Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 6084–6101
Research Article
ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Fixed points for multivalued contractions with respect to a Pompeiu type metric

Iulia Coroian

Department of Land Measurements and Exact Sciences, University Of Agricultural Sciences and Veterinary Medicine, 3-5 Manastur St., 400372 Cluj-Napoca, Romania.

Communicated by A. Petrusel

Abstract

The purpose of this paper is to present a fixed point theory for multivalued H^+-contractions from the following perspectives: existence/uniqueness of the fixed and strict fixed points, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point equation, well-posedness of the fixed point problem, limit shadowing property for a multivalued operator, set-to-set operatorial equations and fractal operator theory.

Keywords: H^+-type multivalued mapping, Lipschitz equivalent metric, multivalued operator, contraction.
2010 MSC: 47H10, 54H25.

1. Introduction

Let (X, d) be a metric space and $\mathcal{P}(X)$ be the set of all subsets of X. We denote

$$\begin{aligned} P(X) &:= \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}, \\ P_{cl}(X) &:= \{Y \in \mathcal{P}(X) \mid Y \text{ is closed }\}, \\ P_{b,cl}(X) &:= \{Y \in \mathcal{P}(X) \mid Y \text{ is bounded and closed }\}, \\ P_{cp}(X) &:= \{Y \in \mathcal{P}(X) \mid Y \text{ is compact }\}. \end{aligned}$$

By $B(x, r)$ and respectively $\tilde{B}(x, r)$ we will denote the open and respectively the closed ball centered at $x \in X$ with radius $r > 0$.

The following (generalized) functionals are used in the main sections of the paper.

1. The gap functional generated by d:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad D_d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

2. The diameter generalized functional:

$$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad \delta(A, B) = \sup\{d(a, b) | a \in A, b \in B\}.$$

Email address: coroian.iulia@gmail.com (Iulia Coroian)
doi:10.22436/jnsa.010.11.43

Received 2015-03-23

3. The excess generalized functional:
$$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad \rho_d(A,B) = \sup\{D_d(a,B) | a \in A\}.$$
4. The Hausdorff-Pompeiu generalized functional:
$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad H_d(A,B) = \max\{\rho_d(A,B), \rho_d(B,A)\}.$$
5. The Pompeiu generalized functional:
$$H_d^+ : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad H_d^+(A,B) := \frac{1}{2}\{\rho_d(A,B) + \rho_d(B,A)\}.$$

We will avoid the subscript $_d$ when we work with just one metric d on X.

Let (X,d) be a metric space. If $T : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for T if and only if $x \in T(x)$. We denote by F_T the fixed point set of T and by $(SF)_T$ the set of all strict fixed points of T, i.e., elements $x \in X$ such that $T(x) = \{x\}$.

Concerning the Pompeiu functional H^+ defined above, we have several properties.

Lemma 1.1 ([13]). *The following conclusions take place:*

(a) H^+ *is a metric on* $P_{b,cl}(X)$;

(b) H^+ *is a generalized metric (in the sense that it can take also infinite values) on* $P_{cl}(X)$.

Using the Pompeiu type functional H^+, the following notion was introduced in [13], see also [12].

Definition 1.2 ([13]). Let (X,d) be a metric space. A multivalued mapping $T : X \to P_{b,cl}(X)$ is called H^+-contraction with constant α, if

1. there exists a fixed real number α, $0 < \alpha < 1$ such that for every $x, y \in X$
$$H^+(T(x), T(y)) \leqslant \alpha d(x,y);$$
2. for every $x \in X$, $y \in T(x)$ and for every $\epsilon > 0$ there exists z in $T(y)$ such that
$$d(y,z) \leqslant H^+(T(x), T(y)) + \varepsilon.$$

Remark 1.3. Let (X,d) be a metric space. A multivalued mapping $T : X \to P_{cl}(X)$ is called (H^+, α)-Lipschitz if $\alpha > 0$ and
$$H^+(T(x), T(y)) \leqslant \alpha d(x,y), \quad \forall x, y \in X.$$
If $0 < \alpha < 1$, then T is called a multivalued (H^+, α)-contraction.

The purpose of this paper is to study different properties of the Pompeiu functional H^+ and of the multivalued operators satisfying a Lipschitz condition with respect to H^+. The connections with some continuity notions for multivalued operators are also given. The second purpose of this paper is to extend the results given in [13], by presenting several properties of the fixed point set of multivalued H^+-contractions. Several other fixed point results and applications of it will be also given.

2. Properties of the Pompeiu type functional

Concerning the functional H^+ defined above, we have some nice properties.

Lemma 2.1 ([13]). *We have the following relations:*
$$\frac{1}{2}H(A,B) \leqslant H^+(A,B) \leqslant H(A,B),$$
(i.e., H *and* H^+ *are strong equivalent metrics).*

Proposition 2.2 ([13]). *Let* $(X, \|\cdot\|)$ *be a normed linear space. For any* λ *(real or complex),* $A, B \in P_{b,cl}(X)$

1. $H^+(\lambda A, \lambda B) = |\lambda| H^+(A,B)$;
2. $H^+(A + a, B + a) = H^+(A,B)$.

Theorem 2.3 ([13]). *If $a, b \in X$ and $A, B \in P_{b,cl}(X)$, then the following relations hold:*

1. $d(a,b) = H^+(\{a\},\{b\})$;
2. $A \subset \overline{S}(B, r_1), B \subset \overline{S}(A, r_2) \Rightarrow H^+(A,B) \leqslant r$ *where* $r = \frac{r_1+r_2}{2}$.

Theorem 2.4 ([13]). *If the metric space (X, d) is complete, then $(P_{cp}(X), H^+)$, $(P_{b,cl}(X), H^+)$ and $(P_{cl}(X), H^+)$ are complete too.*

The following concept was introduced by Nadler jr. as follows.

Definition 2.5. Let (X, d) be a metric space. A mapping $T : X \to P_{cl}(X)$ is called a multivalued α-contraction if $\alpha \in (0, 1)$ and

$$H(T(x), T(y)) \leqslant \alpha d(x, y), \quad \forall x, y \in X.$$

Notice that any multivalued α-contraction is an (H^+, α)-contraction, but the reverse implication does not hold.

We will now introduce a similar concept. For this purpose, we recall now the concept of (strong) comparison function.

Definition 2.6. A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$.

As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0.

Definition 2.7. A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strong comparison function if it is a comparison function and $\sum\limits_{k=0}^{\infty} \varphi^k(t) < \infty$, for any $t > 0$.

With respect to the Pompeiu type functional H^+, we define the following concept.

Definition 2.8. Let (X, d) be a metric space. Then, the multivalued operator $T : X \to P_{b,cl}(X)$ is called a φ-contraction w.r.t. H^+, if

1. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function;
2. for all $x, y \in X$, we have that
$$H^+(T(x), T(y)) \leqslant \varphi(d(x, y)).$$

In particular, if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\varphi(t) := kt$ (for some $k \in [0, 1[$), then φ is a strong comparison function and the multivalued operator T is an (H^+, k)-contraction.

We recall now some useful concepts in the theory of multivalued operators.

Definition 2.9 (see, for example, [1, 14]). Let (X, d) be a metric space and $T : X \to P_{b,cl}(X)$. Then, T is called upper semi-continuous (briefly u.s.c.) in $x \in X$, if for any open subset U of X with $F(x) \subset U$, there exists $\eta > 0$ such that $T(B(x; \eta)) \subset U$. T is u.s.c. on X if it is u.s.c. in each $x \in X$.

