Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 6149–6158
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Wavelet thresholding estimator on $B^s_{p,q}(\mathbb{R}^n)$

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Communicated by Y. H. Yao

Abstract

This paper deals with the convergence of the wavelet thresholding estimator on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$. We show firstly the equivalence of several Besov norms. It seems different with one dimensional case. Then we provide two convergence theorems for the wavelet thresholding estimator, which extend Liu and Wang's work [Y.-M. Liu, H.-Y. Wang, Appl. Comput. Harmon. Anal., **32** (2012), 342–356].

Keywords: Wavelet thresholding estimator, Besov spaces, convergence.
2010 MSC: 42C40, 35Q30, 41A15.

1. Introduction

The convergence of wavelet series is important in both pure and applied mathematics. Kelly et al. [9] studied firstly almost everywhere convergence of wavelet series in 1994. The wavelet thresholding method, proposed by Donoho and Johnstone [7], plays fundamental roles in data compression, signal processing, and statistical problems. Tao and Vidakovic [12, 13], Chen and Meng [2] study the convergence of resulting wavelet series in pointwise and L_p settings, respectively.

As we know, the estimation of density function is important in statistical problems. Wavelet can be successfully applied to the study of this problem. In some statistical models, the error of estimators is measured in L_p norm (e.g., [3]). Besides, Besov spaces contain many functional spaces (e.g., Hölder spaces, Sobolev spaces with non-integer exponents) as special examples. Liu and Wang [10] studied the convergence rate of wavelet thresholding estimators for differential operator in L_p norm over Besov spaces $B^s_{p,q}(\mathbb{R})$. In this paper, we shall study the convergence rate of wavelet thresholding estimators over Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, which is different from $B^s_{p,q}(\mathbb{R})$ ([10]) because of different equivalent Besov-norm theorems in $B^s_{p,q}(\mathbb{R}^n)$.

This paper is organized as follows. Besov space with a standard norm is presented in Section 1.1. The next subsection is devoted to give a wavelet thresholding estimator for ∂f based on non-standard form (NSF) of differential operators. The main results are given in Section 1.3, which will be proved in Section 3. To do that, some auxiliary results are presented in Section 2.

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doi:10.22436/jnsa.010.12.02

Received 2017-09-20

1.1. Besov norms

Let $0 < p, q \leqslant \infty$, $s > 0$ and $[s]$ stands for the largest integer less than or equals to s,

$$B^s_{p,q}(\mathbb{R}^n) := \{f \in L_p(\mathbb{R}^n) : |f|_{B^s_{p,q}(\mathbb{R}^n)} < \infty\}.$$

Here, $|f|_{B^s_{p,q}(\mathbb{R}^n)} := \|(2^{js}\omega^M_p(f, 2^{-j}))_{j\in\mathbb{Z}}\|_{\ell_q}$ with $M \geqslant [s] + 1$ and $\omega^M_p(f, 2^{-j})$ denotes the M-th order smooth modulus of a function f, defined by $\sup\limits_{|h|\leqslant 2^{-j}} \|\Delta^M_h f(\cdot)\|_{L_p(\mathbb{R}^n)}$. The difference operator Δ_h is defined by $\Delta_h f(\cdot) := f(\cdot + h) - f(\cdot)$ and $\Delta^M_h f = \Delta_h(\Delta^{M-1}_h f)$ for a positive integer $M > 1$. The Besov (quasi-)norm (called the standard Besov norm) is given by

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|f\|_{L_p(\mathbb{R}^n)} + |f|_{B^s_{p,q}(\mathbb{R}^n)}$$

and two integers $M, M' > s$ yield equivalent norms ([5, Remark 3.2.2]). Here and after, let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ be the set of positive integers, the set of integers, and the set of real numbers, respectively, as well as $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In addition, we use $A \lesssim B$ to abbreviate that A is bounded by a constant multiple of B, $A \gtrsim B$ is defined as $B \lesssim A$ and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

1.2. Wavelet thresholding estimator

We begin with the concept of multiresolution analysis (MRA, [6]) in this section, which is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of the square integrable function space $L_2(\mathbb{R})$ satisfying the following properties:

(i) $V_j \subset V_{j+1}$, $\forall j \in \mathbb{Z}$;
(ii) $f(\cdot) \in V_0 \Leftrightarrow f(2^j \cdot) \in V_j$;
(iii) $\bigcup_{j\in\mathbb{Z}} V_j = L_2(\mathbb{R})$;
(iv) there exists a function $\phi(x) \in L_2(\mathbb{R})$ called the scaling function such that $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal system and $V_0 = \overline{\operatorname{span}}\{\phi(x-k)\}$.

We can derive a corresponding wavelet function $\psi(x) = \sum_k (-1)^k \overline{h_{1-k}} 2^{\frac{1}{2}} \phi(2x - k)$ with $h_k = \langle \phi(x), 2^{\frac{1}{2}}\phi(2x-k)\rangle$, such that $\{2^{\frac{j}{2}}\psi(2^j x - k)\}_{k\in\mathbb{Z}}$ constitutes an orthonormal basis of W_j, which is the orthogonal complement of V_j in V_{j+1}. Now, we expand these results to $L_2(\mathbb{R}^n)$. Define

$$\Phi(x) = \phi(x_1)\phi(x_2)\cdots\phi(x_n) \quad \text{and} \quad \Psi_e(x) = \xi(x_1)\xi(x_2)\cdots\xi(x_n)$$

for $e \in \{0,1\}^n$ and $e \neq (0, 0, \cdots, 0)$ (if $e_i = 1$, $\xi(x_i) = \psi(x_i)$, else $\xi(x_i) = \phi(x_i)$). With the standard notation $f_{jk}(x) := 2^{\frac{jn}{2}} f(2^j x - k)$, we construct an orthonormal bases $\{\Phi_{J,k}, \Psi_{e:j,k}\}_{j\geqslant J, e\neq(0,0,\cdots,0),k}$ for $L_2(\mathbb{R}^n)$. As usual, let

$$P_j f := \sum_k s_{j,k}\Phi_{j,k} \text{ and } Q_j f = P_{j+1} f - P_j f$$

with $s_{j,k} := \langle f, \Phi_{j,k}\rangle$.

