Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 6056–6061
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Existence and multiplicity of positive solutions for a nonlocal problem

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Communicated by K. Q. Lan

Abstract

In this work, we are interested in considering the following nonlocal problem

$$\begin{cases} -\left(a-b\int_{\Omega}|\nabla u|^2dx\right)\Delta u = f(x)|u|^{p-2}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $a, b > 0, 1 \leqslant p < 2^*$, $f \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is nonzero and nonnegative. By using the variational method, some existence and multiplicity results are obtained.

Keywords: Nonlocal problem, positive solutions, variational method.
2010 MSC: 35B09, 35J60, 35J67.

1. Introduction and main result

In this paper, we consider the following nonlocal problem

$$\begin{cases} -\left(a-b\int_{\Omega}|\nabla u|^2dx\right)\Delta u = f(x)|u|^{p-2}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $a, b > 0, 1 \leqslant p < 2^*$, the weight function $f \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is nonzero and nonnegative. $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, 2^*]$, where $H_0^1(\Omega)$ is a Sobolev space equipped with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2dx\right)^{\frac{1}{2}}$, and $|u|_q = \left(\int_\Omega |u|^q\right)^{\frac{1}{q}}$ denotes the norm of $L^q(\Omega)$.

When $2 < p < 2^*$ and $f(x) \equiv 1$, problem (1.1) was considered by [5] for the first time. By using the mountain pass lemma, they obtained the existence of nontrivial solutions for problem (1.1). One of their

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doi:10.22436/jnsa.010.11.40

Received 2017-07-07

important works is the $(PS)_c$ condition with $c < \frac{a^2}{4b}$. Recently, [3] studied problem (1.1) with $N = 3$ and $1 < p < 2$. When $f \in L^\infty(\Omega)$ changes sign, they got two positive solutions by the variational method and Harnack inequality. Compared with [5], they used a different method to prove the $(PS)_c$ condition with $c < \frac{a^2}{4b}$. While $0 < p < 1$ and $N = 3$, problem (1.1) was researched by [2].

Inspired by the works in [2, 3] and [5], we study the existence and multiplicity of positive solutions for problem (1.1) with $N \geqslant 3$ and $1 \leqslant p < 2^*$. Via the variational method and strong maximum principle, when $1 \leqslant p < 2$, we obtain two positive solutions of problem (1.1); while $2 \leqslant p \leqslant 2^*$, we get the existence of positive solutions of problem (1.1). Our results generalize and complete the results of [3] and [5].

The energy functional corresponding to problem (1.1) is given by

$$I(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_\Omega f(x)|u|^p\,dx, \quad \forall u \in H_0^1(\Omega).$$

In general, a function u is called a weak solution of problem (1.1) if $u \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ it holds

$$(a - b\|u\|^2)\int_\Omega (\nabla u, \nabla\varphi)\,dx - \int_\Omega f(x)|u|^{p-2}u\varphi\,dx = 0. \tag{1.2}$$

Let S be the best Sobolev constant, namely

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}}. \tag{1.3}$$

Now our main result can be described as follows:

Theorem 1.1. *Assume that $a, b > 0$, $1 \leqslant p < 2^*$ and $f \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is nonzero and nonnegative, then*

(1) *when $1 \leqslant p < 2$, there exists $T > 0$ such that for any $|f|_{\frac{2^*}{2^*-p}} < T$, (1.1) has at least two positive solutions u_*, u_{**} with $I(u_*) < 0$ and $I(u_{**}) > 0$;*

(2) *when $p = 2, |f|_{\frac{2^*}{2^*-2}} < aS$ or $2 < p < 2^*$, (1.1) has at least one positive mountain-pass solution u_{**} with $I(u_{**}) > 0$.*

Remark 1.2. Compared with [3] and [5], we consider (1.1) with $p = 1, 2$ and obtain the existence of positive solutions by the strong maximum principle. Particular, compared with [5], we study problem (1.1) with $1 \leqslant p \leqslant 2$ and obtain the existence and multiplicity of positive solutions. Compared with [3], we generalize the dimension $N = 3$ to $N \geqslant 3$.

2. Proof of Theorem 1.1

In this part, we will give the proof of Theorem 1.1. Before proving Theorem 1.1, we give the following lemma.

Lemma 2.1. Assume $a, b > 0, 1 \leqslant p < 2^*$ and $f \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is nonzero and nonnegative, then I satisfies the $(PS)_c$ condition with $c < \frac{a^2}{4b}$.