Definition 2.10 ([1, 14]). Let (X, d) be a metric space and $T : X \to P_{b,cl}(X)$. Then T is called lower semi-continuous (briefly l.s.c.) in $x \in X$, if for all $(x_n)_{n \in \mathbb{N}^*} \subset X$ such that $\lim\limits_{n\to\infty} x_n = x$ and for all $y \in T(x)$, there exists a sequence $(y_n)_{n \in \mathbb{N}^*} \subset X$ such that $y_n \in T(x_n)$, for all $n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} y_n = y$. T is l.s.c. on X if it is l.s.c. in each $x \in X$.

Definition 2.11 ([1, 14]). Let (X, d) be a metric space. $T : X \to P_{b,cl}(x)$ is called H-upper semi-continuous in $x_0 \in X$ (H-u.s.c.) respectively H-lower semi-continuous (H-l.s.c.), if for each sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that

$$\lim_{n\to\infty} x_n = x_0,$$

we have

$$\lim_{n\to\infty} \rho(T(x_n), T(x_0)) = 0, \quad \text{respectively} \quad \lim_{n\to\infty} \rho(T(x_0), T(x_n)) = 0.$$

It is well-known that if T is u.s.c. in $x \in X$, then T is H-u.s.c. in $x \in X$, while if T is H-l.s.c. in $x \in X$ implies that T is l.s.c. in $x \in X$.

Definition 2.12 ([1, 14]). Let (X, d) be a metric space and $T : X \to P(X)$. Then T is said to be with closed graph, if for each $x \in X$ and for all $(x_n)_{n\in\mathbb{N}^*} \subset X$ such that

$$\lim_{n\to\infty} x_n = x,$$

and for all $(y_n)_{n\in\mathbb{N}^*} \subset X$ with $y_n \in T(x_n)$, for all $n \in \mathbb{N}^*$ and

$$\lim_{n\to\infty} y_n = y,$$

we have $y \in T(x)$.

Some properties of a multivalued (H^+, α)-Lipschitz operators are given now.

Theorem 2.13. *Let (X, d) be a metric space and $T : X \to P_{b,cl}(X)$ be (H^+, α)-Lipschitz. Then*

1. *T has closed graph in $X \times X$;*
2. *T is $H-$l.s.c. on X;*
3. *T is $H-$u.s.c. on X;*
4. *If, additionally T has compact values, then T is l.s.c.*

Proof.

(1) Let $(x_n, y_n) \subset X \times X$ such that $(x_n, y_n) \overset{d}{\to} (x, y)$, when $n \to \infty$ and $y_n \in T(x_n)$, for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} D(y, T(x)) &\leqslant d(y, y_n) + D(y_n, T(x)) \\ &\leqslant d(y, y_n) + H(T(x_n), T(x)) \\ &\leqslant d(y, y_n) + 2H^+(T(x_n), T(x)) \\ &\leqslant d(y, y_n) + 2kd(x_n, x), \quad n \in \mathbb{N}. \end{aligned}$$

Let us consider $n \to \infty$ and we obtain

$$D(y, T(x)) \leqslant 0 \Rightarrow y \in \overline{T(x)} = T(x).$$

(2) Let $x \in X$ such that $x_n \to x$. We have

$$\begin{aligned} \rho(T(x), T(x_n)) &\leqslant H(T(x), T(x_n)) \\ &\leqslant 2 \cdot H^+(T(x), T(x_n)) \\ &\leqslant 2k \cdot d(x, x_n) \to 0. \end{aligned}$$

In conclusion, T is H-l.s.c. on X.

(3) Using the relation

$$\begin{aligned} \rho(T(x_n), T(x)) &\leqslant H(T(x_n), T(x)) \\ &\leqslant 2 \cdot H^+(T(x_n), T(x)) \\ &\leqslant 2k \cdot d(x, x_n) \to 0, \end{aligned}$$

we obtain that T is H-u.s.c. on X.

(4) The conclusion follows by the fact that any H-l.s.c. multivalued operator with compact values is l.s.c. (see [11]). □

Lemma 2.14. *Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ such that*

$$H^+(T(x), T(y)) < d(x, y), \quad \forall x, y \in X, \quad x \neq y.$$

Then T is u.s.c. on X.

Proof. Let $Z \subset Y$ be a closed set. We will prove that $T^-(Z)$ is closed in X. Let $x \in \overline{T^-(Z)} \setminus T^-(Z)$ and $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$, when $n \to \infty$, $x_n \neq x$, for all $n \in \mathbb{N}$ and $x_n \in T^-(Z)$, for all $n \in \mathbb{N}$. It follows $T(x_n) \cap H \neq \emptyset$, for all $n \in \mathbb{N}$. Let $(y_n)_{n\in\mathbb{N}} \in T(x_n) \cap Z,\ n \in \mathbb{N}$. Then

$$D(y_n, T(x)) \leqslant H_d(T(x_n), T(x)) \leqslant 2H^+(T(x_n), T(x)) < 2d(x_n, x).$$

If $n \to \infty$ we get that

$$\lim_{n\to\infty} D(y_n, T(x)) = 0.$$

But

$$D(y_n, T(x)) = \inf_{y\in T} d(y_n, y) = d(y_n, x_n^{'}), \quad \text{(using the compactness of the set } T(x)).$$

When $n \to \infty$ we have $d(y_n, x_n^{'}) \to 0$. Because $(x_n^{'})_{n\in\mathbb{N}} \subset T(x)$, we obtain that there exists a subsequence $(x_{n_k}^{'})_{k\in\mathbb{N}}$ which converges to an element $\overline{x} \in T(x)$. Then

$$d(y_{n_k}, \overline{x}) \leqslant d(y_{n_k}, x_{n_k}^{'}) + d(x_{n_k}^{'}, \overline{x}) \ \text{ when } \ k \to \infty.$$

Hence, $y_{n_k}^{'} \to \overline{x} \in T(x),\ n \to \infty$.

Because $(y_{n_k}^{'})_{k\in\mathbb{N}} \subset Z$ and Z is closed, we obtain that $\overline{x} \in Z$. So $T(x) \cap Z \neq \emptyset$, which implies $x \in T^-(Z)$, a contradiction. In conclusion, $\overline{T^-(Z)} = T^-(Z)$ and hence $T^-(Z)$ is closed in X. □

3. MWP operators and multivalued α-contractions w.r.t. H^+

The following concepts appeared in [15].

Definition 3.1. Let (X, d) be a metric space. Then, $T : X \to P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

1. $x_0 = x,\ x_1 = y$;
2. $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;
3. the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Definition 3.2. Let (X, d) be a metric space and $T : X \to P(X)$ be an MWP operator. Then we define the multivalued operator $T^\infty : \text{Graph}(T) \to P(F_T)$ by the formula $T^\infty(x, y) = \{z \in F_T |$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.

Definition 3.3. Let (X, d) be a metric space and $T : X \to P(X)$ an MWP operator. Then T is said to be a c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x, y)) \leqslant c d(x, y), \quad \forall (x, y) \in \text{Graph}(T).$$

We recall now the notion of multivalued Picard operator.

Definition 3.4. Let (X, d) be a complete metric space and $T : X \to P(X)$. By definition, T is called a multivalued Picard operator (briefly MP operator) if and only if

1. $(SF)_T = F_T = \{x^*\}$;
2. $T^n(x) \overset{H}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

Recall that, by definition, for $(A_n)_{n\in\mathbb{N}} \in P_{cl}(X)$, we will write $A_n \overset{H}{\to} A^*$ as $n \to \infty$ if and only if $H(A_n, A^*) \to 0$ as $n \to \infty$. Notice also that

$$A_n \overset{H}{\to} A^* \in P_{cl}(X) \text{ as } n \to \infty \text{ if and only if } A_n \overset{H^+}{\to} A^* \in P_{cl}(X) \text{ as } n \to \infty.$$

The purpose of this section is to study some properties of the fixed point set of H^+-contraction with constant α from the MWP operator theory point of view.

