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On some extensions of Nadler's fixed point theorem

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Abstract

In this paper, we give the notion of the pseudo-fixed point for multi-valued mappings which enable us to extend Nadler's theorem and other well-known results in the literature.

Keywords: Complete metric space, Hausdorff metric, multi-valued mapping, pseudo-fixed point, fixed point, Nadler's fixed point theorem.
2010 MSC: 54H25, 47H10.

1. Introduction and preliminaries

Banach's contraction principle (1922) was the starting point of a growing field which is the fixed point theory. The interest of this field comes from its application to several branches in mathematics, for example the resolution of nonlinear problems involving integral equations and integro-differential equations and the convergence of some iterative processes. Banach's contraction principle has been the subject of several extensions in various types of metric spaces and using some generalized contraction conditions on the self-mappings. In 1969, Nadler [6] generalized this contraction principle to the case of multi-valued mappings. This contribution is of great importance due to the fact that this class of mappings plays a central role in applied sciences (optimization, equilibrium problems, games theory, differential and partial differential equations involving integral inclusions, $\cdots$). For a good reading concerning theses mappings, we can quote for example [1, 3, 4].

Nadler's result is given as follows:

Theorem 1.1 ([6]). *Let (X, d) be a complete metric space and f be a multi-valued mapping on X such that $f(x)$ is a nonempty closed bounded subset for any $x \in X$. If there exists $\alpha \in (0, 1)$ such that*

$$H(f(x), f(y)) \leqslant \alpha d(x, y), \quad \forall x, y \in X,$$

then there exists $z \in X$ such that $z \in f(z)$ (here H is the Hausdorff metric distance defined on the family of the collection of all nonempty closed subsets of X).

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Let (X,d) be a complete metric space, $N(X)$ denotes the collection of all nonempty subsets of X, $C(X)$ is the collection of all nonempty closed subsets of X, $CB(X)$ is the collection of all nonempty closed bounded subsets of X and $K(X)$ is the collection of all nonempty compact subsets of X.

For $A, B \subset C(X)$, let

$$d(A,B) = \inf\{\, d(x,z) | x \in A, z \in B\},$$

and

$$H(A,B) = \max\{\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\}.$$

H is called the generalized Hausdorff distance induced by d.

Let (X,d) be a complete metric space and f be a multi-valued mapping on X such that $f(x)$ is a nonempty closed bounded subset for any $x \in X$ and let $\{A_x, x \in X\} \subset K(X)$ be a family of compact subsets of X such that $x \in A_x$ for all $x \in X$. Let $x \in X$, we denote by $\mathcal{Z}^x = \{z \in X \text{ such that } A_z \bigcap f(x) \neq \emptyset\}$. It is easy to show that for all $x \in X$, $\mathcal{Z}^x$ is nonempty since $\mathcal{Z}^x$ contains the nonempty subset $\{z \in f(x)\}$. In the following assume that the family $\{A_x, x \in X\}$ satisfies the following two assumptions:

($\imath$) $d(A_x, f(x)) = \inf_{z\in\mathcal{Z}^x}\{H(A_x, f(x) \bigcap A_z)\}$.

($\imath\imath$) If $x_n \longrightarrow x$, then $d(A_x, f(x)) \leqslant \liminf_{n\longrightarrow\infty} d(A_{x_n}, f(x_n))$.

For a positive real number $0 < b < 1$, we define the set $\Gamma_b^x \subset X$ as follows:

$$\Gamma_b^x = \{z \in \mathcal{Z}^x | bH(A_x, f(x) \bigcap A_z) \leqslant d(A_x, f(x))\}.$$

Thus $\Gamma_b^x \neq \emptyset$. Indeed, if we assume that for all $z \in \mathcal{Z}^x, d(A_x, f(x)) < bH(A_x, f(x) \bigcap A_z)$, it follows by ($\imath$) that $d(A_x, f(x)) \leqslant b \inf_{z\in\mathcal{Z}^x}\{H(A_x, f(x) \bigcap A_z)\} < d(A_x, f(x))$ which is a contradiction.

In the following, assume that for every $\epsilon > 0$ the set

$$A_\epsilon^x = \{y \in X | d(x,y) - \epsilon \leqslant d(A_x, f(x))\},$$

is nonempty.