Note that $Tf(x) = \int K(x,y) f(y) dy$ ([1]) defines a bounded linear operator from $L(\mathbb{R}^n)$ to $L(\mathbb{R}^n)$ for $K(x,y) \in L(\mathbb{R}^{2n})$. When representing $K(x,y)$ by the basis

$$\{\Phi_{0,k}(x)\Phi_{0,k'}(y), \Psi_{e:j,k}(x)\Phi_{j,k'}(y), \Phi_{j,k}(x)\Psi_{e':j,k'}(y), \Psi_{e:j,k}(x)\Psi_{e':j,k'}(y)\}_{e,e',j,k,k'},$$

we have

$$\begin{aligned} K(x,y) = \sum_{k,k'} r^0_{k,k'}\Phi_{0,k}(x)\Phi_{0,k'}(y) + \sum_{j\geqslant 0,k,k'} [\sum_{e,e'} \alpha^j_{e,e',k,k'}\Psi_{e:j,k}(x)\Psi_{e':j,k'}(y) \\ + \sum_e \beta^j_{e,k,k'}\Psi_{e:j,k}(x)\Phi_{j,k'}(y) + \sum_{e'} \gamma^j_{e',k,k'}\Phi_{j,k}(x)\Psi_{e':j,k'}(y)] \end{aligned}$$

with

$$r^j_{k,k'} := \langle T\Phi_{j,k'}, \Phi_{j,k}\rangle, \qquad \alpha^j_{e,e',k,k'} := \langle T\Psi_{e':j,k'}, \Psi_{e:j,k}\rangle,$$
$$\beta^j_{e,k,k'} := \langle T\Phi_{j,k'}, \Psi_{e:j,k}\rangle, \qquad \gamma^j_{e',k,k'} := \langle T\Psi_{e':j,k'}, \Phi_{j,k}\rangle.$$

So the NSF of T is defined by

$$\mathcal{T}f = \sum_k \Phi_{0,k}(x)\sum_{k'} r^0_{k,k'}\Phi_{0,k'}(y) + \sum_{j\geqslant 0}\sum_{e,k}[\Psi_{e:j,k}(x)\sum_{e',k'}\alpha^j_{e,e',k,k'}\Psi_{e':j,k'}(y)$$
$$+\Psi_{e:j,k}(x)\sum_{k'}\beta^j_{e,k,k'}\Phi_{j,k'}(y) + \Phi_{j,k}(x)\sum_{e',k'}\gamma^j_{e',k,k'}\Psi_{e':j,k'}(y)].$$

Then NSF of T for differential operator $\partial := \frac{\partial}{\partial x_i}$ is rewritten by

$$\begin{aligned}\mathcal{T}f = \sum_k \Phi_{0,k}(x)\sum_l r_l s_{0,k-l} + \sum_{j\geqslant 0}\sum_{e,k}[\Psi_{e:j,k}(x)2^{nj}\sum_{e',l}\alpha_{e,e',l}d_{e',j:k-l}\\ +\Psi_{e:j,k}(x)2^{nj}\sum_l \beta_{e,l}s_{j,k-l} + \Phi_{j,k}(x)2^{nj}\sum_{e',l}\gamma_{e',l}d_{e':j,k-l}],\end{aligned} \tag{1.1}$$

where $\alpha_{e,e',l} := \int \Psi_e(x-l)\partial\Psi_{e'}(x)dx$, $\beta_{e,l} := \int \Psi_e(x-l)\partial\Phi(x)dx$, $\gamma_{e',l} := \int \Phi_e(x-l)\partial\Psi_{e'}(x)dx$, $r_l := \int \Phi(x-l)\partial\Phi(x)dx$ and $d_{e':j,k} := \langle f, \Psi_{e':j,k}\rangle$. With the definitions of P_j and Q_j, (1.1) leads to $\mathcal{T}f = P_0\partial P_0 f + \sum_{j=0}^{\infty}(Q_j\partial Q_j f + Q_j\partial P_j f + P_j\partial Q_j f)$. Since $P_j + Q_j = P_{j+1}$,

$$\mathcal{T}f(x) = \lim_{j\to\infty} P_j\partial P_j f := \lim_{j\to\infty} P'_j f(x).$$

1.3. Main results

The main work of this paper is to study the convergence of corresponding wavelet thresholding estimator in $B^s_{p,q}(\mathbb{R}^n)$. Throughout this paper, C stands for some positive constant which may change from place to place and

$$\varepsilon_{j,q} := \begin{cases} o(1), & 1 \leqslant q < \infty, \\ O(1), & q = \infty \end{cases} \quad \text{as } j \to \infty.$$

We also need a classical notation (e.g., [10]): A scaling function φ is called r-regular, if φ has r continuous (partial) derivatives and accuracy r, i.e., there exist finitely many $c_{l,k}$ such that for each fixed $x \in \mathbb{R}$,

$$x^k = \sum_l c_{l,k}\varphi(x+l) \quad \text{for } k = 0, 1, \cdots, r-1.$$

Theorem 1.1. *Let $\Phi(x)$ be an r-regular, compactly supported, and orthonormal scaling function. If $f \in B^{s+1}_{p,q}(\mathbb{R}^n)$ with $1 < p < \infty$, $1 \leqslant q < \infty$ and $s > 0$ such that $\frac{n}{p} < s < r-1$, then $P'_J f(x) \in B^s_{p,q}(\mathbb{R}^n)$ and*

(i) $2^{J(s-\frac{1}{p})}\|P'_J f - \partial f\|_\infty = \varepsilon_{J,q}$ *as* $J \to +\infty$;
(ii) $2^{Js}\|P'_J f - \partial f\|_p = \varepsilon_{J,q}$ *as* $J \to +\infty$;
(iii) $\|\partial f - P'_J f(x)\|_{B^s_{p,q}(\mathbb{R}^n)} = \varepsilon_{J,q}$.