We will start by presenting some auxiliary results.

Lemma 3.5 (see, for example, [14]). *Let (X, d) be a metric space and $A, B \in P_{cl}(X)$. Suppose that there exists $\eta > 0$ such that for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leqslant \eta$ and for each $b \in B$ there exists $a \in A$ such that $d(a, b) \leqslant \eta$. Then $H(A, B) \leqslant \eta$.*

Lemma 3.6 ([14]). *Let (X,d) be a metric space, $A, B \in P(X)$ and $q > 1$. Then, for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leqslant qH(A, B)$.*

Lemma 3.7 ([11]). *Let (X,d) be a metric space and $A, B \in P_{cp}(X)$. Then for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leqslant H(A, B)$.*

Lemma 3.8 ([16]). *Let (X, d) be a metric space. If $A, B \in P(X)$ and $\epsilon > 0$ then for every $a \in A$ there exists $b \in B$ such that*

$$d(a, b) \leqslant H(A, B) + \epsilon.$$

Lemma 3.9. *Let (X, d) be a metric space, $A, B \in P_{cl}(X)$ and $\varepsilon > 0$. If $H^+(A, B) < \varepsilon$, then*

1. *for all $a \in A$ there exists $b \in B$ such that $d(a, b) < \varepsilon$; or*
2. *for all $b \in B$ there exists $a \in A$ such that $d(a, b) < \varepsilon$.*

Proof. Suppose, by reductio ad absurdum, that

(i) there exists $a_0 \in A$, for all $b \in B$ such that $d(a_0, b) \geqslant \varepsilon$;

(ii) there exists $b_0 \in B$, for all $a \in A$ such that $d(a, b_0) \geqslant \varepsilon$.

Then, taking $\inf_{b\in B}$ in (i) and $\inf_{a\in A}$ in (ii), we obtain $D(a_0, B) \geqslant \varepsilon$. Since $\rho(A, B) \geqslant D(a_0, B)$, we get

$$\rho(A, B) \geqslant \varepsilon.$$

On the other hand, we also have $D(b_0, A) \geqslant \varepsilon$. Since $\rho(B, A) \geqslant D(b_0, A)$, we get

$$\rho(B, A) \geqslant \varepsilon.$$

Adding the above relations and then dividing by 2, we obtain $H^+(A, B) \geqslant \varepsilon$, which is a contradiction with $H^+(A, B) < \varepsilon$. □

Lemma 3.10 (Cauchy, see [17]). *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of non-negative real numbers, such that*

$$\sum_{k=0}^{+\infty} a_k < +\infty, \quad \textit{and} \quad \lim_{n\to+\infty} b_n = 0.$$

Then,

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{n-k} b_k = 0.$$

Theorem 3.11. *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued H^+-contraction with constant α. Then we have*

(i) $F_T \neq \emptyset$.

(ii) *T is a $\dfrac{1}{1-\alpha}$-MWP operator.*

(iii) *Let* $S : X \to P_{cl}(X)$ *be an* H^+*-contraction with constant* α *and* $\eta > 0$ *such that* $H^+(S(x), T(x)) \leqslant \eta$*, for each* $x \in X$*. Then* $H^+(F_S, F_T) \leqslant \dfrac{2 \cdot \eta}{1-\alpha}$.

(iv) *Let* $T_n : X \to P_{cl}(X), n \in \mathbb{N}$ *be a sequence of multivalued* H^+*-contraction with constant* α *such that* $T_n(x) \overset{H^+}{\to} T(x)$ *as* $n \to \infty$*, uniformly with respect to* $x \in X$*. Then,* $F_{T_n} \overset{H^+}{\to} F_T$ *as* $n \to \infty$*.*

If, additionally $T(x) \in P_{cp}(X)$ *for each* $x \in X$*, then we also have*

(v) *(Ulam-Hyers stability of the inclusion* $x \in T(x)$*) Let* $\epsilon > 0$ *and* $x \in X$ *be such that* $D(x, T(x)) \leqslant \epsilon$*. Then there exists* $x^* \in F_T$ *such that* $d(x, x^*) \leqslant \dfrac{\epsilon}{1-\alpha}$.

(vi) *The fractal operator* $\hat{T} : P_{cp}(X) \to P_{cp}(X),\ \hat{T}(Y) := \bigcup_{x \in Y} T(x)$ *is a* 2α*-contraction.*

(vii) *If, additionally,* $\alpha \in [0, \frac{1}{2}[$*, then* $F_{\hat{T}} = \{A_T^*\}$ *and* $T^n(x) \overset{H^+}{\to} A_T^*$ *as* $n \to \infty$*, for each* $x \in X$*. Moreover,* $F_T \subset A_T^*$*,* F_T *is compact and*

$$A_T^* = \bigcup_{n \in \mathbb{N}^*} T^n(x)\ , \quad \forall x \in F_T.$$

Proof.

(i) Let $\epsilon > 0$ be given. Let $x_0 \in X$ be arbitrary. Fix an element $x_1 \in T(x_0)$. From the definition of H^+-contraction with constant α it follows that we can choose $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leqslant H^+(T(x_0), T(x_1)) + \epsilon.$$

In general, if x_n is chosen, then we choose $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) \leqslant H^+(T(x_{n-1}), T(x_n)) + \epsilon.$$

Suppose $H^+(T(x_{n-1}), T(x_n)) > 0$ for each $n \in \mathbb{N}^*$ (if not, i.e., if there is $k \in \mathbb{N}^*$ such that

$$H^+(T(x_{k-1}), T(x_k)) = 0,$$

then $x_k \in T(x_{k-1}) = T(x_k)$ is a fixed point for T and we are done). Let $1 < q < \frac{1}{\alpha}$ and set

$$\epsilon_n := (q-1)H^+(T(x_{n-1}), T(x_n)).$$

Then, from the above relation it follows that

$$d(x_n, x_{n+1}) \leqslant qH^+(T(x_{n-1}), T(x_n)).$$

Thus, if we set $\beta := q\alpha < 1$, we have

$$d(x_n, x_{n+1}) \leqslant qH^+(T(x_{n-1}), T(x_n)) \leqslant q\alpha d(x_{n-1}, x_n) = \beta d(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. Repeating the same argument n-times we get a sequence of successive approximations for T starting from $(x_0, x_1) \in \text{Graph}(T)$ such that, for each $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) \leqslant \beta^n d(x_0, x_1).$$

Then,

$$d(x_n, x_{n+p}) \leqslant \beta^n \frac{1-\beta^p}{1-\beta} d(x_0, x_1), \quad \forall n \in \mathbb{N}^*, \quad p \in \mathbb{N}^*. \tag{3.1}$$

This implies that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy and hence convergent in (X, d) to some $x^* \in X$. Notice that, by the contraction condition, we immediately get that $\text{Graph}(T)$ is closed in $X \times X$. Hence $x^* \in F_T$.

(ii) By (3.1), letting $p \to \infty$, we get that

$$d(x_n, x^*) \leqslant \beta^n \frac{1}{1-\beta} d(x_0, x_1), \quad \forall n \in \mathbb{N}^*.$$

For $n = 1$ we get

$$d(x_1, x^*) \leqslant \frac{\beta}{1-\beta} d(x_0, x_1).$$

Then

$$d(x_0, x^*) \leqslant d(x_0, x_1) + d(x_1, x^*) \leqslant \frac{1}{1-\beta} d(x_0, x_1) = \frac{1}{1-q\alpha} d(x_0, x_1).$$

Letting $q \searrow 1$ we get that for each $(x_0, x_1) \in \mathrm{Graph}(T)$, there exists $x^* := t^\infty(x_0, x_1) \in F_T$ such that

$$d(x_0, t^\infty(x_0, x_1)) \leqslant \frac{1}{1-\alpha} d(x_0, x_1),$$

proving that T is a $\frac{1}{1-\alpha}$-multivalued weakly Picard operator.