Definition 1.2. Under the above notations, we say that $f : X \longrightarrow N(X)$ has a pseudo-fixed point if there exists $x_0 \in X$ such that $A_{x_0} \bigcap f(x_0) \neq \emptyset$.

Remark 1.3. Noting that Definition 1.2 is new and extends strictly the classical one in the case of single-valued mappings or multi-valued mappings at the same time, indeed, here x_0 is not necessarily an element of the set $A_{x_0} \bigcap f(x_0)$.

Remark 1.4. If $A_x = \{x\}$ for all $x \in X$, the precedent definition is reduced to the well-known definition of fixed points concerning multi-valued mappings.

Example 1.5. Let $X = [0, +\infty[$ and $f : [0, +\infty[\longrightarrow N([0, +\infty[)$ be a mapping defined by $f(x) = [0, \frac{x}{2}]$, we denote by $F_p(f)$ the set of its pseudo-fixed points. Thus

1. If $A_x = [\frac{x}{2}, x]$, then $F_p(f) = [0, +\infty[$ since $f(x) \bigcap A_x = \{\frac{x}{2}\}$ for all $x \in [0, +\infty[$.
2. If $A_x = \{x\}$, then $F_p(f) = \{0\}$ since $f(x) \bigcap A_x \neq \emptyset$ is just satisfied for $x = 0$.

2. Main results

Theorem 2.1. *Let (X,d) be a complete metric space and $f : X \longrightarrow C(X)$ a multi-valued mapping. Assume that the assumptions ($\imath$) and ($\imath\imath$) are satisfied. Let $0 < b < 1$. If there exists $0 < c < 1$ such that for all $x \in X$ and for all $\epsilon > 0$, there exists $y \in A_\epsilon^x \bigcap \Gamma_b^x$ satisfying that*

$$d(A_y, f(y)) \leqslant cH(A_x, A_y \bigcap f(x)),$$

then f has a pseudo-fixed point provided that $c < b$.

Proof. For an initial point $x_0 \in X$, there exists $x_1 \in A_1^{x_0} \bigcap \Gamma_b^{x_0}$ such that

$$d(A_{x_1}, f(x_1)) \leqslant cH(A_{x_0}, A_{x_1} \bigcap f(x_0)).$$

Thus, the fact that the set $A_{\frac{1}{2}}^{x_1}$ is nonempty and by assumptions, there exists $x_2 \in A_{\frac{1}{2}}^{x_1} \bigcap \Gamma_b^{x_1}$ for which we have the following inequalities

$$d(x_1, x_2) - \frac{1}{2} \leqslant d(A_{x_1}, f(x_1)) \leqslant cH(A_{x_0}, A_{x_1} \bigcap f(x_0)) \leqslant \frac{c}{b} d(A_{x_0}, f(x_0)),$$

and consequently by induction on k, we can deduce that there exists $x_{k+1} \in A_{\frac{1}{2^k}}^{x_k} \bigcap \Gamma_b^{x_k}$ such that

$$d(x_k, x_{k+1}) - \frac{1}{2^k} \leqslant d(A_{x_k}, f(x_k)) \leqslant cH(A_{x_{k-1}}, A_{x_k} \bigcap f(x_{k-1})) \leqslant \frac{c^k}{b^k} d(A_{x_0}, f(x_0)).$$

It follows that

$$\sum_{k=1}^{\infty} d(x_k, x_{k+1}) \leqslant 1 + (\sum_{k=1}^{\infty} \frac{c^k}{b^k}) d(A_{x_0}, f(x_0)).$$

Hence the series $\sum_{k=1}^{\infty} d(x_k, x_{k+1})$ is convergent and consequently by Cauchy's criterion, the sequence $\{x_n\}_n$ is a Cauchy sequence in X. The completeness of X implies the existence of $\overline{x} \in X$ such that $x_n \longrightarrow \overline{x}$. Moreover, it is easy to observe that $\lim_{k \longrightarrow \infty} d(A_{x_k}, f(x_k)) = 0$. Then by $(\mathfrak{u})$, we get that

$$d(A_{\overline{x}}, f(\overline{x})) \leqslant \liminf_{k \longrightarrow \infty} d(A_{x_k}, f(x_k)) = 0.$$

Hence $d(A_{\overline{x}}, f(\overline{x})) = 0$. The fact that $f(\overline{x})$ is closed and $A_{\overline{x}}$ is compact implies that $A_{\overline{x}} \bigcap f(\overline{x}) \neq \emptyset$ which gives the result. □

Corollary 2.2 (see [2, Theorem 3.1]). *Let* (X, d) *be a complete metric space,* $f : X \longrightarrow C(X)$ *a multi-valued mapping. Let* $0 < b < 1$*. If there exists* $0 < c < 1$ *such that for any* $x \in X$*, there exists* $y \in \Gamma_b^x$ *satisfying*

$$d(y, f(y)) \leqslant cd(x, y),$$

and the function $h(x) = d(x, f(x))$ *is lower semi-continuous. Then there exists* $x_0 \in X$ *such that* $x_0 \in f(x_0)$ *provided that* $c < b$.