To present next theorem, we shall import a concept [7]: A function $\delta(x,\lambda) : \mathbb{R}^n \times \mathbb{R}^+$ is called a thresholding rule, if there exists $C > 0$ such that for all $\lambda > 0$,

$$|x - \delta(x,\lambda)| \leqslant C\lambda \text{ and } |\delta(x,\lambda)| \leqslant C|x|\chi_{\{|x|>\lambda\}},$$

where

$$\chi_{\{|x|>\lambda\}} = \begin{cases} 1, & |x| > \lambda, \\ 0, & |x| \leqslant \lambda. \end{cases}$$

Hard and soft thresholding are two well-known examples. Then the wavelet thresholding estimator of $f \in L_p$ is given by

$$T_\lambda f(x) = \sum_k s_{0,k}\Phi_{0,k}(x) + \sum_{j=0}^{\infty}\sum_{e,k}\delta(d_{e:j,k},\lambda)\Psi_{e:j,k},$$

where $d_{e:j,k} := \langle f, \Psi_{e:j,k}\rangle$. Meanwhile, its NSF on differential operator is $\mathcal{T}_\lambda f := \mathcal{T}(T_\lambda f)$. Similar to $\varepsilon_{j,q}$, define

$$\varepsilon_{\lambda,q} := \begin{cases} o(1), & 1 \leqslant q < \infty, \\ O(1), & q = \infty, \end{cases} \quad \text{as } \lambda \to 0.$$

The next theorem studies the convergence of $\mathcal{T}_\lambda f$ to ∂f.

Theorem 1.2. *Let $\Phi(x)$ be an r-regular, compactly supported, and orthonormal scaling function. If $f \in B^{s+1}_{p,q}(\mathbb{R}^n)$ with $1 < p < \infty$, $1 \leqslant q < \infty$ and $s > 0$ such that $\frac{n}{p} < s < r-1$, $s' = s - \frac{n}{p}$, then $\mathcal{T}_\lambda f(x) \in B^s_{p,q}(\mathbb{R}^n)$ and*

(i) $\lambda^{-\frac{2s'}{2s'+n+2}} \|\mathcal{T}_\lambda f - f'\|_\infty = \varepsilon_{\lambda,q}$;

(ii) $\lambda^{-\frac{2s'}{2s'+3n+2}} \|\mathcal{T}_\lambda f - f'\|_p = \varepsilon_{\lambda,q}$ *when f has compact support;*

(iii) $\|\mathcal{T}_\lambda f - f'\|_{B^s_{p,q}(\mathbb{R}^n)} = \varepsilon_{\lambda,q}$.

2. Some auxiliary results

In order to prove our main results in Section 3, we will give some auxiliary results in this part. Before that, some equivalent Besov norms are presented below.

Let $\varphi \in C^\infty(\mathbb{R}^n)$ (real infinite differential functional spaces), supp $\varphi \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$ and $\varphi(\xi) = 1$ if $|\xi| \leqslant 1$. Write $\varphi_j(\xi) := \varphi(2^{-j}\xi) - \varphi(2^{-j+1}\xi)$ with $j \in \mathbb{N}$. Then define ([14, Page 92])

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|(2^{ks}\|\varphi_k(D)f\|_{L_p})_{k\in\mathbb{N}_0}\|_{l_q} < \infty,$$

where $\varphi_k(D)f := (\varphi_k \hat{f})^\vee$ and $\hat{f}$, $f^\vee$ are the classical Fourier transform and the inverse Fourier transform, respectively.

Besides, let $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Using $W^k_p(\mathbb{R}^n)$ (the famous L_p Sobolev space with integer exponents k), we give the left needed Besov norms below.

Assume $s < M \in \mathbb{N}$ ([14, Page 140]),

$$\|f\|_{{}_1B^s_{p,q}(\mathbb{R}^n)} := \|f\|_{L_p} + \Big(\int_0^1 t^{-sq} \sup_{0\leqslant|h|\leqslant t} \|\Delta^M_h f\|_p \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty$$

and using the usual modification when $q = \infty$.

Let $k < s < m + k$ with $m \in \mathbb{N}, k \in \mathbb{N}_0$ ([14, Page 8]),

$$\|f\|_{{}_2B^s_{p,q}(\mathbb{R}^n)} := \|f\|_{W^k_p} + \sum_{|\alpha|=k} \Big(\int_{\mathbb{R}^n} |h|^{-(s-k)q} \|\Delta^m_h D^\alpha f\|_p \frac{dh}{|h|^n}\Big)^{\frac{1}{q}} < \infty$$

and also using the usual modification when $q = \infty$. When $m = 2$, $k = [s]$ for $s \notin \mathbb{N}$ and $k = [s] - 1$ for $s \in \mathbb{N}$, $\beta = s - k$ $(0 < \beta \leqslant 1)$, we denote $\|f\|_{{}_2B^s_{p,q}(\mathbb{R}^n)}$ by ([10])

$$\|f\|_{HB^s_{p,q}(\mathbb{R}^n)} := \|f\|_{W^k_p} + \sum_{|\alpha|=k} \Big(\int_{\mathbb{R}^n} |h|^{-\beta q} \|\Delta^2_h D^\alpha f\|_p \frac{dh}{|h|^n}\Big)^{\frac{1}{q}} < \infty.$$

First, we present these Besov norms are equivalent.

Theorem 2.1. *If $k < s < 2 + k$ $(m = 2)$, $k \in \mathbb{N}_0$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, then $\|f\|_{HB^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{B^s_{p,q}(\mathbb{R}^n)}$.*

Proof. By Definitions (1.2.5/1), Theorem 1.2.5 (3) and Theorem 1.3.4 in [14] ($k < s < 2 + k$, $k \in \mathbb{N}_0$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$), we have $\|f\|_{HB^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{B^s_{p,q}(\mathbb{R}^n)}$. On the other hand, Remak 9.13 in [15] tells us that $\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{{}_1B^s_{p,q}(\mathbb{R}^n)}$ for $0 < p, q \leqslant \infty$, $\sigma_p < s < M \in \mathbb{N}_0$ ($\sigma_p = n\max\{\frac{1}{p} - 1, 0\}$). Meanwhile, when $0 < p, q \leqslant \infty$, $0 < s < M \in \mathbb{N}_0$, simple calculations lead to $\|f\|_{{}_1B^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{B^s_{p,q}(\mathbb{R}^n)}$. In fact,

$$\int_0^1 t^{-sq} \sup_{0\leqslant|h|\leqslant t} \|\Delta^M_h f\|_p \frac{dt}{t} = \sum_{j=0}^\infty \int_{2^{-j-1}}^{2^{-j}} t^{-sq} \sup_{0\leqslant|h|\leqslant t} \|\Delta^M_h f\|_p \frac{dt}{t} \sim \sum_{j=0}^\infty 2^{jsq} \sup_{0\leqslant|h|\leqslant 2^{-j}} \|\Delta^M_h f\|_p \ln 2, \quad (2.1)$$

where the equivalence part of (2.1) is from Lemma 9.1 (iv) in [8]. Then the desired conclusion follows. □

By Theorem 2.1, we have an important corollary.