(iii) Let $x_0 \in S(x_0)$ and $q > 1$. Then, by Lemma 3.6, there exists $x_1 \in T(x_0)$ such that

$$d(x_0, x_1) \leqslant qH(S(x_0), T(x_0)) \leqslant 2qH^+(S(x_0), T(x_0)) \leqslant 2q\eta.$$

Then, by (ii) and the above relation, we have proved that for each $x_0 \in F_S$ there exists $t^\infty(x_0, x_1) \in F_T$ such that

$$d(x_0, t^\infty(x_0, x_1)) \leqslant \frac{1}{1-\alpha} d(x_0, x_1) \leqslant \frac{1}{1-\alpha} 2q\eta.$$

Now Lemma 3.5 tells us that

$$\rho(F_S, F_T) \leqslant \frac{2q\eta}{1-\alpha}. \tag{3.2}$$

By a similar procedure we can prove that for each $y_0 \in T(y_0)$ there exists $y_1 \in S(y_0)$ such that

$$d(y_0, y_1) \leqslant qH(T(y_0), S(y_0)) \leqslant 2qH^+(T(y_0), S(y_0)) \leqslant 2q\eta.$$

Thus, we have proved that for each $y_0 \in F_T$ there exists $s^\infty(y_0, y_1) \in F_S$ such that

$$d(y_0, s^\infty(y_0, y_1)) \leqslant \frac{1}{1-\alpha} 2q\eta.$$

Again, Lemma 3.5 gives that

$$\rho(F_S, F_T) \leqslant \frac{2q\eta}{1-\alpha}. \tag{3.3}$$

Adding (3.2) and (3.3), and then dividing by 2, we get

$$H^+(F_S, F_T) \leqslant \frac{2q\eta}{1-\alpha}, \quad \forall q > 1.$$

Letting $q \searrow 1$, we get the conclusion.

(iv) Let $\epsilon > 0$ be given and choose $N_\epsilon \in \mathbb{N}$ such that for $n \geqslant N_\epsilon$ we have

$$\sup_{x \in X} H^+(T_n(x), T(x)) < \epsilon, \quad n \geqslant N_\epsilon.$$

Then, from (iii), we have

$$H^+(F_{T_n}, F_T) < \frac{2\epsilon}{1-\alpha}, \text{ for all } n \geqslant N_\epsilon.$$

Thus, $F_{T_n} \xrightarrow{H^+} F_T$ as $n \to \infty$.

(v) Let $\epsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leqslant \epsilon$. Then, since $T(x)$ is compact, there exists $y \in T(x)$ such that $d(x, y) \leqslant \epsilon$. By the proof of (i), we have that

$$d(x, t^\infty(x, y)) \leqslant \frac{1}{1-\alpha} d(x, y).$$

Since $x^* := t^\infty(x, y) \in F_T$, we get the conclusion $d(x, x^*) \leqslant \frac{\epsilon}{1-\alpha}$.

(vi) By the contraction condition with respect to H^+, one obtains (see Theorem 2.13) that the operator T is H-u.s.c. Since $T(x)$ is compact, for each $x \in X$, we obtain that T is upper semicontinuous. Thus T is u.s.c.

We will prove now that

$$H^+(T(A), T(B)) \leqslant 2\alpha H^+(A, B).$$

For this purpose, let $u \in T(A)$. Then there exists $a \in A$ such that $u \in T(a)$. From Lemma 3.7 there exists $b \in T(B)$ such that

$$d(a, b) \leqslant H_d(A, B).$$

Since

$$D(u, T(B)) \leqslant D(u, T(b)) \leqslant \rho(T(a), T(b)),$$

taking $\sup_{u \in T(A)}$, we get

$$\rho(T(A), T(B)) \leqslant \rho(T(a), T(b)).$$

Interchanging the roles of A and B, we get

$$\rho(T(B), T(A)) \leqslant \rho(T(b), T(a)).$$

Adding the above relations and then dividing by 2, we get

$$H^+(T(A), T(B)) \leqslant H^+(T(a), T(b)).$$

Thus,

$$H(T(A), T(B)) \leqslant 2H^+(T(A), T(B)) \leqslant 2\alpha d(a, b) \leqslant 2\alpha H(A, B), \quad \forall A, B \in P_{cp}(X).$$

(vii) By (vi) it follows that $\hat{T}$ is a self-contraction (with constant $2\alpha < 1$) on the complete metric space $(P_{cp}(X), H)$. By the contraction principle, we obtain that

$$F_{\hat{T}} = \{A_T^*\} \text{ and } \hat{T}^n(A) \stackrel{H}{\to} A_T^*, \text{ as } n \to \infty, \quad \text{for each } A \in P_{cp}(X).$$

As a consequence of Lemma 2.1, we also get that $\hat{T}^n(A) \stackrel{H^+}{\to} A_T^*$ as $n \to \infty$, for each $A \in P_{cp}(X)$. In particular, if $A := \{x\}$, we get that $T^n(x) = \hat{T}^n(x) \stackrel{H^+}{\to} A_T^*$ as $n \to +\infty$, for each $x \in X$. Let $x \in F_T$ be arbitrary. Then $x \in T(x) \subset T^2(x) \subset \cdots \subset T^n(x) \subset \cdots$. Hence $x \in T^n(x)$, for each $n \in \mathbb{N}^*$. Moreover, $\lim_{n \to +\infty} T^n(x) = \bigcup_{n \in \mathbb{N}^*} T^n(x)$. By (vi), we immediately get that $A_T^* = \bigcup_{n \in \mathbb{N}^*} T^n(x)$. Hence

$$F_T \subset \bigcup_{n \in \mathbb{N}^*} T^n(x) = A_T^*.$$

Since F_T is closed subset of the compact A_T^*, it follows that F_T is compact, too. □

Some new conclusions with respect to the fixed point and the strict fixed point sets are given in our next result.

Theorem 3.12. *Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued* (H^+, α)*-contraction with* $(SF)_T \neq \emptyset$. *Then, the following assertions hold:*

(i) $(SF)_T = \{x^*\}$.

If additionally, $\alpha \in [0, \frac{1}{2}[$, *then*

(ii) $F_T = (SF)_T = (SF_{T^n}) = \{x^*\}$, *for* $n \in \mathbb{N}^*$.

(iii) $T^n(x) \overset{H^+}{\to} \{x^*\}$ *as* $n \to \infty$, *for each* $x \in X$.

(iv) *Let* $S : X \to P_{cl}(X)$ *be a multivalued operator such that* $F_S \neq \emptyset$ *and suppose there exists* $\eta > 0$ *such that*

$$H^+(S(x), T(x)) \leqslant \eta, \quad \forall x \in X.$$

Then

$$H^+(F_S, F_T) \leqslant \frac{2\eta}{1-2\alpha}.$$

(v) *(Well-posedness of the fixed point problem w.r.t. to* H^+*) If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that*

$$H^+(x_n, T(x_n)) \to 0 \text{ as } n \to \infty,$$

then $x_n \overset{d}{\to} x^*$ *as* $n \to \infty$.

(vi) *(Limit shadowing property of the multivalued operator) If* $(y_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that*

$$D(y_{n+1}, T(y_n)) \to 0 \;\text{ as }\; n \to \infty,$$

then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ *of successive approximations for* T, *such that* $d(x_n, y_n) \to 0$ *as* $n \to \infty$.

Proof.