Proof. Suppose $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_c$ sequence for I, that is,

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to \infty. \tag{2.1}$$

By the Hölder inequality and (1.3), one has

$$\int_\Omega f(x)|u|^p\,dx \leqslant |f|_{\frac{2^*}{2^*-p}}|u|_{2^*}^p \leqslant |f|_{\frac{2^*}{2^*-p}} S^{-\frac{p}{2}}\|u\|^p. \tag{2.2}$$

When $1 \leqslant p < 2$, it follows from (2.1) and (2.2) that

$$\begin{aligned} 1 + c + o(\|u_n\|) &\geqslant I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{a}{4}\|u_n\|^2 - \frac{4-p}{4p}\int_\Omega f(x)|u_n|^p\,dx \\ &\geqslant \frac{a}{4}\|u_n\|^2 - \frac{(4-p)|f|_{\frac{2^*}{2^*-p}}}{4pS^{\frac{p}{2}}}\|u_n\|^p, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. When $2 \leqslant p < 2^*$, it follows from (2.1) that

$$\begin{aligned} 1 + c + o(\|u_n\|) &\geqslant I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\ &= \frac{b}{4}\|u_n\|^4 + \frac{p-2}{2p}\int_\Omega f(x)|u_n|^p\,dx \\ &\geqslant \frac{b}{4}\|u_n\|^4, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u, & \text{strongly in } L^s(\Omega),\ 1 \leqslant s < 2^*, \\ u_n(x) \to u(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.3}$$

as $n \to \infty$. Moreover, by the Vitali Theorem, one obtains

$$\lim_{n\to\infty}\int_\Omega f(x)|u_n|^p\,dx = \int_\Omega f(x)|u|^p\,dx.$$

Set $w_n = u_n - u$, then $\|w_n\| \to 0$. Otherwise, there exists a subsequence, still denoted by $\{w_n\}$, such that

$$\lim_{n\to\infty}\|w_n\| = l > 0.$$

From (2.1), for every $\phi \in H_0^1(\Omega)$, it holds

$$(a - b\|u_n\|^2)\int_\Omega (\nabla u_n, \nabla\phi)\,dx - \int_\Omega f(x)|u_n|^{p-2}u_n\phi\,dx = o(1).$$

Letting $n \to \infty$, by using (2.3), we have

$$(a - bl^2 - b\|u\|^2)\int_\Omega (\nabla u, \nabla\phi)\,dx - \int_\Omega f(x)|u|^{p-2}u\phi\,dx = 0. \tag{2.4}$$

Taking $\phi = u$ in (2.4), one has

$$(a - bl^2 - b\|u\|^2)\|u\|^2 - \int_\Omega f(x)|u|^p\,dx = 0. \tag{2.5}$$

Note that $\langle I'(u_n), u_n\rangle \to 0$ as $n \to \infty$, it holds

$$a\|w_n\|^2 + a\|u\|^2 - b\|w_n\|^4 - 2b\|w_n\|^2\|u\|^2 - b\|u\|^4 - \int_\Omega f(x)|u|^p\,dx = o(1). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$a\|w_n\|^2 - b\|w_n\|^4 - b\|w_n\|^2\|u\|^2 = o(1). \tag{2.7}$$

Consequently, one has $l^2(a - b\|u\|^2 - bl^2) = 0$, that is,

$$l^2 = \frac{a}{b} - \|u\|^2. \tag{2.8}$$

On the one hand, from (2.5) and (2.8), we have

$$\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_\Omega f(x)|u|^p\,dx \\
&= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\left(a\|u\|^2 - bl^2\|u\|^2 - b\|u\|^4\right) \\
&= \frac{a(p-2)}{2p}\|u\|^2 + \frac{b(4-p)}{4p}\|u\|^4 + \frac{b}{p}\|u\|^2\frac{a - b\|u\|^2}{b} \\
&= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4.
\end{aligned} \tag{2.9}$$

On the other hand, by (2.1), (2.7) and (2.8), it follows from $c < \frac{a^2}{4b}$ that

$$\begin{aligned}
I(u) &= \lim_{n\to\infty}\left[I(u_n) - \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u\|^2\right] \\
&= \lim_{n\to\infty}\left[I(u_n) - \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{1}{2}\left(a\|w_n\|^2 - b\|w_n\|^4\right)\right] \\
&= c - \frac{b}{4}l^4 \\
&= c - \frac{a^2}{4b} + \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 \\
&< \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4,
\end{aligned}$$

which contradicts (2.9). Hence, $l \equiv 0$, that is, $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$. Therefore, I satisfies the $(PS)_c$ condition for $c < \frac{a^2}{4b}$. This completes the proof of Lemma 2.1. □

Now, we give the following two important propositions.