Corollary 2.2. *If* $s > 0$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, *then* $f \in B^{s+1}_{p,q}(\mathbb{R}^n)$ *if and only if* $\partial f \in B^s_{p,q}(\mathbb{R}^n)$.

In addition, we can get the convergence of projection operators in Besov spaces.

Theorem 2.3. *Let* $\Phi(x)$ *be an* r*-regular function,* $0 < s < r$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$. *Then*

$$\lim_{j\to\infty} \|f - P_j f\|_{B^s_{p,q}(\mathbb{R}^n)} = 0$$

for $f \in B^s_{p,q}(\mathbb{R}^n)$. *Moreover,*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|(s_{0,\cdot})\|_{l_p} + \|\{2^{j(s+\frac{n}{2}-\frac{n}{p})}\|d_{e:j,\cdot}\|_{l_p}\}_{j\geqslant 0}\|_{l_q}.$$

Proof. Note that $\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{DB^s_{p,q}(\mathbb{R}^n)} := \|P_0 f\|_{L_p} + \|(2^{js}\|Q_j f\|_{L_p})_{j\geqslant 0}\|_{l_q}$ by Theorem 3.6.1 in [5]. It is easy to show that $\lim\limits_{j\to\infty} \|f - P_j f\|_{DB^s_{p,q}(\mathbb{R}^n)} = 0$, where $q \neq \infty$ is needed. Then

$$\lim_{j\to\infty} \|f - P_j f\|_{B^s_{p,q}(\mathbb{R}^n)} = 0$$

holds for $f \in B^s_{p,q}(\mathbb{R}^n)$.

Similar to Theorem 1.1 in [11], we can easily get the proof of equivalence. □

Remark 2.4. Theorem 2.3 says that $f = P_J f + \sum_{j\geqslant J} Q_j f$ in Besov norm. Moreover, $f \in B^s_{p,q}(\mathbb{R}^n)$ can be characterized by wavelet coefficients norm $\|(s_{0,\cdot})\|_{l_p} + \|\{2^{j(s+\frac{n}{2}-\frac{n}{p})}\|d_{e:j,\cdot}\|_{l_p}\}_{j\geqslant 0}\|_{l_q}$.

The last two auxiliary results are the following lemmas.

Lemma 2.5 ([9])**.** *Let* Φ *be an orthonormal scaling function and* Ψ_e *be the corresponding wavelets. If* Φ *and* Ψ_e *are bounded in absolute value by a Lebesgue integrable function* $L(x)$ *with* $L(x) \leqslant L(y)$ *for* $|x| \geqslant |y|$, *then the scaling function and wavelet expansion* $\sum_k s_{0,k}\Phi_{0,k}(x) + \sum_{j=0}^{\infty}\sum_{e,k} d_{e:j,k}\Psi_{e:j,k}$ ($1 \leqslant p \leqslant \infty$) *of* $f \in L_p(\mathbb{R}^n)$ *converges to* $f(x)$ *pointwise almost everywhere.*

Lemma 2.6. *Let* $\Phi(x)$ *be 1-regular and* $\Psi(x)$ *be the corresponding wavelet. If* $f \in L^p(\mathbb{R}^n)\bigcap C(\mathbb{R}^n)$ *with* $1 \leqslant p \leqslant +\infty$ *and* $s \in \mathbb{R}$ *such that* $sp > n$, *then the following two identities hold uniformly on* $\mathbb{R}^n$.

(i) $f(x) = P_0 f(x) + \sum_{j=0}^{\infty}\sum_{e,k} d_{e:j,k}\Psi_{e:j,k}(x)$, *when* $|d_{e:j,k}| \lesssim 2^{-j(s+\frac{n}{2}-\frac{n}{p})}$;

(ii) $\partial f(x) = \partial(P_0 f)(x) + \sum_{j=0}^{\infty}\sum_{e,k} d_{e:j,k}\partial\Psi_{e:j,k}(x)$, *when* $|d_{e:j,k}| \lesssim 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}$.

Proof.

(i). When $1 \leqslant p \leqslant \infty$, Lemma 2.5 says

$$f(x) = P_0 f(x) + \sum_{j=0}^{\infty}\sum_{e,k} d_{e:j,k}\Psi_{e:j,k}(x)$$

almost everywhere. On the other hand, when $|d_{e:j,k}| \lesssim 2^{-j(s+\frac{n}{2}-\frac{n}{p})}$,

$$|\sum_{e,k} d_{e:j,k}\Psi_{e:j,k}(x)| \leqslant \sum_{e,k} |d_{e:j,k}| 2^{\frac{jn}{2}} |\Psi(2^j x - k)| \lesssim 2^{-j(s-\frac{n}{p})}.$$

Hence, $\sum\limits_{j\geqslant 0,e,k} d_{e:j,k}\Psi_{e:j,k}(x)$ converges uniformly for $sp > n$, which implies the continuity of $\sum\limits_{j=0}^{\infty}\sum\limits_{e,k} d_{e:j,k}\Psi_{e:j,k}(x)$. Because $P_0 f(x)$ and $f(x)$ are continuous, the proof of (i) is completed.

(ii). Similar to (i), $|d_{e:j,k}| \lesssim 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}$ implies the uniform convergence and the continuity of

$$\sum_{j=0}^{\infty}\sum_{e,k} d_{e:j,k}\partial\Psi_{e:j,k}(x).$$ □

3. Proof of main results

Based on auxiliary results of Section 2, this section is devoted to prove Theorem 1.1 and Theorem 1.2. In fact, the proofs of these theorems are very similar to Theorem 1.2.a and Theorem 1.3.a in [10]. But the characterization of Besov spaces by wavelet coefficients is different from [10], and even the definition of Besov space is different. For the sake of understanding the proof of main results easily, we give the proofs in detail.

We begin with the proof of Theorem 1.1 firstly.

