+n f[(1-\lambda)A+\lambda B]\right.\\
&\quad\left.+2(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+2\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\right\}\\
&\leqslant \frac{1}{2}\{f[(1-\lambda)A+\lambda B]+(1-\lambda)f(B)+\lambda f(A)\}\leqslant\frac{f(A)+f(B)}{2}.
\end{aligned}\tag{2.10}$$

Proof. Using the Hermite-Hadamard inequality (2.6) we have

$$\begin{aligned}
f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right] &\leqslant \frac{1}{2}\left\{f\left[\frac{(2^n+1)(1-\lambda)A+[(2^n+1)\lambda+(2^n-1)]B}{2^{n+1}}\right]\right.\\
&\quad\left.+f\left[\frac{(2^n-1)(1-\lambda)A+[(2^n-1)\lambda+(2^n+1)]B}{2^{n+1}}\right]\right\}\\
&\leqslant \int_0^1 f[(1-t)((1-\lambda)A+\lambda B)+tB]dt\\
&\leqslant \frac{1}{2(n+1)}\left\{n f[(1-\lambda)A+\lambda B]+2f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+n f(B)\right\}\\
&\leqslant \frac{f[(1-\lambda)A+\lambda B]+f(B)}{2}.
\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}
f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] &\leqslant \frac{1}{2}\left\{f\left[\frac{[(2^n-1)(1-\lambda)+(2^n+1)]A+(2^n-1)\lambda B}{2^{n+1}}\right]\right.\\
&\quad\left.+f\left[\frac{[(2^n+1)(1-\lambda)+(2^n-1)]A+(2^n+1)\lambda B}{2^{n+1}}\right]\right\}\\
&\leqslant \int_0^1 f[(1-t)A+t((1-\lambda)A+\lambda B)]dt\\
&\leqslant \frac{1}{2(n+1)}\left\{n f(A)+2f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]+n f[(1-\lambda)A+\lambda B]\right\}\\
&\leqslant \frac{f(A)+f[(1-\lambda)A+\lambda B]}{2}.
\end{aligned}\tag{2.12}$$

If we multiply inequality (2.11) by $1-\lambda$ and (2.12) by λ, add the obtained inequalities, and use representation (2.9), then we have

$$
\begin{aligned}
&(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\\
&\leqslant \frac{1-\lambda}{2}\left\{f\left[\frac{(2^n+1)(1-\lambda)A+[(2^n+1)\lambda+(2^n-1)]B}{2^{n+1}}\right]\right.\\
&\quad\left.+f\left[\frac{(2^n-1)(1-\lambda)A+[(2^n-1)\lambda+(2^n+1)]B}{2^{n+1}}\right]\right\}\\
&\quad+\frac{\lambda}{2}\left\{f\left[\frac{[(2^n-1)(1-\lambda)+(2^n+1)]A+(2^n-1)\lambda B}{2^{n+1}}\right]\right.\\
&\quad\left.+f\left[\frac{[(2^n+1)(1-\lambda)+(2^n-1)]A+(2^n+1)\lambda B}{2^{n+1}}\right]\right\}\\
&\leqslant \int_0^1 f((1-t)A+tB)dt\\
&\leqslant \frac{1}{2(n+1)}\left\{\lambda nf(A)+(1-\lambda)nf(B)+nf[(1-\lambda)A+\lambda B]\right.\\
&\quad\left.+2(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+2\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\right\}\\
&\leqslant (1-\lambda)\frac{f[(1-\lambda)A+\lambda B]+f(B)}{2}+\lambda\frac{f(A)+f[(1-\lambda)A+\lambda B]}{2},
\end{aligned}
$$

which proves the inequality in (2.10) except the first and last inequalities. By the operator convexity of f we have