Proposition 2.2. Assume $1 \leqslant p < 2$ and $f \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is nonzero and nonnegative. There exists $T > 0$ such that for any $|f|_{\frac{2^*}{2^*-p}} < T$, (1.1) has at least one positive local minimal solution u_* with $I(u_*) < 0$.

Proof. We claim that there exist $T, R, \rho > 0$ such that for every $|f|_{\frac{2^*}{2^*-p}} < T$, I satisfies

$$I(u)|_{u\in S_R} \geqslant \rho, \quad \inf_{u\in B_R} I_\lambda(u) < 0,$$

where $B_R = \{u \in H_0^1(\Omega) : \|u\| \leqslant R\}$ is a closed ball and $S_R = \{u \in H_0^1(\Omega) : \|u\| = R\}$. It follows from (2.2) that

$$\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_\Omega f(x)|u|^p\,dx \\
&\geqslant \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{|f|_{\frac{2^*}{2^*-p}}}{pS^{\frac{p}{2}}}\|u\|^p \\
&= \|u\|^p\left(\frac{a}{2}\|u\|^{2-p} - \frac{b}{4}\|u\|^{4-p} - \frac{|f|_{\frac{2^*}{2^*-p}}}{pS^{\frac{p}{2}}}\right).
\end{aligned}$$

For any $t \geqslant 0$, $g(t)$ is defined by

$$g(t) = \frac{a}{2}t^{2-p} - \frac{b}{4}t^{4-p} - \frac{|f|_{\frac{2^*}{2^*-p}}}{pS^{\frac{p}{2}}},$$

then

$$g'(t) = t^{1-p}\left[\frac{a(2-p)}{2} - \frac{b(4-p)}{4}t^2\right].$$

Consequently, let $g'(t) = 0$, we can easily get $t_{\max} = \left[\frac{2a(2-p)}{b(4-p)}\right]^{\frac{1}{2}}$ such that

$$\max_{t\geqslant 0} g(t) = g(t_{\max}) = \frac{a}{4-p}\left[\frac{2a(2-p)}{b(4-p)}\right]^{\frac{2-p}{2}} - \frac{|f|_{\frac{2^*}{2^*-p}}}{pS^{\frac{p}{2}}}.$$

Choosing

$$T = \frac{apS^{\frac{p}{2}}}{4-p}\left[\frac{2a(2-p)}{b(4-p)}\right]^{\frac{2-p}{2}}, \quad R = \left[\frac{2a(2-p)}{b(4-p)}\right]^{\frac{1}{2}},$$

then there exists $\rho > 0$ such that for all $|f|_{\frac{2^*}{2^*-p}} < T$, one has $I(u)|_{u\in S_R} \geqslant \rho$. Moreover, fixing $u_0 \in H_0^1(\Omega)$ and $u_0 \neq 0$, one gets

$$\lim_{t\to 0^+} \frac{I(tu_0)}{t^p} = -\frac{1}{p}\int_\Omega f(x)|u_0|^p\, dx < 0.$$

Thus, one has $\inf_{u\in B_R} I(u) < 0$. Therefor, our claim is true. Without loss of generality, we denote

$$m = \inf_{u\in B_R} I(u).$$

For this minimization problem, there exists a minimization sequence $\{u_n\}$ such that $\lim_{n\to\infty} I(u_n) = m$. Moreover, by [4, Proposition 9], we can take a subsequence from $\{u_n\}$, still denotes by $\{u_n\}$, such that $\{u_n\}$ is a $(PS)_m$ sequence of I in $H_0^1(\Omega)$. Thus, by Lemma 2.1, there exists $u_* \in H_0^1(\Omega)$ such that $u_n \to u_*$ in $H_0^1(\Omega)$ as $n \to \infty$ and $I(u_*) = m < 0$. Consequently, u_* is a nonzero solution of problem (1.1). Since $I(u) = I(|u|)$, we can assume that $u_* \geqslant 0$ in Ω. From (1.2), choosing $\varphi = u_*$, we have

$$(a - b\|u_*\|^2)\int_\Omega |\nabla u_*|^2 dx - \int_\Omega f(x)|u_*|^p\, dx = 0.$$