Proof of Theorem 1.1. Note that $d_{e:j,k} := \langle f, \Psi_{e:j,k}\rangle$, then $f \in B^{s+1}_{p,q}(\mathbb{R}^n)$ implies

$$(2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\|d_{e:j\cdot}\|_p)_{j\geqslant 0} \in l^q, \tag{3.1}$$

according to Theorem 2.3. By $|d_{jk}| \lesssim 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}$ and Lemma 2.6, one knows that

$$f(x) = P_0f(x) + \sum_{j=0}^{\infty}\sum_{k} d_{e:j,k}\Psi_{e:j,k}(x), \quad \partial f(x) = \partial(P_0f)(x) + \sum_{j=0}^{\infty}\sum_{e,k} d_{e:j,k}\partial\Psi_{e:j,k}(x) \tag{3.2}$$

hold uniformly. Note that $P_Jf(x) = P_0f(x) + \sum\limits_{j=0}^{J-1}\sum\limits_{k} d_{e:j,k}\Psi_{e:j,k}(x)$ for $J > 0$. Then $|s^J_{0,k}| := |\langle P_Jf,\ \varphi_{0k}\rangle| = |s_{0,k}| := |\langle f,\ \varphi_{0,k}\rangle|$ and $|d^J_{e:j,k}| := |\langle P_Jf,\ \Psi_{e:j,k}\rangle| \leqslant |d_{jk}|$ for $j \geqslant 0$. Hence,

$$\|s^J_{0,\cdot}\|_p + \|\{2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\|d^J_{e:j,\cdot}\|_p\}_{j\geqslant 0}\|_{l^q} \leqslant \|s_{0,\cdot}\|_p + \|\{2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\|d_{e:j,\cdot}\|_p\}_{j\geqslant 0}\|_{l^q} \lesssim \|f\|_{B^{s+1}_{p,q}(\mathbb{R}^n)}.$$

Now, $P_Jf \in B^{s+1}_{p,q}(\mathbb{R}^n)$ follows from the fact $\|P_Jf\|_p \lesssim \|f\|_p$ and Theorem 2.3. Hence, $(P_Jf) \in B^s_{p,q}(\mathbb{R}^n)$ (Corollary 2.2). This argument also shows the boundedness of P_J on $B^s_{p,q}(\mathbb{R}^n)$. Therefore, $P'_Jf := P_J\partial(P_Jf) \in B^s_{p,q}(\mathbb{R}^n)$.

(i) By the representation of P_Jf, $\partial(P_Jf)(x) = \partial(P_0f)(x) + \sum\limits_{j=0}^{J-1}\sum\limits_{e,k} d_{e:j,k}\partial\Psi_{e:j,k}(x)$. This with (3.2) leads to $\partial(P_Jf)(x) - \partial f(x) = \sum\limits_{j=J}^{\infty}\sum\limits_{k} d_{e:j,k}\partial\Psi_{e:j,k}(x)$. Using (3.1), one has

$$\|\partial(P_Jf)(x) - \partial f(x)\|_\infty = \|\sum_{j=J}^{\infty}\sum_{k} d_{e:j,k}\partial\Psi_{e:j,k}(x)\|_\infty \lesssim \sum_{j=J}^{\infty}\|d_{e:j,\cdot}\|_\infty 2^{(\frac{n}{2}+1)j} = 2^{-J(s-\frac{n}{p})}\varepsilon_{J,q}. \tag{3.3}$$

Similarly, because $f \in B^{s+1}_{p,q}(\mathbb{R}^n)$, $\partial f \in B^s_{p,q}(\mathbb{R}^n)$ and $d'_{e:j,k} := \langle \partial f,\ \Psi_{e:j,k}\rangle$ satisfies $|d'_{e:j,k}| \lesssim 2^{-j(s+\frac{n}{2}-\frac{n}{p})}\varepsilon_{j,q}$ due to Theorem 2.3. Then Lemma 2.6 says $\partial f(x) = P_0(\partial f)(x) + \sum\limits_{j=0}^{\infty}\sum\limits_{k} d'_{e:j,k}\Psi_{e:j,k}(x)$ and $P_J(\partial f)(x) = P_0(\partial f)(x) + \sum\limits_{j=0}^{J-1}\sum\limits_{k} d'_{e:j,k}\Psi_{e:j,k}(x)$. Moreover, $P_J(\partial f)(x) - (\partial f)(x) = \sum_{j=J}^{\infty}\sum_k d'_{e:j,k}\Psi_{e:j,k}(x)$ and

$$\|P_J(\partial f)(x) - (\partial f)(x)\|_\infty \lesssim \sum_{j=J}^{\infty}\|d_{e:j\cdot}\|_\infty 2^{\frac{nj}{2}} = 2^{-J(s-\frac{n}{p})}\varepsilon_{J,q}. \tag{3.4}$$

Note that $P'_Jf := P_J(\partial(P_Jf))$ and $\|P_Jf\|_\infty \lesssim \|f\|_\infty$. Then $\|P'_Jf - (\partial f)\|_\infty = \|P_J(P_Jf)' - f'\|_\infty \leqslant \|P_J\partial(P_Jf) - P_J(\partial f)\|_\infty + \|P_J(\partial f) - (\partial f)\|_\infty \lesssim \|\partial(P_Jf) - (\partial f)\|_\infty + \|P_J(\partial f) - (\partial f)\|_\infty$, which leads to the desired $\|P'_Jf - (\partial f)\|_\infty \lesssim 2^{-J(s-\frac{n}{p})}\varepsilon_{J,q}$ from (3.3) and (3.4).

(ii) By the assumption $f \in B^{s+1}_{p,q}(\mathbb{R}^n)$, one knows $\partial f \in B^s_{p,q}(\mathbb{R}^n)$. In addition, the proved fact says $P'_Jf \in B^s_{p,q}(\mathbb{R}^n)$. Hence, $P'_Jf - \partial f \in B^s_{p,q}(\mathbb{R}^n)$, and $(2^{js}\|(P'_Jf - \partial f) - P_j(P'_Jf - \partial f)\|_p)_{j\geqslant J} \in l^q$ thanks to Theorem 2.3. Clearly,

$$P_j(P'_Jf) = P_jP_J\partial(P_Jf) = P_J\partial(P_Jf) = P'_Jf \ \text{ for } \ j \geqslant J,$$

then $(2^{js}\|P_j(\partial f)-\partial f\|_p)_{j\geqslant J}\in l^q$. Namely, $\|P_J(\partial f)-(\partial f)\|_p=2^{-Js}\varepsilon_{J,q}$. Since $\|P'_Jf-(\partial f)\|_p\leqslant\|P_J\partial(P_Jf)-P_J(\partial f)\|_p+\|P_J(\partial f)-\partial f\|_p\lesssim\|\partial(P_Jf)-\partial f\|_p+\|P_J(\partial f)-\partial f\|_p$, it remains to show

