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Some new integral inequalities for n-times differentiable convex and concave functions

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Abstract

In this work, by using an integral identity together with both the Hölder and the Power-mean integral inequalities we establish several new inequalities for n-times differentiable convex and concave mappings. ©2017 All rights reserved.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a concave function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$\frac{f(a)+f(b)}{2} \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant f\left(\frac{a+b}{2}\right),$$

is well-known in the literature as Hermite-Hadamard's inequality for concave functions [16]. Both inequalities hold in the reserved direction if f is convex. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex or concave function. Hadamard's inequality for convex or concave functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; for example see [1–16].

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y),$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval

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$I \neq \emptyset$. This definition is well-known in the literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. For some inequalities, generalizations and applications concerning convexity see [3, 5–11, 15–18]. Recently, in the literature there are so many papers about n-times differentiable functions on several kinds of convexities. In references [2–4, 6, 13, 17, 19], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex functions see for instance the recent papers [1, 5, 7–12, 14] and the references within these papers.

Let $0 < a < b$, throughout this paper we will use

$$\begin{aligned} A(a,b) &= \frac{a+b}{2}, \\ G(a,b) &= \sqrt{ab}, \\ L_a(a,b) &= \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad a \neq b, \ p \in \mathbb{R}, \ p \neq -1, 0, \end{aligned}$$

for the arithmetic, geometric, generalized logarithmic means, respectively.

2. Main results

We will use the following for obtain our main results.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable mapping on I° for $n \in \mathbb{N}$ and $f^{(n)} \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. We have the identity*

$$\sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx = \frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx, \tag{2.1}$$

where an empty sum is understood to be nil.

Proof. To prove, we shall use the induction method. For $n = 1$, by integration by parts we have

$$f(b)b - f(a)a - \int_a^b f(x)dx = \int_a^b xf'(x)dx.$$

This coincides with the equality (2.1) for $n = 1$. Similarly for $n = 2$ and using integration by parts as above, we have

$$\sum_{k=0}^{1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx = \frac{(-1)^{2+1}}{2!}\int_a^b x^2 f''(x)dx,$$

$$\frac{f(b)b - f(a)a}{1!} - \frac{f'(b)b^2 - f'(a)a^2}{2!} - \int_a^b f(x)dx = -\frac{1}{2!}\int_a^b x^2 f''(x)dx,$$

$$\begin{aligned} \int_a^b x^2 f''(x)dx &= f'(b)b^2 - f'(a)a^2 - 2\left[xf(x)|_a^b - \int_a^b f(x)dx\right] \\ &= f'(b)b^2 - f'(a)a^2 - 2[bf(b) - af(a)] + 2\int_a^b f(x)dx. \end{aligned} \tag{2.2}$$

Equation (2.2) coincides with the equality (2.1) for $n = 2$.

Suppose (2.1) holds for $n = t$. That is

$$\sum_{k=0}^{t-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx = \frac{(-1)^{t+1}}{t!}\int_a^b x^t f^{(t)}(x)dx. \tag{2.3}$$

Using the integration by parts we have

$$\begin{aligned}\frac{(-1)^{t+2}}{(t+1)!}\int_a^b x^{t+1}f^{(t+1)}(x)dx &= \frac{(-1)^{t+2}}{(t+1)!}\left\{x^{t+1}f^{(t)}(x)\Big|_a^b-(t+1)\int_a^b x^t f^{(t)}(x)dx\right\}\\ &= \frac{(-1)^{t}}{(t+1)!}\left[f^{(t)}(b)b^{t+1}-f^{(t)}(a)a^{t+1}\right]-(t+1)\frac{(-1)^{t+2}}{(t+1)!}\int_a^b x^t f^{(t)}(x)dx\\ &= \frac{(-1)^{t+2}}{(t+1)!}\left[f^{(t)}(b)b^{t+1}-f^{(t)}(a)a^{t+1}\right]+\frac{(-1)^{t+1}}{t!}\int_a^b x^t f^{(t)}(x)dx,\end{aligned}$$

$$\frac{(-1)^{t+1}}{t!}\int_a^b x^t f^{(t)}(x)dx = \frac{(-1)^{t+2}}{(t+1)!}\int_a^b x^{t+1}f^{(t+1)}(x)dx-\frac{(-1)^{t}}{(t+1)!}\left[f^{(t)}(b)b^{t+1}-f^{(t)}(a)a^{t+1}\right]. \tag{2.4}$$