(i) Let $x^* \in (SF)_T$. Notice first that $(SF)_T = \{x^*\}$. Indeed, if $z \in (SF)_T$ with $z \neq x^*$, then $0 < d(x^*, z) = H^+(T(x^*), T(z)) \leqslant \alpha d(x^*, z)$, which is a contradiction. Thus $(SF)_T = \{x^*\}$.

(ii) Suppose that $y \in F_T$. Then,

$$\begin{aligned} d(y, x^*) &= D(y, T(x^*)) \\ &\leqslant \rho(T(y), T(x^*)) \\ &\leqslant H(T(y), T(x^*)) \\ &\leqslant 2H^+(T(y), T(x^*)) \\ &\leqslant 2\alpha d(y, x^*). \end{aligned}$$

Hence, $y = x^*$ and $F_T \subset (SF)_T$. Since $(SF)_T \subset F_T$, we get that $(SF)_T = F_T$.

Notice now that $x^* \in (SF)_{T^n}$, for each $n \in \mathbb{N}^*$. Consider $y \in (SF)_{T^n}$, for arbitrary $n \in \mathbb{N}^*$. Then

$$\begin{aligned} d(x^*, y) &= H(T^n(x^*), T^n(y)) \\ &\leqslant 2\alpha H(T^{n-1}(x^*), T^{n-1}(y)) \\ &\leqslant (2\alpha)^2 H(T^{n-2}(x^*), T^{n-2}(y)) \\ &\;\;\vdots \\ &\leqslant (2\alpha)^n d(x^*, y). \end{aligned}$$

Thus, $y = x^*$ and hence $(SF)^n_T = \{x^*\}$.

(iii) Let $x \in X$ be arbitrarily chosen. Then we have

$$\begin{aligned}H^+(T^n(x), x^*) &= H^+(T^n(x), T^n(x^*))\\ &\leqslant H(T^n(x), T^n(x^*))\\ &\leqslant (2\alpha)H(T^{n-1}(x), T^{n-1}(x^*))\\ &\ \ \vdots\\ &\leqslant (2\alpha)^n d(x, x^*) \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

(iv) Let $y \in F_S$. Then

$$\begin{aligned}d(y, x^*) &\leqslant H(S(y), x^*)\\ &\leqslant 2H^+(S(y), x^*)\\ &\leqslant 2(H^+(S(y), T(y)) + H^+(T(y), x^*))\\ &\leqslant 2(\eta + \alpha d(y, x^*)).\end{aligned}$$

Thus, $d(y, x^*) \leqslant \dfrac{2\eta}{1-2\alpha}$. The conclusion follows by the relations

$$H^+(F_S, F_T) \leqslant \sup_{y\in F_S} d(y, x^*) \leqslant \frac{2\eta}{1-2\alpha}.$$

(v) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $H^+(x_n, T(x_n)) \to 0$ as $n \to \infty$. Then,

$$\begin{aligned}d(x_n, x^*) &\leqslant D(x_n, T(x_n)) + H_d(T(x_n), T(x^*))\\ &\leqslant H_d(x_n, T(x_n)) + 2H^+(T(x_n), T(x^*))\\ &\leqslant 2H^+(x_n, T(x_n)) + 2kd(x_n, x^*).\end{aligned}$$

Then

$$d(x_n, x^*) \leqslant \frac{2}{1-2k} H^+(x_n, T(x_n)) \to 0 \text{ as } n \to \infty.$$

(vi) Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that

$$D(y_{n+1}, T(y_n)) \to 0 \text{ as } n \to \infty.$$

Then, there exists $u_n \in T(y_n), n \in \mathbb{N}$ such that

$$d(y_{n+1}, u_n) \to 0 \text{ as } n \to \infty.$$

We shall prove that $d(y_n, x^*) \to 0$ as $n \to \infty$.
We successively have

$$\begin{aligned}d(x^*, y_{n+1}) &\leqslant H(x^*, T(y_n)) + D(y_{n+1}, T(y_n))\\ &\leqslant 2H^+(x^*, T(y_n)) + D(y_{n+1}, T(y_n))\\ &\leqslant 2\alpha d(x^*, y_n) + D(y_{n+1}, T(y_n))\\ &\leqslant 2\alpha[2\alpha d(x^*, y_{n-1}) + D(y_n, T(y_{n-1}))] + D(y_{n+1}, T(y_n))\\ &\ \ \vdots\\ &\leqslant (2\alpha)^{n+1} d(x^*, y_0) + (2\alpha)^n D(y, T(y_0)) + \cdots + D(y_{n+1}, T(y_n)).\end{aligned}$$

By Lemma 3.10, the right hand side tends to 0 as $n \to \infty$. Thus, $d(x^*, y_{n+1}) \to 0$ as $n \to \infty$.

On the other hand, by the fact that T is an MWP operator, we know that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from arbitrary $(x_0, x_1) \in \mathrm{Graph}(T)$ which converges to a fixed point $x^* \in X$ of the operator T. Since the fixed point is unique, we get that $d(x_n, x^*) \to 0$ as $n \to \infty$. Hence, for such a sequence $(x_n)_n \in \mathbb{N}$, we have

$$d(y_n, x_n) \leqslant d(y_n, x^*) + d(x^*, x_n) \to 0, \text{ as } n \to \infty. \qquad \square$$

Remark 3.13. Similar results can be given for the case of multivalued φ-contraction w.r.t. H^+. The results of this type can be viewed as generalizations of some theorems given in [7].

We now give an application of the above results to the continuous dependence of the solution set for a Cauchy problem associated to a differential inclusion, with respect to the initial condition. The existence of a solution to the initial value problem

$$\begin{cases} \dot{x}(t) \in T(t, x(t)), \\ x(0) = b, \end{cases} \tag{3.4}$$

was proved by Filippov [3] and Castaing [2] under certain conditions on T.

In [10], Markin proved a stability theorem on the set of solutions to (3.4) using the L^2 norm, while Lim [9] proved a stability result in terms of the Hausdorff-Pompeiu functional. We will prove now a similar theorem using the sup norm and the Pompeiu functional generated by it.

We recall first the concept of solution.

Definition 3.14. Let $D = [0, a] \times \mathbb{R}^n$ and $T : D \to P(\mathbb{R}^n)$ be a continuous operator. Then, a mapping $x : [0, a] \to \mathbb{R}^n$ is said to be a solution of the differential inclusion (3.4), if x is an absolutely continuous mapping and $x'(t) \in T(t, x(t))$, a.e. on $[0, a]$.

Let B be an origin-centered closed ball in R^n and $P_{cl,cv}(B)$ endowed with the H^+ metric generated by the Euclidean norm $\|\cdot\|$ of R^n. Let $C[0, a]$ be the set of the continuous maps of $[0, a]$ into R^n with the sup norm $\|\cdot\|_C$.

Assume that T is a continuous map of $[0, a] \times B$ into $P_{cl,cv}(B)$ satisfying, for some $k > 0$, the condition

$$H^+(T(t, u), T(t, v)) \leqslant k\|u - v\|_C, \quad \forall t \in [0, a], \quad u, v \in B.$$

For $b \in B$, we will denote $S(b)$ the set of solutions of (3.4) on $[0, a]$. $S(b)$ is nonempty and compact, by [3] and [2].