$$\|\partial(P_Jf)-(\partial f)\|_p=2^{-Js}\varepsilon_{J,q}. \tag{3.5}$$

From (3.2), $\partial(P_Jf)(x)-(\partial f)(x)=\sum_{j=J}^{\infty}\sum_k d_{e:j,k}\partial\Psi_{e:j,k}(x)$. Because Ψ_e is compact supported and bounded, one obtains

$$\|\partial(P_Jf)-(\partial f)\|_p\leqslant\sum_{j=J}^{\infty}\|\sum_{e,k}d_{e:j,k}\partial\Psi_{e:j,k}(x)\|_p\lesssim\sum_{j=J}^{\infty}2^{(\frac{n}{2}+1)j}2^{-\frac{jn}{p}}\|d_{e:j,\cdot}\|_p=\sum_{j=J}^{\infty}2^{-js}\|d_{e:j,\cdot}\|_p2^{j(\frac{n}{2}+s+1-\frac{n}{p})}.$$

When $q=1$, $\|\partial(P_Jf)-(\partial f)\|_p=2^{-Js}\varepsilon_{J,q}$ follows easily from (3.1); when $1<q<\infty$, assume $\frac{1}{q}+\frac{1}{q'}=1$, then by the Hölder inequality and (3.1),

$$\|\partial(P_Jf)-\partial f\|_p\lesssim(\sum_{j=J}^{\infty}2^{-jsq'})^{\frac{1}{q'}}\|\{\|d_{e:j,\cdot}\|_p2^{j(\frac{3}{2}+s-\frac{1}{p})}\}_{j\geqslant J}\|_q\lesssim 2^{-Js}\varepsilon_q.$$

This reaches (3.5) and the proof of (ii) is completed.

(iii) As shown in the first paragraph of this proof, when $f\in B^{s+1}_{p,q}(\mathbb{R}^n)$, $P_Jf\in B^{s+1}_{p,q}(\mathbb{R}^n)$, $s^J_{0,k}:=\langle P_Jf,\ \Phi_{0,k}\rangle=s_{0,k}:=\langle f,\ \Phi_{0,k}\rangle$ for $k\in\mathbb{Z}$, and $d^J_{e:j,k}:=\langle P_Jf,\ \Psi_{e:j,k}\rangle=d_{e:j,k}:=\langle f,\ \Psi_{e:j,k}\rangle$ for $j<J$. Then $P_Jf-f\in B^{s+1}_{p,q}(\mathbb{R}^n)$. With the help of Theorem 2.3,

$$\|P_Jf-f\|_{B^{s+1}_{p,q}}\sim\|s^J_{0,\cdot}-s_{0,\cdot}\|_p+\|\{\|d^J_{e:j,k}-d_{e:j,k}\|_p2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\}_{j\geqslant0}\|_q=\|\{\|d_{e:j,\cdot}\|_p2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\}_{j\geqslant J}\|_q.$$

Because $\|f\|_{B^{s+1}_{p,q}}\sim\|s_{0,\cdot}\|_p+\|\{\|d_{e:j,k}\|_p2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\}_{j\geqslant0}\|_q$, $\lim_{J\to+\infty}\|P_Jf-f\|_{B^{s+1}_{p,q}}=0$ for $1\leqslant q<\infty$. Similarly, $\lim_{J\to+\infty}\|P_J(\partial f)-(\partial f)\|_{B^s_{p,q}}=0$. Note that $\|P'_Jf-P_J(\partial f)\|_{B^s_{p,q}}=:\|P_J\partial(P_Jf)-P_J(\partial f)\|_{B^s_{p,q}}\lesssim\|\partial(P_Jf)-(\partial f)\|_{B^s_{p,q}}\lesssim\|P_Jf-f\|_{B^{s+1}_{p,q}}$. Then, $\lim_{J\to+\infty}\|P'_Jf-P_J(\partial f)\|_{B^s_{p,q}}=0$, and finally the desired conclusion $\lim_{J\to+\infty}\|P'_Jf-(\partial f)\|_{B^s_{p,q}}=0$ follows from

$$\|P'_Jf-(\partial f)\|_{B^s_{p,q}}\leqslant\|P'_Jf-P_J(\partial f)\|_{B^s_{p,q}}+\|P_J(\partial f)-(\partial f)\|_{B^s_{p,q}}.$$ □

Next, we prove Theorem 1.2.

Proof of Theorem 1.2. By $\mathcal{T}_\lambda f(x)=:\mathcal{T}(T_\lambda f)(x)$, it is sufficient to show $\partial(T_\lambda f)\in B^s_{p,q}(\mathbb{R})$ or $T_\lambda f\in B^{s+1}_{p,q}(\mathbb{R})$ by Theorem 1.1 (i), in order to conclude $\mathcal{T}_\lambda f\in B^s_{p,q}(\mathbb{R}^n)$. Note that $d_{e:j,k}:=\langle f,\Psi_{e:j,k}\rangle$ satisfies $|\delta(d_{e:j,k},\lambda)|\lesssim|d_{e:j,k}|$ and $\|d_{e:j,\cdot}\|_p\lesssim 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}\varepsilon_q$ due to Theorem 1.1. Because Ψ_e are compactly supported,

$$\|\sum_k\delta(d_{e:j,k},\lambda)\Psi_{e:j,k}(x)\|_p\lesssim 2^{(\frac{n}{2}-\frac{n}{p})j}\|\delta(d_{e:j,\cdot},\lambda)\|_p\lesssim 2^{(\frac{n}{2}-\frac{n}{p})j}\|d_{e:j,\cdot}\|_p\lesssim 2^{-(s+1)j}.$$

In fact, if a bounded function $g\in L^2(\mathbb{R}^n)$ has compact support, then for $1\leqslant p\leqslant\infty$, $\frac{1}{p}+\frac{1}{p'}=1$ ([8]),

$$\|\sum_k c_k2^{\frac{j}{2}}g(2^jx-k)\|_p\leqslant\|\sum_k|g(x-k)|\|_\infty^{\frac{1}{p'}}\|c\|_p2^{(\frac{n}{2}-\frac{n}{p})j}\|g\|_1^{\frac{1}{p}}.$$