Substituting (2.4) in (2.3) we obtain

$$\sum_{k=0}^{t-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx=\frac{(-1)^{t+1}}{t!}\int_a^b x^t f^{(t)}(x)dx,$$

$$\begin{aligned}\sum_{k=0}^{t-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx &= \frac{(-1)^{t+2}}{(t+1)!}\int_a^b x^{t+1}f^{(t+1)}(x)dx\\ &\quad-\frac{(-1)^{t}}{(t+1)!}\left[f^{(t)}(b)b^{t+1}-f^{(t)}(a)a^{t+1}\right],\end{aligned}$$

that is,

$$\begin{aligned}&\sum_{k=0}^{t-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx+\frac{(-1)^{t}}{(t+1)!}\left[f^{(t)}(b)b^{t+1}-f^{(t)}(a)a^{t+1}\right]\\ &\quad=\frac{(-1)^{t+2}}{(t+1)!}\int_a^b x^{t+1}f^{(t+1)}(x)dx.\end{aligned}$$

This completes the proof of Lemma. □

Theorem 2.2. *For* $n \in \mathbb{N}$, *let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be* n*-times differentiable function on* I° *and* $a, b \in I^\circ$ *with* $a < b$. *If* $f^{(n)} \in L[a,b]$ *and* $\left|f^{(n)}\right|^q$ *for* $q > 1$ *is convex on* $[a,b]$, *then the following inequality holds:*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\\ &\qquad\leqslant \frac{1}{n!}(b-a)L_{np}^n(a,b)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right).\end{aligned}$$

Proof. If $\left|f^{(n)}\right|^q$ for $q > 1$ is convex on $[a,b]$, using Lemma 2.1, the Hölder integral inequality and

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q\leqslant\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^q+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^q,$$

we have

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$
\begin{aligned}
&\leqslant \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx \\
&\leqslant \frac{1}{n!}\left(\int_a^b x^{np}\,dx\right)^{\frac{1}{p}}\left(\int_a^b \left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\leqslant \frac{1}{n!}\left(\int_a^b x^{np}\,dx\right)^{\frac{1}{p}}\left(\int_a^b \left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^q+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^q\right] dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)^{\frac{1}{q}}(b-a)^{\frac{1}{p}}\left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}}\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}} \\
&= \frac{1}{n!}(b-a)L_{np}^n(a,b)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right).
\end{aligned}
$$

This completes the proof of theorem. □

Corollary 2.3. *Under the conditions of Theorem 2.2 for $n = 1$, we have the following inequality:*

$$
\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant L_p(a,b)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.
$$

Proposition 2.4. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in (-\infty, 0] \cup [1,\infty)\backslash\{-2q,-q\}$, we have*

$$
L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leqslant L_p(a,b)A^{\frac{1}{q}}(a^m,b^m).
$$

Proof. Under the assumption of the proposition, let $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. Then

$$
\left|f'(x)\right|^q = x^m,
$$

is convex on $(0,\infty)$ and the result follows directly from Corollary 2.3. □

Theorem 2.5. *For $n \in \mathbb{N}$, let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q \geqslant 1$ is convex on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \\
&\quad\leqslant \frac{1}{n!}(b-a)^{1-\frac{1}{q}}L_{np}^{n\left(\frac{q-1}{q}\right)}(a,b) \\
&\quad\quad\times\left\{\left|f^{(n)}(b)\right|^q\left[L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right]+\left|f^{(n)}(a)\right|^q\left[bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right]\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 2.1 and power-mean integral inequality, we obtain