Theorem 3.15. *If the following conditions hold:*

1. $T : [0, a] \times B \to P_{cl,cv}(B)$ *is continuous;*
2. *there exists* $k > 0$ *such that*

$$H^+(T(t, u), T(t, v)) \leqslant k\|u - v\|_C, \quad \forall t \in [0, a], \quad \forall u, v \in B \subseteq \mathbb{R}^n;$$

3. $2ka < 1$,

then $S(b)$ *is continuous from* B *into the family of nonempty compact subsets of* $C[0, a]$ *equipped with the* H^+ *metric.*

Proof. Suppose $b_n \to b_0$. For $x \in C[0, a]$, define

$$F(b, x) = \{y \in [0, a] : y(t) = b + \int_0^t z(s)ds, \ z(s) \in T(s, x(s))\}.$$

Let $F_n(x) = F(b_n, x)$, $n = 0, 1, 2, \cdots$. Since $F_0(x) = b_n - b_0 + F_n(x)$ it is obvious that $F_n(x)$ converges uniformly to $F_0(x)$. $F_n(x)$ is compact convex for each x and n. Given any pair $x_1, x_2 \in C[0, a]$ and $y_1 \in F(b, x_1)$, let

$$y_1(t) = b + \int_0^t r_1(s)ds, \ r_1(s) \in T(s, x_1(s)).$$

Define $r_2(s)$ to be the point in $T(s, x_2(s))$ nearest to $r_1(s)$, i.e., $r_2 \in T(s, x_2(s))$ and

$$\|r_1(s) - r_2(s)\| = \min\{\|r_1(s) - z\| \; |z \in T(s, x_2(s))\}.$$

It follows from the measurability of $r_1(s)$ and the continuity of $T(s, x_2(s))$ and the nearest point projection that $r_2(s)$ is measurable.

Setting

$$y_2(t) = b + \int_0^t r_2(s)ds, \quad r_2(s) \in T(s, x_2(s)),$$

we have

$$\begin{aligned}\|y_2 - y_1\| &\leqslant \int_0^a \|r_1(s) - r_2(s)\|ds \\ &= \int_0^a \min\{\|r_1(s) - z\|, z \in T(s, x_2(s))\} \\ &= \int_0^a D_{\|\cdot\|}\Big(r_1(s), T(s, x_2(s))\Big)ds \\ &\leqslant \int_0^a H_d\Big(T(s, x_1(s)), T(s, x_2(s))\Big)ds \\ &\leqslant \int_0^a 2H^+\Big(T(s, x_1(s)), T(s, x_2(s))\Big)ds \leqslant 2k\int_0^a \|x_1(s) - x_2(s)\|ds \\ &\leqslant 2ka\|x_1 - x_2\|_C.\end{aligned}$$

Thus F_n are λ-contraction with $\lambda = 2ka < 1$.

By (iv) of Theorem 3.11, $F(T_n) \overset{H^+}{\to} F(T_0)$ i.e., $S(b_n) \overset{H^+}{\to} S(b_0)$. □

More generally, we have the following result.

Theorem 3.16. *For each* $n = 0, 1, 2, \cdots$, *let* T_n *be a continuous map of* $[0, a] \times B$ *into* $C(B)$ *satisfying, for some* $k > 0$, *the condition*

$$H^+(T_n(t, u), T_n(t, v)) \leqslant k\|u - v\|_C, \quad \forall t \in [0, a], \; \forall u, v \in B.$$

Assume that $T_n \to T_0$ *uniformly on* $[0, a] \times B$. *For each* $b \in B$ *and* $n = 0, 1, 2, \cdots$. *Let* $S_n(b)$ *be the set of solutions of*

$$\begin{cases} \dot{x}(t) \in T_n(t, x(t)), \\ x(0) = b. \end{cases}$$

If $2ka < 1$ *and* $b_n \to b_0$ *in* B, *then* $S_n(b_n) \to S_0(b_0)$.

Proof. Let $b_n \to b_0$. For $x_n \in C[0, a]$, define

$$F(b, x_n) = \{y_n \in [0, a] : y_n(t) = b + \int_0^t z_n(s)ds, \; z_n(s) \in T_n(s, x(s))\}.$$

Let $F_n(x_n) = F(b_n, x_n)$, $n = 0, 1, 2, \cdots$. Since $F_0(x) = b_n - b_0 + F_n(x)$ it is obvious that $F_n(x)$ converges uniformly to $F_0(x)$. $F_n(x)$ is compact convex for each x_n and n.

Given any pair $x_n^{(1)}, x_n^{(2)} \in C[0, a]$ and $y_n^{(1)} \in F(b, x_n^{(1)})$, let

$$y_n^{(1)}(t) = b + \int_0^t r_n^{(1)}(s), \; r_n^{(1)}(s) \in T_n(s, x_n^{(1)}(s))ds.$$

Define $r_n^{(2)}(s)$ to be the point in $T_n(s, x_n^{(2)}(s))$ nearest to $r_n^{(1)}(s)$, i.e., $r_n^{(2)} \in T_n(s, x_n^{(2)}(s))$ and

$$\|r_n^{(1)}(s) - r_n^{(2)}(s)\| = \min\{\|r_n^{(1)}(s) - z_n\| \; |z_n \in T_n(s, x_n^{(2)}(s))\}.$$

It follows from the measurability of $r_n^{(1)}(s)$ and the continuity of $T_n(s, x_n^2(s))$ and the nearest point projection that $r_n^{(2)}(s)$ is measurable.

Setting

$$y_n^{(2)}(t) = b + \int_0^t r_n^{(2)}(s)ds, \quad r_n^{(2)}(s) \in T_n(s, x_n^{(2)}(s)),$$

we have

$$\|y_n^{(2)} - y_n^{(1)}\| \leqslant 2ka\|x_n^{(1)} - x_n^{(2)}\|_C.$$

Thus F_n are λ-contraction with $\lambda = 2ka < 1$.

By (iv) of Theorem 3.11, we have $F(T_n) \overset{H^+}{\to} F(T_0)$, i.e., $S_n(b_n) \overset{H^+}{\to} S_n(b_0)$. □

Next, an application to Ulam-Hyers stability of the inclusion $x \in T(x)$ ((v) of Theorem 3.11) is given. The notion of Ulam-Hyers stability for a differential inclusion is defined as follows.

Definition 3.17. Let

$$x' \in T(t, x(t)), \quad t \in [0, a], \tag{3.5}$$

and $T : [0, a] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ be a continuous operator. We say that (3.5) is Ulam-Hyers stable if for any $\epsilon > 0$, any $y \in C[0, a]$ and any ϵ-solution of (3.5) (which means that

$$D\Big(y(t), y(0) + \int_0^t T(s, y(s))ds\Big) \leqslant \epsilon, \quad t \in [0, a]),$$

there exists a solution x^* of (3.5) and $c > 0$ such that $\|x^* - y\| \leqslant c \cdot \epsilon$.

Definition 3.18. Let $F : [0, a] \to P_cl(\mathbb{R}^n)$ be a measurable multivalued operator. If $L^1([0, a], \mathbb{R}^n)$ denotes the set of all measurable and integrable mappings from $[0, a]$ to $\mathbb{R}^n$, then S_F will denote the set of all integrable selections of F, i.e.,

$$S_F := \{f \in L^1([0, a], \mathbb{R}^n) | f(t) \in F(t), \ \text{ a.e. } \ t \in [0, a]\}.$$

Remark 3.19. In particular, if $x : [0, a] \to \mathbb{R}^n$ and $T : [0, a] \times \mathbb{R}^n \to P_{cl}(\mathbb{R}^n)$, then the set of all integrable selections of T will be denoted by

$$S_{T(\cdot, x(\cdot))} := \{f \in L^1([0, a], \mathbb{R}^n) \mid f(t) \in T(t, x(t)) \text{ a.e. } t \in [0, a]\}.$$

Theorem 3.20. *Let us consider the inclusion (3.5). We assume:*

(a) $T : [0, a] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *is a continuous, measurable and integrably bounded multivalued operator.*

(b) *There exists* $L > 0$ *such that*

$$H^+(T(t, u_1), T(t, u_2)) \leqslant L\|u_1 - u_2\|, \ \ \forall (t, u_1), (t, u_2) \in [0, a] \times \mathbb{R}^n.$$

Then the differential inclusion (3.5) with initial condition $x(0) = x^0$ *has at least one solution. Moreover the differential inclusion (3.5) is Ulam-Hyers stable.*

Proof. Let us define $U : C[0, a] \to P(C[0, a])$, $u \to Ux$ and $Ux(t) := b + \int_0^t T(s, x(s))ds, \ t \in [0, a]$. We notice that, since T is u.s.c., (3.5) is equivalent with the fixed point problem

$$x \in Ux. \tag{3.6}$$

We will show that the fixed point problem (3.6) is Ulam-Hyers stable.