Moreover, $\sum_{j\geqslant0}\|\sum_{e,k}\delta(d_{e:j,k},\lambda)\Psi_{e:j,k}(x)\|_p\lesssim\sum_{j\geqslant0}2^{-(s+1)j}<\infty$, which means

$$\sum_{j\geqslant0,e,k}\delta(d_{e:j,k},\lambda)\Psi_{e:j,k}(x)\in L^p(\mathbb{R}^n).$$

Now,

$$T_\lambda f(x) =: \sum_k s_{0,k}\Phi_{0,k}(x) + \sum_{j\geqslant 0}\sum_{e,k} \delta(d_{e:j,k},\lambda)\Psi_{e:j,k}(x) \in L^p(\mathbb{R}^n)$$

with $s_{0,k} = \langle f, \Phi_{0,k}\rangle$. On the other hand, assume $\hat{s}_{0,k} = \langle T_\lambda f, \Phi_{0,k}\rangle$ and $\hat{d}_{e:j,k} = \langle T_\lambda f, \Psi_{e:j,k}\rangle$, then $\hat{s}_{0,k} = s_{0,k}$ and $|\hat{d}_{e:j,k}| = |\delta(d_{e:j,k},\lambda)| \lesssim |d_{e:j,k}|$. Hence,

$$\|\hat{s}_{0,\cdot}\|_p + \|\{2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\|\hat{d}_{e:j,\cdot}\|_p\}_{j\geqslant 0}\|_q \lesssim \|s_{0,\cdot}\|_p + \|\{2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\|d_{e:j,\cdot}\|_p\}_{j\geqslant 0}\|_q \lesssim \|f\|_{B^{s+1}_{p,q}(\mathbb{R}^n)}.$$

This shows $T_\lambda f \in B^{s+1}_{p,q}(\mathbb{R}^n)$ thanks to Theorem 2.3.

(i) As in the first paragraph, one knows $|d_{e:j,k}| := |\langle f, \Psi_{e:j,k}\rangle| \lesssim \|d_{e:j,\cdot}\|_p \lesssim 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}$. This with Lemma 2.6 (ii) leads to

$$\partial f(x) = \partial(P_0 f)(x) + \sum_{j=0}^{\infty}\sum_{e,k} d_{e:j,k}\partial\Psi_{e:j,k}(x). \tag{3.6}$$

Recall that $T_\lambda f(x) := P_0 f(x) + \sum_{j=0}^{\infty}\sum_{e,k}\delta(d_{e:j,k},\lambda)\Psi_{e:j,k}(x)$ and $T_\lambda f \in B^{s+1}_{p,q}(\mathbb{R}^n)$ implies $\mathcal{T}_\lambda f(x) = \partial(T_\lambda f)(x) \in B^s_{p,q}(\mathbb{R}^n)$, according to Corollary 2.2. Then $\mathcal{T}_\lambda f(x) = \partial(P_0 f)(x) + \sum_{j=0}^{\infty}\sum_{e,k}\delta(d_{e:j,k},\lambda)\cdot\partial\Psi_{e:j,k}(x)$. Since $|\delta(d_{e:j,k},\lambda)| \lesssim |d_{e:j,k}| \lesssim 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}$, $\sum_{j\geqslant 0,e,k}\delta(d_{e:j,k},\lambda)\Psi_{e:j,k}(x)$ and $\sum_{j\geqslant 0,e,k}\delta(d_{e:j,k},\lambda)\partial\Psi_{e:j,k}(x)$ converge uniformly on $\mathbb{R}^n$. Therefore $\mathcal{T}_\lambda f(x) = \partial(P_0 f)(x) + \sum_{j\geqslant 0,k}\delta(d_{e:j,k},\lambda)\partial\Psi_{jk}(x)$. Combining this with (3.6), one has

$$\mathcal{T}_\lambda f(x) - \partial f(x) = \sum_{j\geqslant 0}\sum_{e,k}[\delta(d_{e:j,k},\lambda) - d_{e:j,k}]\partial\Psi_{jk}(x). \tag{3.7}$$

Let $J := \max\{1, [\frac{2}{2s'+n+2}\log_2\frac{\varepsilon_q}{\lambda}]\}$, because $\|d_{e:j,\cdot}\|_p \lesssim 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}\varepsilon_q$, $\delta(d_{e:j,k},\lambda) = 0$ when $j \geqslant J$. Here $s' := s - \dfrac{n}{p}$. Then, $|\mathcal{T}_\lambda f(x) - \partial f(x)| \leqslant \sum_{j=0}^{J}\sum_{e,k}|\delta(d_{e:j,k},\lambda) - d_{e:j,k}||\partial\Psi_{e:j,k}(x)| + \sum_{j=J+1}^{\infty}\sum_{e,k}|d_{e:j,k}||\partial\Psi_{e:j,k}(x)|$. By $|\delta(d_{e:j,k},\lambda) - d_{e:j,k}| \lesssim \lambda$ and $|d_{e:j,k}| \lesssim 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}\varepsilon_q$,

$$|\mathcal{T}_\lambda f(x) - \partial f(x)| \lesssim \sum_{j=0}^{J}\lambda 2^{(\frac{n}{2}+1)j} + \sum_{j=J+1}^{\infty} 2^{-j(s+1+\frac{n}{2}-\frac{n}{p})}\cdot\varepsilon_q 2^{(\frac{n}{2}+1)j} \lesssim \lambda 2^{\frac{3}{2}J} + 2^{-s'J}\varepsilon_q.$$

This with the choice of J leads to $|\mathcal{T}_\lambda f(x) - \partial f(x)| \leqslant 2\lambda^{\frac{2s'}{2s'+n+2}}\varepsilon_q^{\frac{n+2}{2s'+n+2}}$. Note that $J \to +\infty$ if and only if $\lambda \to 0$, then the conclusion (i) holds.