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \\
&\quad\leqslant \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right| dx \\
&\quad\leqslant \frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left[\frac{x-a}{b-a}\left|f^{(n)}(b)\right|^q+\frac{b-x}{b-a}\left|f^{(n)}(a)\right|^q\right]dx\right)^{\frac{1}{q}}\\
&=\frac{1}{n!}(b-a)^{1-\frac{1}{q}}\left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{1-\frac{1}{q}}\\
&\quad\times\left\{\left|f^{(n)}(b)\right|^q\left[\frac{b^{n+2}-a^{n+2}}{(n+2)(b-a)}-a\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]\right.\\
&\quad\left.+\left|f^{(n)}(a)\right|^q\left[b\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}-\frac{b^{n+2}-a^{n+2}}{(n+2)(b-a)}\right]\right\}^{\frac{1}{q}}\\
&=\frac{1}{n!}(b-a)^{1-\frac{1}{q}}L_{np}^{n\left(\frac{q-1}{q}\right)}(a,b)\\
&\quad\times\left\{\left|f^{(n)}(b)\right|^q\left[L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right]+\left|f^{(n)}(a)\right|^q\left[bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right]\right\}^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof of theorem. □

Corollary 2.6. *Under the conditions of Theorem 2.5 for $n=1$, we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leqslant 6^{-\frac{1}{q}}\left(\frac{a+b}{2}\right)^{1-\frac{1}{q}}\left[(2b+a)\left|f'(b)\right|^q+(b+2a)\left|f'(a)\right|^q\right]^{\frac{1}{q}}.$$

Proposition 2.7. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in(-\infty,0]\cup[1,\infty)\backslash\{-2q,-q\}$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b)\leqslant 3^{-\frac{1}{q}}A^{1-\frac{1}{q}}(a,b)\left[2A\left(a^{m+1},b^{m+1}\right)+G^2(a,b)A\left(a^{m-1},b^{m-1}\right)\right]^{\frac{1}{q}}.$$

Proof. The result follows directly from Corollary 2.6 for the function

$$f(x)=\frac{q}{m+q}x^{\frac{m}{q}+1},\quad x\in(0,\infty).$$

This completes the proof of proposition. □

Corollary 2.8. *Using Proposition 2.7 for $m=1$, we have the following inequality:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b)\leqslant 3^{-\frac{1}{q}}A^{1-\frac{1}{q}}(a,b)\left[2A\left(a^2,b^2\right)+G^2(a,b)\right]^{\frac{1}{q}}.$$

Corollary 2.9. *Using Proposition 2.7 for $q=1$, we have the following inequality:*

$$L_{m+1}^{m+1}(a,b)\leqslant\frac{1}{3}\left[2A\left(a^{m+1},b^{m+1}\right)+G^2(a,b)A\left(a^{m-1},b^{m-1}\right)\right].$$

Corollary 2.10. *Using Corollary 2.9 for $m=1$, we have the following inequality:*

$$L_2^2(a,b)\leqslant\frac{1}{3}\left[2A\left(a^2,b^2\right)+G^2(a,b)\right].$$

Corollary 2.11. *Under the conditions of Theorem 2.5 for $q=1$, we have the following inequality:*

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\\
&\qquad\leqslant\frac{1}{n!}\left\{\left|f^{(n)}(b)\right|\left[L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right]+\left|f^{(n)}(a)\right|\left[bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right]\right\}.
\end{aligned}
$$

Theorem 2.12. *For $n \in \mathbb{N}$, let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q > 1$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\leqslant \frac{1}{n!}(b-a)^{\frac{1}{p}}\left\{\left|f^{(n)}(b)\right|^q\left[L_{nq+1}^{nq+1}(a,b)-aL_{nq}^{nq}(a,b)\right]+\left|f^{(n)}(a)\right|^q\left[bL_{nq}^{nq}(a,b)-L_{nq+1}^{nq+1}(a,b)\right]\right\}^{\frac{1}{q}}.$$

Proof. Since $\left|f^{(n)}\right|^q$ for $q > 1$ is convex on $[a,b]$, using Lemma 2.1 and the Hölder integral inequality, we have the following inequality:

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\leqslant \frac{1}{n!}\int_a^b 1.x^n\left|f^{(n)}(x)\right|dx$$
$$\leqslant \frac{1}{n!}\left(\int_a^b 1^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$= \frac{1}{n!}\left(\int_a^b 1dx\right)^{\frac{1}{p}}\left(\int_a^b x^{nq}\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q dx\right)^{\frac{1}{q}}$$
$$\leqslant \frac{1}{n!}\left(\int_a^b 1dx\right)^{\frac{1}{p}}\left(\int_a^b\left[\frac{x-a}{b-a}x^{nq}\left|f^{(n)}(b)\right|^q+\frac{b-x}{b-a}x^{nq}\left|f^{(n)}(a)\right|^q\right]dx\right)^{\frac{1}{q}}$$
$$= \frac{1}{n!}(b-a)^{\frac{1}{p}}\left\{\left|f^{(n)}(b)\right|^q\left[\frac{b^{nq+2}-a^{nq+2}}{(nq+2)(b-a)}-a\frac{b^{nq+1}-a^{nq+1}}{(nq+1)(b-a)}\right]\right.$$
$$\left.+\left|f^{(n)}(a)\right|^q\left[b\frac{b^{nq+1}-a^{nq+1}}{(nq+1)(b-a)}-\frac{b^{nq+2}-a^{nq+2}}{(nq+2)(b-a)}\right]\right\}^{\frac{1}{q}}$$
$$= \frac{1}{n!}(b-a)^{\frac{1}{p}}\left\{\left|f^{(n)}(b)\right|^q\left[L_{nq+1}^{nq+1}(a,b)-aL_{nq}^{nq}(a,b)\right]+\left|f^{(n)}(a)\right|^q\left[bL_{nq}^{nq}(a,b)-L_{nq+1}^{nq+1}(a,b)\right]\right\}^{\frac{1}{q}}.$$

This completes the proof of theorem. □

Corollary 2.13. *Under the conditions of Theorem 2.12 for $n = 1$, we have the following inequality:*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant\left\{\frac{|f'(b)|^q}{b-a}\left[L_{q+1}^{q+1}(a,b)-aL_q^q(a,b)\right]+\frac{|f'(a)|^q}{b-a}\left[bL_q^q(a,b)-L_{q+1}^{q+1}(a,b)\right]\right\}^{\frac{1}{q}}.$$

Proposition 2.14. *Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in (-\infty, 0] \cup [1,\infty)\backslash\{-2q,-q\}$, then we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leqslant (b-a)^{-\frac{1}{q}}\left\{\left(b^m-a^m\right)L_{q+1}^{q+1}(a,b)-G^2(a,b)\left(b^{m-1}-a^{m-1}\right)L_q^q(a,b)\right\}^{\frac{1}{q}}.$$

Proof. The result follows directly from Corollary 2.13 for the function

$$f(x)=\frac{q}{m+q}x^{\frac{m}{q}+1},\quad x\in(0,\infty).$$

This completes the proof of proposition. □

Corollary 2.15. *For* $m = 1$ *from Proposition 2.14, we obtain the following inequality:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b) \leqslant \left[L_{q+1}^{q+1}(a,b)\right]^{\frac{1}{q}} = L_{q+1}^{\frac{q+1}{q}}(a,b).$$

Theorem 2.16. *For* $n \in \mathbb{N}$, *let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be* n*-times differentiable function on* I° *and* $a, b \in I^{\circ}$ *with* $a < b$. *If* $f^{(n)} \in L[a,b]$ *and* $\left|f^{(n)}\right|^q$ *for* $q > 1$ *is concave on* $[a,b]$, *then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \leqslant \frac{b-a}{n!}L_{np}^{n}(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.$$

Proof. Since $\left|f^{(n)}\right|^q$ for $q > 1$ is concave on $[a,b]$, with respect to Hermite-Hadamard inequality we get

$$\int_a^b \left|f^{(n)}(x)\right|^q dx \leqslant (b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q.$$

Using Lemma 2.1 and the Hölder integral inequality we have

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \\
&\leqslant \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right| dx \\
&\leqslant \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^{bn}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\leqslant \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left((b-a)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
&= \frac{b-a}{n!}\left[\frac{b^{np+1} - a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right| \\
&= \frac{b-a}{n!}L_{np}^{n}(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 2.17. *Under the conditions Theorem 2.16 for* $n = 1$, *we have the following inequality:*

$$\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant L_p(a,b)\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Proposition 2.18. *Let* $a, b \in (0,\infty)$ *with* $a < b$, $q > 1$ *and* $m \in [0,1]$, *then we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leqslant L_p(a,b)A^{\frac{m}{q}}(a,b).$$

Proof. Under the assumption of the proposition, let $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. Then

$$\left|f'(x)\right|^q = x^m,$$

is concave on $(0,\infty)$ and the result follows directly from Corollary 2.17 . □

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