Proof. This result follows from Theorem 2.1 by taking $A_x = \{x\}$ for all $x \in X$. Indeed, in this case we have $\mathcal{Z}^x = f(x)$ and $H(A_x, A_y \bigcap f(x)) = d(x, y)$ for any $x, y \in X$. Moreover, we have $A_\epsilon^x \neq \emptyset$ for all $x \in X$ and $\epsilon > 0$ by the definition of the infimum and $A_\epsilon^x \subseteq \Gamma_b^x$ $(\epsilon \leqslant \frac{1-b}{b} d(x, f(x)))$. On the other hand, the fact that the family $A_x = \{x\}, x \in X$ satisfies $(\mathfrak{u})$ follows from the lower semi-continuity of the function $h(x) = d(x, f(x))$ [2]. □

Corollary 2.3 (see [2, Theorem 3.1]). *Let* (X, d) *be a complete metric space,* $f : X \longrightarrow C(X)$ *a multi-valued mapping. If there exists* $0 < c < 1$ *such that for any* $x, y \in X$*, we have*

$$H(f(x), f(y)) \leqslant cd(x, y).$$

Then there exists $x_0 \in X$ *such that* $x_0 \in f(x_0)$.

Proof. By [2, Remark 1(2)], this result follows directly from Corollary 2.2. Hence it can be seen as a particular case of our general framework given by Theorem 2.1. □

Example 2.4. Let $X = \{\frac{1}{2}, \frac{1}{4}, \cdots\} \cup \{0,1\}$, $d(x,y) = |x-y|$, for $x, y \in X$. It is easy to observe that (X,d) is a complete metric space. Let $f : X \longrightarrow C(X)$ defined as follows:

$$f(x) = \begin{cases} \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\}, & x = \frac{1}{2^n}, \quad n = 0,1,2,\cdots, \\ \{\frac{1}{2}\}, & x = 0, \end{cases}$$

and let $A_x = \{x\}$ for any $x \in X$. It is easy to observe that $A_x \cap f(x) = \emptyset$ for any $x \in X$. Hence necessarily f is not a contraction mapping. This fact can be deduced by observing that f is not continuous at 0.

2.1. Extension to the Hardy-Rogers mappings

We start this subsection by giving the following main result which can be regarded as an extension of Hardy-Rogers result and consequently Nadler's result.

Theorem 2.5. *Let (X,d) be a complete metric space, $f : X \longrightarrow CB(X)$ a multi-valued mapping. Let*

$$\{A_x, x \in X\} \subset K(X),$$

be a family of compact subsets of X such that $x \in A_x$ for all $x \in X$ satisfying that:

(ı) *$\{A_x, x \in X\}$ is closed in $(CB(X), H)$.*

(ıı) *If $A, B \in CB(X)$, then for all A_x such that $A_x \bigcap A \neq \emptyset$ and for all $\epsilon > 0$, there exists $A_y \subset B$ such that*

$$H(A_x, A_y) \leqslant H(A,B) + \epsilon.$$

(ııı) *If $A_{x_n} \longrightarrow A_x$ in $(CB(X), H)$, then there exists $\mu > 0$ such that $d(A_x, f(x)) \leqslant \mu \liminf_{n\longrightarrow\infty} d(A_{x_n}, f(x_n))$.*

If f satisfies that

$$\begin{aligned} H(f(x), f(y)) \leqslant\ & \alpha_1 H(A_x, A_y) + \alpha_2(d(A_x, f(x)) + d(A_y, f(y))) \\ & + \alpha_3(d(A_x, f(y)) + d(A_y, f(x))), \end{aligned} \tag{2.1}$$

with $\alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$, then there exists $z \in X$ such that z is a pseudo-fixed point for f.