Consequently, one has

$$(a - b\|u_*\|^2)\int_\Omega |\nabla u_*|^2 dx = \int_\Omega f(x)|u_*|^p\, dx \geqslant 0,$$

which implies that

$$a - b\|u_*\|^2 \geqslant 0. \tag{2.10}$$

Obviously, we have

$$-\left(a - b\|u_*\|^2\right)\Delta u_* = f(x)u_*^{p-1}.$$

Combining with (2.10), we get

$$-\Delta u_* = \frac{f(x)u_*^{p-1}}{a - b\|u_*\|^2} \geqslant 0.$$

Hence, by the strong maximum principle, one has $u_* > 0$, that is, u_* is a positive local minimal solution of problem (1.1). Thus, the proof of Proposition 2.2 is completed. □

Proposition 2.3. Assume $1 \leqslant p < 2$, $2 < p < 2^*$ or $p = 2$, $|f|_{\frac{2^*}{2^*-2}} < aS$, and $f \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is nonzero and nonnegative. Then (1.1) has at least one positive mountain-pass solution u_{**} with $I(u_{**}) > 0$.

Proof. We claim that I satisfies the mountain-pass geometry in $H_0^1(\Omega)$. Firstly, we should prove that there exists $e \in H_0^1(\Omega)$ with $\|e\| > R$ such that $I(e) < 0$. In fact, fixing $u \in H_0^1(\Omega)$ and $u \neq 0$, as $t \to +\infty$, one has

$$I(tu) = \frac{at^2}{2}\|u\|^2 - \frac{bt^4}{4}\|u\|^4 - \frac{t^p}{p}\int_\Omega f(x)|u|^p\,dx \to -\infty,$$

which implies that there exists $e \in H_0^1(\Omega)$ with $\|e\| > R$ such that $I(e) < 0$. Secondly, we prove that there exist $R, \rho > 0$ such that $I(u)|_{u \in S_R} \geqslant \rho$. When $1 \leqslant p < 2$, from Proposition 2.2, there exist $T, R, \rho > 0$, for every $|f|_{\frac{2^*}{2^*-p}} < T$ such that $I(u)|_{u\in S_R} \geqslant \rho$. While $p = 2$, $|f|_{\frac{2^*}{2^*-2}} < aS$, we can easily obtain this conclusion by the similar way. When $2 < p < 2^*$, obviously, 0 is a local minimizer of I with $I(0) = 0$. In fact, fixing $u_0 \in H_0^1(\Omega)$ and $u_0 \neq 0$, we have

$$\lim_{t\to 0^+} \frac{I(tu_0)}{t^2} = \frac{a}{2}\|u_0\|^2 > 0,$$

which implies that there exist $R, \rho > 0$ such that $I(u)|_{u\in S_R} \geqslant \rho$. Hence, our claim is proved to be true. Taking

$$c_0 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}$. We claim that $c_0 \leqslant \frac{a^2}{4b}$. In fact,

$$\begin{aligned}\max_{t\in[0,1]} I(te) &= \max_{t\in[0,1]} \left(\frac{at^2}{2}\|e\|^2 - \frac{bt^4}{4}\|e\|^4 - \frac{t^p}{p}\int_\Omega f(x)|e|^p\,dx\right)\\ &\leqslant \max_{t\in[0,1]} \left(\frac{at^2}{2}\|e\|^2 - \frac{bt^4}{4}\|e\|^4\right)\\ &\leqslant \frac{a^2}{4b}.\end{aligned}$$

Therefore, by Lemma 2.1, applying the mountain-pass lemma, there exists $u_{**} \in H_0^1(\Omega)$ such that $I(u_{**}) = c_0 > 0$, that is, u_{**} is a nonzero mountain-pass solution of (1.1). Since $I(u) = I(|u|)$, from [1, Theorem 10], one has $u_{**} \geqslant 0$ in Ω. Similar to Proposition 2.2, by the strong maximum principle, we can prove that u_{**} is a positive mountain-pass solution of (1.1). This completes the proof of Proposition 2.3. □

According to Proposition 2.2 and Proposition 2.3, we can obtain the proof of Theorem 1.1.

Acknowledgment

This work is supported by the Natural Science Foundation of Education of Guizhou Province (No. KY[2016]281, KY[2017]297); Science and Technology Foundation of Guizhou Province (No. LH[2015]7595, LH[2016]7054).

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