Let $y, z \in C[a, b]$ and $u_1 \in Ux$. Then $u_1 \in C[0, a]$ and

$$u_1(t) \in x(0) + \int_0^t T(s, x(s))ds \ \text{ a.e. on } [0, a].$$

It follows that there is a mapping $k_y \in S_{T(\cdot,y(\cdot))}$ such that

$$u_1(t) = x(0) + \int_0^t k_x(s)ds \ \text{ a.e. on } [0, a].$$

Since

$$H^+(T(t, x(t)), T(t, y(t))) \leqslant L\|x(t) - y(t)\|,$$

one obtains that there exists $w \in T(t, y(t))$ such that

$$\|k_x(t) - w\| \leqslant H(T(t, x(t)), T(t, y(t))) \leqslant 2H^+(T(t, x(t)), T(t, y(t))) \leqslant 2L\|x(t) - y(t)\|.$$

Thus the multivalued operator G defined by $G(t) = T_y(t) \cup K(t)$ (where $T_y(t) = T(t, y(t))$ and

$$K(t) = \{w \mid \|k_x(t) - w\| \leqslant 2L\|x(t) - y(t)\|\},$$

has nonempty values and is measurable.

Let k_z be a measurable selection for G (which exists by Kuratowski and Ryll Nardzewski's selection theorem). Then $k_y(t) \in T(t, y(t))$ and

$$\|k_x(t) - k_y(t)\| \leqslant 2L\|x(t) - y(t)\|, \ \ \text{a.e. on } [0, a].$$

Define $u_2 = x(0) + \int_0^t k_y(s)ds$. It follows that $u_2 \in Uy$ and

$$\begin{aligned}
\|u_1(t) - u_2(t)\| &\leqslant \int_0^t \|k_x(s) - k_y(s)\|ds \\
&\leqslant 2L \int_0^t \|x(s) - y(s)\|ds \\
&= 2L \int_0^t \|x(s) - y(s)\| \cdot e^{-\tau(s-a)} \cdot e^{\tau(s-a)}ds \\
&\leqslant 2L\|x - y\|_B \int_0^t e^{\tau(s-a)}ds \\
&\leqslant \frac{2L}{\tau} e^{\tau(s-a)}\|x - y\|_B.
\end{aligned}$$

Here $\|\cdot\|_B$ denotes the Bielecki-type norm on $C[0, a]$. Finally, we have that

$$\|u_1 - u_2\|_B \leqslant \frac{2L}{\tau}\|x - y\|_B.$$

From this and the analogous inequality obtained by interchanging the roles x and y and adding them and then dividing by 2 we get that

$$H^+(Ux, Uy) \leqslant \frac{2L}{\tau}\|x - y\|_B, \quad \forall x, y \in C[0, a].$$

Taking $\tau > 2L$, it follows that U is multivalued (H^+, α)-contraction. □

4. Continuation results for multivalued (H^+, α)-contractions

In this section, we present a local result and a continuation result for a special kind of multivalued (H^+, α)-contractions. Following Kirk and Shahzad ([6]), we will replace the second condition of the Definition 1.2:

$(*)$ for every $x \in X$, $y \in T(x)$ and for every $\epsilon > 0$ there exists z in $T(y)$ such that

$$d(y,z) \leqslant H^+(T(x),T(y)) + \varepsilon,$$

with the following one:

$(**)$ for every $x \in X$ and every $y \in T(x)$ we have that

$$D(y,T(y)) \leqslant H^+(T(x),T(y)).$$

Notice that $(**)$ implies $(*)$. Moreover if we consider the following condition:

$(***)$ for every $\epsilon > 0$, for every $x \in X$, $y \in T(x)$ there exists z in $T(y)$ such that

$$d(y,z) \leqslant H^+(T(x),T(y)) + \varepsilon.$$

Then it is easy to see that $(**)$ is equivalent with $(***)$. In this last case, we also notice that for each $x \in X$ and every $y \in T(x)$ we have $\rho(T(x),T(y)) \leqslant H^+(T(x),T(y))$. As a consequence, $\rho(T(x),T(y)) \leqslant \rho(T(y),T(x))$ and so

$$H^+(T(x),T(y)) \leqslant \rho(T(y),T(x)), \quad \forall x \in X, \ \ y \in T(x).$$

Homotopy results for multivalued operators of contractive types are well-known in the literature, see [4, 5, 8]. This approach is applied in all cases on a local fixed point theorem. The first result of this section is the following local fixed point theorem.

Theorem 4.1. *Let* (X,d) *be a complete metric space,* $x_0 \in X$, $r > 0$ *and* $T : \tilde{B}(x_0,r) \to P_{cl}(X)$ *is a multivalued operator. We suppose that:*

(i) *T is a multivalued* (H^+,α)*-contraction, i.e.,* $\alpha \in]0,1[$ *and*

$$H^+(T(x),T(y)) \leqslant \alpha d(x,y), \quad \forall x,y \in X;$$

(ii) *for every* $x \in X$ *and every* $y \in T(x)$ *we have that* $D(y,T(y)) \leqslant H^+(T(x),T(y))$*;*

(iii) $D(x_0,T(x_0)) < (1-\alpha)r$.

Then, there exists $x^* \in \tilde{B}(x_0,r)$ *such that* $x^* \in T(x^*)$.

Proof. Notice first that, by (iii), we can find an element $x_1 \in T(x_0)$ such that $d(x_0,x_1) < (1-\alpha)r$. Clearly $x_1 \in \tilde{B}(x_0,r)$. Now, for arbitrary $\epsilon > 0$, by (ii) and (i), it follows that we can choose $x_2 \in T(x_1)$ such that

$$d(x_1,x_2) < D(x_1,T(x_1)) + \epsilon \leqslant H^+(T(x_0),T(x_1)) + \epsilon \leqslant \alpha d(x_0,x_1) + \epsilon.$$

If we take $\epsilon := \alpha[(1-\alpha)r - d(x_0,x_1)] > 0$, then we get that

$$d(x_1,x_2) < \alpha(1-\alpha)r.$$

Moreover $d(x_2,x_0) \leqslant d(x_0,x_1) + d(x_1,x_2) < (1-\alpha)r + \alpha(1-\alpha)r = (1-\alpha^2)r$, proving that $x_2 \in \tilde{B}(x_0,r)$. Using this procedure (taking at each step $k \geqslant 2$, for the construction of x_k, the value of ϵ as $\epsilon_k := \alpha[\alpha^{k-2}(1-\alpha)r - d(x_{k-2},x_{k-1})]$), we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ having the properties:

(1) $d(x_0,x_n) < (1-\alpha^n)r$, for each $n \in \mathbb{N}^*$ (i.e., $x_n \in \tilde{B}(x_0,r)$, for each $n \in \mathbb{N}^*$);

(2) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;

(3) $d(x_n,x_{n+1}) < \alpha^n(1-\alpha)r$, for each $n \in \mathbb{N}$.