Proof. Let $x_0 \in X$ and A_{x_1} such that $A_{x_1} \bigcap f(x_0) \neq \emptyset$. We define $\gamma = \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - (\alpha_2 + \alpha_3)}$. If $\gamma = 0$, then the result is trivial. Now, assume that $\gamma > 0$, thus $0 < \gamma < 1$. Following the assumption (ıı), there exists A_{x_2} such that $A_{x_2} \subseteq f(x_1)$ with

$$H(A_{x_1}, A_{x_2}) \leqslant H(f(x_0), f(x_1)) + \gamma.$$

By (ıı), we obtain a family of compact subsets $\{A_{x_n}\}_n$ such that $A_{x_n} \subseteq f(x_{n-1})$ and

$$H(A_{x_n}, A_{x_{n+1}}) \leqslant H(f(x_{n-1}), f(x_n)) + \gamma^n.$$

Hence, we have

$$\begin{aligned} H(A_{x_n}, A_{x_{n+1}}) &\leqslant H(f(x_{n-1}), f(x_n)) + \gamma^n \\ &\leqslant \alpha_1 H(A_{x_{n-1}}, A_{x_n}) + \alpha_2(d(A_{x_n}, f(x_n)) + d(A_{x_{n-1}}, f(x_{n-1}))) \\ &\quad + \alpha_3(d(A_{x_n}, f(x_{n-1})) + d(A_{x_{n-1}}, f(x_n))) + \gamma^n, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Also by (ıı), we get

$$\begin{aligned} H(A_{x_n}, A_{x_{n+1}}) &\leqslant \alpha_1 H(A_{x_{n-1}}, A_{x_n}) + \alpha_2(H(A_{x_n}, A_{x_{n+1}}) + H(A_{x_{n-1}}, A_{x_n})) \\ &\quad + \alpha_3(H(A_{x_n}, A_{x_n}) + H(A_{x_{n-1}}, A_{x_{n+1}})) + \gamma^n, \qquad \forall n \in \mathbb{N} \\ &\leqslant \alpha_1 H(A_{x_{n-1}}, A_{x_n}) + \alpha_2(H(A_{x_n}, A_{x_{n+1}}) + H(A_{x_{n-1}}, A_{x_n})) \\ &\quad + \alpha_3(H(A_{x_{n-1}}, A_{x_n}) + H(A_{x_n}, A_{x_{n+1}})) + \gamma^n, \qquad \forall n \in \mathbb{N}. \end{aligned}$$

Consequently,

$$H(A_{x_n}, A_{x_{n+1}}) \leqslant \gamma H(A_{x_{n-1}}, A_{x_n}) + \frac{\gamma^n}{1-(\alpha_2+\alpha_3)}, \quad \forall n \in \mathbb{N}.$$

Thus,

$$H(A_{x_n}, A_{x_{n+1}}) \leqslant \gamma^n H(A_{x_0}, A_{x_1}) + \frac{n\gamma^n}{1-(\alpha_2+\alpha_3)}, \quad \forall n \in \mathbb{N}.$$

Since $\gamma < 1$, then $\sum_{n=1}^{\infty} H(A_{x_n}, A_{x_{n+1}}) < \infty$. This shows that the sequence $(A_{x_n})_n$ is a Cauchy sequence in $(CB(X), H)$. The completeness of $(CB(X), H)$ (see [5]) and the closeness of the set $\{A_x, x \in X\}$ implies that there exists $\overline{x} \in X$ such that $A_{x_n} \longrightarrow A_{\overline{x}}$ in $(CB(X), H)$. To prove that $A_{\overline{x}} \bigcap f(\overline{x}) \neq \emptyset$, by the assertion $(\iota\iota\iota)$, it suffices to get that $\liminf_{n\longrightarrow\infty} d(A_{x_n}, f(x_n)) = 0$. We have

$$d(A_{x_n}, f(x_n)) \leqslant H(A_{x_n}, A_{x_{n+1}}), \quad \forall n \in \mathbb{N},$$

which gives that

$$\liminf_{n\longrightarrow\infty} d(A_{x_n}, f(x_n)) \leqslant \liminf_{n\longrightarrow\infty} H(A_{x_n}, A_{x_{n+1}}) = 0.$$