(ii) Applying Bernstein inequality ([8]) to (3.7), one obtains

$$\|\mathcal{T}_\lambda f - \partial f\|_p \leqslant \sum_{j\geqslant 0} 2^j \|\sum_{e,k}[\delta(d_{e:j,k},\lambda) - d_{e:j,k}]\Psi_{e:j,k}(x)\|_p. \tag{3.8}$$

Because both f and Ψ_e have compact supports, the number of non-zero wavelet coefficients $d_{e:j,k}$ is $O(2^{jn})$ on level j. This with $|\delta(d_{e:j,k},\lambda) - d_{e:j,k}| \lesssim \lambda$ implies that for fixed $J > 0$,

$$\sum_{j=0}^{J} 2^j\|\sum_{e,k}[\delta(d_{e:j,k},\lambda) - d_{e:j,k}]\Psi_{e:j,k}(x)\|_p \lesssim \sum_{j=0}^{J} 2^{nj}\cdot 2^j\lambda\|\Psi_{e:j,k}\|_p = \sum_{j=0}^{J} 2^{j(\frac{3n+2}{2}-\frac{n}{p})}\lambda \lesssim 2^{J(\frac{3n+2}{2}-\frac{n}{p})}\lambda. \tag{3.9}$$

On the other hand (because Ψ_e are compact supported and bounded),

$$\begin{aligned}\|\sum_k[\delta(d_{e:j,k},\lambda) - d_{e:j,k}]\Psi_{e:j,k}\|_p &\lesssim 2^{(\frac{n}{2}-\frac{n}{p})j}\|\delta(d_{e:j,\cdot},\lambda) - d_{e:j,\cdot}\|_p \\ &\leqslant 2^{(\frac{n}{2}-\frac{n}{p})j}(\|\delta(d_{e:j,\cdot},\lambda)\|_p + \|d_{e:j,\cdot}\|_p) \lesssim 2^{(\frac{n}{2}-\frac{n}{p})j}\|d_{e:j,\cdot}\|_p.\end{aligned}$$

By $\|d_{e:j,\cdot}\|_p \lesssim 2^{-(s+1+\frac{n}{2}-\frac{n}{p})j}\varepsilon_q$ and

$$\|\sum_{e,k}[\delta(d_{e:j,k},\lambda)-d_{e:j,k}]\Psi_{e:j,k}(x)\|_p \lesssim 2^{(\frac{n}{2}-\frac{n}{p})j}2^{-(s+1+\frac{n}{2}-\frac{n}{p})j}\varepsilon_q = 2^{-(s+1)j}\varepsilon_q,$$

we have

$$\sum_{j=J+1}^{\infty} 2^j\|\sum_{e,k}[\delta(d_{e:j,k},\lambda)-d_{e:j,k}]\Psi_{e:j,k}(x)\|_p \lesssim \sum_{j=J+1}^{\infty} 2^{-sj}\varepsilon_q = 2^{-sJ}\varepsilon_q. \tag{3.10}$$

The combination of (3.8), (3.9), and (3.10) tells $\|\mathcal{T}_\lambda f-\partial f\|_p \lesssim 2^{J(\frac{3n+2}{2}-\frac{n}{p})}\lambda+2^{-Js}\varepsilon_q$. Similar to (i), taking $J := \max\{1,[\frac{2}{2s'+3n+2}\log_2(\lambda^{-1}\varepsilon_q)]\}$ ($s' := s-\frac{n}{p}$), then $\|\mathcal{T}_\lambda f-\partial f\|_p = \lambda^{\frac{2s}{2s'+3n+2}}\varepsilon_q$.

(iii) Let $d_{e:j,k} = \langle f,\Psi_{e:j,k}\rangle$ and $\hat{d}_{e:j,k} = \langle T_\lambda f,\Psi_{e:j,k}\rangle$. Then

$$\|T_\lambda f-f\|_{B^{s+1}_{p,q}} \lesssim \|\{\|\hat{d}_{e:j,\cdot}-d_{e:j,\cdot}\|_p 2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\}_{j\geqslant 0}\|_q$$

due to Theorem 2.3. Since $\mathcal{T}_\lambda f(x) = \partial(T_\lambda f)(x)$, $\mathcal{T}_\lambda f-\partial f = \partial(T_\lambda f-f)$ and $\|\mathcal{T}_\lambda f-\partial f\|_{B^s_{p,q}} \leqslant \|T_\lambda f-f\|_{B^{s+1}_{p,q}}$. Moreover,

$$\|\mathcal{T}_\lambda f-\partial f\|_{B^s_{p,q}} \lesssim \|\{\|\hat{d}_{e:j,\cdot}-d_{e:j,\cdot}\|_p 2^{j(s+1+\frac{n}{2}-\frac{n}{p})}\}_{j\geqslant 0}\|_q. \tag{3.11}$$

Recall that $T_\lambda f(x) := P_0 f(x)+\sum_{j\geqslant 0,k}\delta(d_{e:j,k},\lambda)\Psi_{e:j,k}(x)$, then

$$|\hat{d}_{e:j,k}| = |\delta(d_{e:j,k},\lambda)| \lesssim |d_{e:j,k}| \text{ and } |\hat{d}_{e:j,k}-d_{e:j,k}| \lesssim |d_{e:j,k}|.$$

By $f\in B^{s+1}_{p,q}(\mathbb{R}^n)$ and Theorem 2.3,

$$\|(\|d_{e:j,\cdot}\|_p 2^{j(s+1+\frac{n}{2}-\frac{n}{p})})_{j\geqslant 0}\|_q \lesssim \|f\|_{B^{s+1}_{p,q}}.$$

Hence,

$$\lim_{J\to+\infty}\sum_{j=J+1}^{\infty}(\|d_{e:j,\cdot}\|_p 2^{j(s+1+\frac{n}{2}-\frac{n}{p})})^q = 0 \quad \text{for} \quad 1\leqslant q<\infty. \tag{3.12}$$

Because $\lim_{\lambda\to 0}\hat{d}_{e:j,k} = d_{e:j,k}$ for each $1\leqslant j\leqslant J$,

$$\lim_{\lambda\to 0}\sum_{j=0}^{J}[\|\hat{d}_{e:j,\cdot}-d_{e:j,\cdot}\|_p 2^{j(s+1+\frac{n}{2}-\frac{n}{p})}]^q = 0.$$

This with (3.11) and (3.12) lead to $\lim\limits_{\lambda\to 0}\|\mathcal{T}_\lambda f-\partial f\|_{B^s_{p,q}} = 0$. □

Remark 3.1. Theorems 1.1 and 1.2 can be used to study the smoothness estimation of n-dimensional density functions in statistical problems (e.g., [4]), and this is the next work we will focus on.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grant No. 11401433, 11661024).

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