Then, by (3), we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d) and hence it converges in (X, d) to some $x^* \in \tilde{B}(x_0, r)$. By Theorem 2.13, we have that $T : \tilde{B}(x_0, r) \to P_{cl}(X)$ has closed graph, thus we immediately get, by (2), that $x^* \in T(x^*)$ as $n \to \infty$.

□

Theorem 4.2. *Let* (X, d) *be a complete metric space. Let* U *be an open subset of* (X, d)*. Let* $G : U \times [0, 1] \to P(X)$ *be a multivalued operator such that the following conditions are satisfied:*

1. $x \neq G(x, t)$ *for each* $x \in \partial B$ *and each* $t \in [0, 1]$*;*
2. *there exists* $\alpha \in [0, 1[$ *such that for each* $t \in [0, 1]$ *and each* $x, y \in U$ *we have*
$$H^+(G(x, t), G(y, t)) \leqslant \alpha d(x, y);$$
3. *there exists a continuous increasing function* $\phi : [0, 1] \to \mathbb{R}$ *such that*
$$H^+(G(x, t), G(x, s)) < |\phi(t) - \phi(s)|, \quad \forall t, s \in [0, 1], \quad t \neq s, \quad \forall x \in U;$$
4. $G : U \times [0, 1] \to P((X, d))$ *is closed.*

Then $G(\cdot, 0)$ *has a fixed point if and only if* $G(\cdot, 1)$ *has a fixed point.*

Proof. Let us consider the set $Q = \{(t, x) \in [0, 1] \times U : x \in G(x, t)\}$. Clearly $Q \neq \emptyset$, since $(0, z) \in Q$ where $z \in G(z, 0)$. On Q we define the partial order

$$(t, x) \leqslant (s, y) \text{ if and only if } t \leqslant s \text{ and } d(x, y) \leqslant \frac{2}{1-\alpha}(\phi(s) - \phi(t)).$$

Let P be a totally ordered subset of Q. Define $t^* = \sup\{t : (t, x) \in P\}$.

Taking a sequence $\{(t_n, x_n)\}$ in P such that

$$(t_n, x_n) \leqslant (t_{n+1}, x_{n+1}) \quad \text{and} \quad t_n \to t^* \text{ as } n \to \infty.$$

We have

$$d(x_m, x_n) \leqslant \frac{2}{1-\alpha}(\phi(t_m) - \phi(t_n)), \qquad \text{for } m > n, \quad m, n \in \mathbb{N}^*.$$

Thus, $\{x_n\}$ is a Cauchy sequence, and hence converges to some $x \in \tilde{U}$. Since $x_n \in G(x_n, t_n)$, $n \in \mathbb{N}^*$ and G is closed, we have $x^* \in G(x^*, t^*)$. Thus $(t^*, x^*) \in Q$.

Since P is totally ordered we get $(t, x) \leqslant (t^*, x^*)$ for each $(t, x) \in P$. That means that (t^*, x^*) is a bound of P. It follows from Zorn's Lemma that Q admits a maximal element $(t_0, x_0) \in Q$.

To complete the proof, we have to show that $t_0 = 1$. Suppose this is false. Then, we can choose $r > 0$ and $t \in (t_0, 1]$ such that $\tilde{B}(x_0, r)$ and $r := \dfrac{2}{1-\alpha}(\phi(t) - \phi(t_0))$. It follows that

$$\begin{aligned} D(x_0, G(x_0, t)) &\leqslant \rho(G(x_0, t_0), G(x_0, t)) \\ &\leqslant H(G(x_0, t_0), G(x_0, t)) \\ &\leqslant 2H^+(G(x_0, t_0), G(x_0, t)) \\ &< \phi(t) - \phi(t_0) = (1-\alpha)r. \end{aligned}$$

Since $\tilde{B}(x_0, r) \subset U$, the multivalued operator $G(\cdot, t) : \tilde{B}(x_0, r) \to P_{cl}(X)$ satisfies, for all $t \in [0, 1]$ the assumptions of Theorem 4.1. Hence, for all $t \in [0, 1]$, there exists $x \in \tilde{B}(x_0, r)$ such that $x \in G(x, t)$. Thus $(t, x) \in Q$.

Since $d(x_0, x) \leqslant r = \dfrac{2}{1-\alpha}(\phi(t) - \phi(t_0))$ we immediately get $(t_0, x_0) < (t, x)$. This is a contradiction with the maximality of (t_0, x_0).

Conversely, if $G(\cdot, 1)$ has a fixed point, then putting $t := 1 - t$ and using the first part of the proof we get the conclusion. □

Acknowledgment

This paper is a result of a doctoral research made possible by the financial support of the Sectoral Operational Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the project POSDRU/159/1.5/S/137750 - " Doctoral and postdoctoral programs - support for increasing research competitiveness in the field of exact sciences ".

References

[1] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, (1990). 2.9, 2.10, 2.11, 2.12
[2] C. Castaing, *Sur les equations differentielles multivoques*, C. R. Acad. Sci. Paris, **263** (1966), 63–66. 3, 3
[3] A. F. Filippov, *Classical solutions of differential equations with multivalued right-hand side*, SIAM J. Control, **5** (1967), 609–621. 3, 3
[4] M. Frigon, *On continuation methods for contractive and nonexpansive mappings.Recent advances on metric fixed point theory*, Univ. Sevilla, Seville, (1996). 4
[5] M. Frigon, A. Granas, *Résultats de type Leray-Schauder pour des contractions sur des espaces de Frchet*, Ann. Sci. Math. Qubec, **22** (1998), 161–168. 4
[6] W. A. Kirk, N. Shahzad, *Remarks on metrics transform and fixed point theorems*, Fixed Point Theory Appl., **2013** (2013), 11 pages. 4
[7] V. L. Lazăr, *Fixed point theory for multivalued φ-contractions*, Fixed Point Theory Appl., **2011** (2011), 12 pages. 3.13
[8] T. Lazăr, D. O'Regan, A. Petruşel, *Fixed points and homotopy results for Ćirić-type multivalued operators on a set with two metrics*, Bull. Korean Math. Soc., **45** (2008), 67–73. 4
[9] T.-C. Lim, *On fixed point stability for se-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl., **110** (1985), 436–441. 3
[10] J. T. Markin, *Stability of solution sets for generalized differential equations*, J. Math. Anal. Appl., **46** (1974), 289–291. 3
[11] G. Moţ, A. Petruşel, G. Petruşel, *Topics in Nonlinear Analysis and Applications to Mathematical Economics*, Casa Cărţii de Ştiinţă, Cluj-Napoca, (2007). 2, 3.7
[12] H. K. Pathak, N. Shahzad, *A new fixed point result and its application to existence theorem for nonconvex Hammerstein type integral inclusions*, Electron. J. Qual. Theory Differ. Equ., **2012** (2012), 13 pages. 1
[13] H. K. Pathak, N. Shahzad, *A generalization of Nadler's fixed point theorem and its application to nonconvez integral inclusions*, Topol. Methods Nonlinear Anal., **41** (2013), 207–227. 1.1, 1, 1.2, 1, 2.1, 2.2, 2.3, 2.4
[14] A. Petruşel, *Multivalued weakly Picard operators and applications*, Sci. Math. Jpn., **59** (2004), 169–202. 2.9, 2.10, 2.11, 2.12, 3.5, 3.6
[15] A. Petruşel, I. A. Rus, *The theory of a metric fixed point theorem for multivalued operators*, Yokohama Publ., Yokohama, (2010). 3
[16] I. A. Rus, *Generalized Contraction and Applications*, Cluj Univ. Press, Cluj-Napoca, (2001). 3.8
[17] I. A. Rus, M.-A. Şerban, *Some generalizations of a Cauchy lemma and applications*, Presa Univ. Clujeană, Cluj-Napoca, (2008). 3.10