By the condition $(\iota\iota\iota)$ and the fact that $A_{x_n} \longrightarrow A_{\overline{x}}$ in $(CB(X), H)$, it follows that $d(A_{\overline{x}}, f(\overline{x})) = 0$ which implies that $A_{\overline{x}} \bigcap f(\overline{x}) \neq \emptyset$ and achieves the result. □

Proposition 2.6. Let (X, d) be a complete metric space and let $f : X \longrightarrow CB(X)$ a multi-valued mapping with the following condition:

$$H(f(x), f(y)) \leqslant \alpha_1(d(x, f(x)) + d(y, f(y))) + \alpha_2(d(x, f(y)) + d(y, f(x))) + \alpha_3 d(x, y), \quad \forall x, y \in X,$$

with $2\alpha_1 + 2\alpha_2 + \alpha_3 < 1$. Then, there exists $z \in X$ such that $z \in f(z)$.

Proof. This result follows immediately from Theorem 2.5 by taking $A_x = \{x\}$ for all $x \in X$. In this case, it is easy to check that assumption (ι) is satisfied, for the assumption $(\iota\iota)$, (see [6]) while the assumption $(\iota\iota\iota)$ is deduced from the following:

If $x_n \longrightarrow x$ then we have for all $n \in \mathbb{N}$

$$\begin{aligned} d(x, f(x)) &\leqslant d(x, x_n) + d(x_n, f(x_n)) + H(f(x_n), f(x)) \\ &\leqslant d(x, x_n) + d(x_n, f(x_n)) + \alpha_1 d(x_n, f(x_n)) + \alpha_1 d(x, f(x)) \\ &\quad + \alpha_2 d(x_n, f(x)) + \alpha_2 d(x, f(x_n)) + \alpha_3 d(x, x_n) \\ &\leqslant d(x, x_n) + (1+\alpha_1) d(x_n, f(x_n)) + \alpha_1 d(x, f(x)) \\ &\quad + \alpha_2 d(x_n, x) + \alpha_2 d(x, f(x)) + \alpha_2 d(x, x_n) + \alpha_2 d(x_n, f(x_n)) + \alpha_3 d(x, x_n). \end{aligned}$$

Hence

$$(1-(\alpha_1+\alpha_2)) d(x, f(x)) \leqslant (1 + 2\alpha_2 + \alpha_3) d(x, x_n) + (1 + \alpha_1 + \alpha_2) d(x_n, f(x_n)).$$

By taking the limit $n \longrightarrow +\infty$, we get

$$d(x, f(x)) \leqslant \frac{1+\alpha_1+\alpha_2}{1-(\alpha_1+\alpha_2)} \liminf_{n\longrightarrow\infty} d(x_n, f(x_n)).$$

Thus $(\iota\iota\iota)$ is satisfied by taking $\mu = \dfrac{1+\alpha_1+\alpha_2}{1-(\alpha_1+\alpha_2)}$. □

Corollary 2.7 (see [6]). *Let (X, d) be a complete metric space and let $f : X \longrightarrow CB(X)$ a multi-valued mapping satisfying that*

$$H(f(x), f(y)) \leqslant \alpha d(x, y),$$

for all $x, y \in X$, where $0 \leqslant \alpha < 1$. Then there exists $z \in X$ such that $z \in f(z)$.

Proof. The result follows from Proposition 2.6 by taking $\alpha_1 = \alpha_2 = 0$. □

By the same principle, we can also derive the following results:

Corollary 2.8 (see [7, 8]). *Let (X, d) be a complete metric space and let f be a mapping from (X, d) into $(CB(X), H)$ satisfying*

$$H(f(x), f(y)) \leqslant \alpha(d(x, f(x)) + d(y, f(y)))$$

for all $x, y \in X$, where $0 \leqslant \alpha < \frac{1}{2}$. Then there exists $z \in X$ such that $z \in f(z)$.

Corollary 2.9. *Let (X, d) be a complete metric space and let f be a mapping from (X, d) into $(CB(X), H)$ satisfying*

$$H(f(x), f(y)) \leqslant \alpha(d(x, f(y)) + d(y, f(x)))$$

for all $x, y \in X$, where $0 \leqslant \alpha < \frac{1}{2}$. Then there exists $z \in X$ such that $z \in f(z)$.

Corollary 2.10. *Let (X, d) be a complete metric space and let f be a mapping from (X, d) into $(CB(X), H)$ satisfying*

$$H(f(x), f(y)) \leqslant \alpha_1 d(x, y) + \alpha_2(d(x, f(x)) + d(y, f(y)))$$

for all $x, y \in X$, where $\alpha_1 + 2\alpha_2 < 1$. Then there exists $z \in X$ such that $z \in f(z)$.