$$
\begin{aligned}
&(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\\
&\geqslant f\left[(1-\lambda)\frac{(1-\lambda)A+(1+\lambda)B}{2}+\lambda\frac{(2-\lambda)A+\lambda B}{2}\right]=f\left(\frac{A+B}{2}\right)
\end{aligned}
$$

and

$$
\frac{1}{2}\{f[(1-\lambda)A+\lambda B]+(1-\lambda)f(B)+\lambda f(A)\}\leqslant\frac{1}{2}\{(1-\lambda)f(A)+\lambda f(B)+(1-\lambda)f(B)+\lambda f(A)\}=\frac{f(A)+f(B)}{2},
$$

which proves the first and last inequalities in (2.10). So Theorem 2.9 is proved. □

Corollary 2.10. *Under the assumptions of Theorem 2.9, if* $n=1$, *then*

$$
\begin{aligned}
f\left(\frac{A+B}{2}\right)&\leqslant(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\\
&\leqslant\frac{1-\lambda}{2}\left\{f\left[\frac{3(1-\lambda)A+(1+3\lambda)B}{4}\right]+f\left[\frac{(1-\lambda)A+(3+\lambda)B}{4}\right]\right\}\\
&\quad+\frac{\lambda}{2}\left\{f\left[\frac{(4-\lambda)A+\lambda B}{4}\right]+f\left[\frac{(4-3\lambda)A+3\lambda B}{4}\right]\right\}\\
&\leqslant\int_0^1 f((1-t)A+tB)dt
\end{aligned}
$$

$$\leqslant \frac{1}{4}\Bigg\{\lambda f(A)+(1-\lambda)f(B)+f[(1-\lambda)A+\lambda B]+2\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]$$
$$+2(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]\Bigg\}$$
$$\leqslant \frac{1}{2}\{f[(1-\lambda)A+\lambda B]+(1-\lambda)f(B)+\lambda f(A)\}\leqslant \frac{f(A)+f(B)}{2}.$$

Remark 2.11. Theorem 2.9 and Corollary 2.10 are refinements of Theorem 1.2.

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References

[1] V. Bacak, R. Türkmen, *Refinements of Hermite-Hadamard type inequalities for operator convex functions*, J. Inequal. Appl., **2013** (2013), 10 pages. 1
[2] N. S. Barnett, P. Cerone, S. S. Dragomir, *Some new inequalities for Hermite-Hadamard divergence in information theory, Stochastic analysis and applications*, Stoch. Anal. Appl., **3** 2002, 7–19. 1
[3] A. Burqan, *Improved Cauchy-Schwarz norm inequality for operators*, J. Math. Inequal., **10** (2016), 205–211. 2.2
[4] S. S. Dragomir, *Hermite-Hadamard type inequalities for operator convex functions*, Appl. Math. Comput., **218** (2011), 766–772. 1
[5] S. S. Dragomir, *Some Hermite-Hadamard Type Inequalities for Operator Convex Functions and Positive Maps*, RGMIA Research Report Collection, **2016** (2016), 15 pages. 1, 2.8
[6] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities*, RGMIA Monographs, Victoria University, (2000). 1
[7] Y. Feng, *Refinements of the Heinz inequalities*, J. Inequal. Appl., **2012** (2012), 6 pages. 2.1
[8] A. G. Ghazanfari, A. Barani, *Some Hermite-Hadamard type inequalities for the product of two operator preinvex functions*, Banach J. Math. Anal., **9** (2015), 9–20. 1
[9] J. Hadamard, *Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann (French)*, ?J. Math. Pures Appl., **58** (1893), 171–216. 1
[10] D. S. Mitrinović, I. B. Lacković, *Hermite and convexity*, Aequationes Math., **28** (1985), 229–232. 1
[11] M. S. Moslehian, *Matrix Hermite-Hadamard type inequalities*, Houston J. Math., **39** (2013), 177–189. 1
[12] C. P. Niculescu, L.-E. Persson, *Convex functions and their applications: A contemporary approach*, Springer, New York, (2006). 1