For any nonempty subsets A, B of X, we define

$$\delta(A, B) = \sup\{d(a, b) | a \in A, b \in B\}.$$

In the following, we denote by $g : X \longrightarrow CB(X)$ for which $g(x) = A_y$ ($A_y \subseteq f(x)$ given in the assumption $(\imath\imath)$ of Theorem 2.5). As an extension of Reich and Iseki results (see [3, 8]), we have:

Lemma 2.11. *Let (X, d) be a complete metric space. If $f : X \longrightarrow CB(X)$ is a multi-valued function for which the assumptions $(\imath), (\imath\imath)$ and $(\imath\imath\imath)$ of Theorem 2.5 are satisfied and*

$$\begin{aligned}\delta(f(x), f(y)) \leqslant\ & \alpha_1(d(A_x, f(x)) + d(A_y, f(y))) + \alpha_2(d(A_x, f(y)) + d(A_y, f(x))) \\ & + \alpha_3 H(A_x, A_y)\end{aligned}$$

for every x, y in X, where $\alpha_1, \alpha_2, \alpha_3$ are nonnegative and $2\alpha_1 + 4\alpha_2 + \alpha_3 < 1$, then $g : X \longrightarrow CB(X)$ satisfies a contraction condition of the form (2.1).

Proof. By the definition of g, since $g(x) \subseteq f(x)$ and $g(y) \subseteq f(y)$, we have

$$\begin{aligned}H(g(x), g(y)) &\leqslant \delta(f(x), f(y)) \\ &\leqslant \alpha_1(d(A_x, f(x)) + d(A_y, f(y))) + \alpha_2(d(A_x, f(y)) + d(A_y, f(x))) + \alpha_3 H(A_x, A_y) \\ &\leqslant \alpha_1(d(A_x, g(x)) + d(A_y, g(y))) + \alpha_2(2d(A_x, A_y) + d(A_x, g(x)) + d(A_y, g(y))) \\ &\quad + \alpha_3 H(A_x, A_y) \\ &\leqslant (\alpha_1 + \alpha_2)(d(A_x, g(x)) + d(A_y, g(y))) + (2\alpha_2 + \alpha_3) H(A_x, A_y).\end{aligned}$$

The fact that $2\alpha_1 + 4\alpha_2 + \alpha_3 < 1$ gives the result.

□

Theorem 2.12. *Let (X, d) be a complete metric space. Under the assumptions of Lemma 2.11, if the mapping g satisfies $(\imath\imath\imath)$ of Theorem 2.5, then there exists $\overline{x} \in X$ such that $\overline{x}$ is a pseudo-fixed point for f.*

Proof. By taking account of Theorem 2.5, the mapping g has a pseudo-fixed point $\overline{x} \in X$, then $A_{\overline{x}} \bigcap g(\overline{x}) \neq \emptyset$. But by definition, we have $g(\overline{x}) \subseteq f(\overline{x})$, hence we obtain that $A_{\overline{x}} \bigcap f(\overline{x}) \neq \emptyset$ which gives the result. □

Remark 2.13. Noting that the uniqueness of fixed points is not ensured in general in the case of multi-valued contraction or generalized contraction mappings, however we have the following result of existence and uniqueness.

Corollary 2.14. *Let* (X, d) *be a complete metric space. If* $f : X \longrightarrow CB(X)$ *is a multi-valued mapping satisfying that*

$$\delta(f(x), f(y)) \leqslant \alpha_1(d(x, f(x)) + d(y, f(y))) + \alpha_2(d(x, f(y)) + d(y, f(x))) + \alpha_3 d(x, y),$$

for every x, y *in* X*, where* $\alpha_1, \alpha_2, \alpha_3$ *are nonnegative and* $2\alpha_1 + 4\alpha_2 + \alpha_3 < 1$*, then there exists a unique point* $\overline{x} \in X$ *such that* $f(\overline{x}) = \{\overline{x}\}$*.*

Proof. The existence can be deduced from Theorem 2.12 by taking $A_x = \{x\}, x \in X$ and the properties of the single-valued mapping $g : X \longrightarrow CB(X)$ in this case. For the uniqueness see [3]. □

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