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Inequalities for new class of fractional integral operators

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Abstract

The applications of fractional order integrals have promoted the study of inequalities. In this paper, we utilize recently introduced left- and right-fractional conformable integrals (FCI) for a class of decreasing n positive functions such that $n \in \mathbb{N}$, for the generalization of existing integral inequalities. Our results have the potentials to be used for the investigation of positive solutions of different classes of fractional differential equations.

Keywords: Fractional integral inequalities, left-fractional conformable integral, right-fractional conformable integral.
2010 MSC: 26D10, 26A33.

1. Introduction and preliminaries

The applications of fractional calculus have attracted more and more attention of scientists in different application fields in last two decades including control theory, viscoelastic theory, fluid dynamics, image processing, biology, hydrodynamics, signals, computer networking, and many others [4–13, 17, 18, 22, 28, 31, 33, 34].

Recently, several authors have worked on the generalization of existing inequalities through different ways. One of the most popular ways is the use of fractional order integrals. For example, Agarwal et al. [2] produced Hermite-Hadamard type inequalities by considering the generalized k-fractional integrals. Set et al. [25] obtained an integral identity for the generalized fractional integral operators and with help of the identity. They proved some Hermite-Hadamard type for a class of functions whose absolute values of derivatives are convex. Their work generalized Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral. Sarikaya and Budak [24] derived a generalized inequality for local fractional integrals.

In existing literature, there are numerous applications of inequalities in the applied sciences, for instance see [3, 21, 23, 32]. The applications of inequalities in the mathematics are widely investigated. One

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of the recently highly attracted areas is the existence of non-trivial solutions of differential equations via inequalities. For example, Jleli et al. [15] considered a fractional differential equation (FDE) involving fractional derivative with respect to another function and established a Hartman-Winter-type inequality. There are several results for the existence of non-trivial solution of eigenvalue problem by using the inequalities, see [23, 32].

In this paper we utilize the newly introduced fractional integral operators that are left- and right-FCI introduced in [14] for the generalization of integral inequalities for a class of n decreasing positive functions. Our new inequalities generalize the work given in [21] and many others in the previous literature. For the details about the work related to the inequalities, applications, and stabilities, we refer the readers to [16, 19, 20, 27].

Abdeljawad [1] introduced the notion of left and right conformable derivatives for a differentiable function f whose are expressed as below

$$ {}_a\mathcal{T}^{\alpha}f(\tau)=(\tau-a)^{1-\alpha}f'(\tau),\ \mathcal{T}_b^{\alpha}f(\tau)=(b-\tau)^{1-\alpha}f'(\tau), $$

and the corresponding left and right integrals for $0<\alpha<1$, by

$$ {}_a\mathcal{I}^{\alpha}f(\tau)=\int_a^{\tau}f(x)\frac{dx}{(x-a)^{1-\alpha}},\quad \mathcal{I}_b^{\alpha}f(\tau)=\int_{\tau}^{b}f(x)\frac{dx}{(x-a)^{1-\alpha}}. $$

Definition 1.1 ([14])**.** Left-FCI operator for any $\beta\in\mathbb{C}$, $\mathrm{Re}(\beta)>0$, is defined as

$$ {}_a^{\beta}\mathfrak{J}^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}f(t)\frac{dt}{(t-a)^{1-\alpha}}. $$

Definition 1.2 ([14])**.** Right-FCI operator for any $\beta\in\mathbb{C}$, $\mathrm{Re}(\beta)>0$, is defined as

$$ {}^{\beta}\mathfrak{J}_b^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}f(t)\frac{dt}{(b-t)^{1-\alpha}}, $$

where the Gamma function is defined for β as

$$ \Gamma(\beta)=\int_0^{+\infty}e^{-s}s^{\beta-1}ds. $$

It is worthy to note that Riemann-Liouville, Hadamad, and generalized fractional integrals are special cases of Definitions 1.1 and 1.2; see in [14].

Integral inequalities have been studied for different purposes. One of the most useful applications of the integral inequalities is the existence of non-trivial solutions of FDEs. In literature, there are applications of several fractional integral operators in the generalizations of pre-existing inequalities. For instance one can see and apply the integral inequalities given in [26, 29, 30].

2. Main results

Theorem 2.1. *Let $(f_j)_{j=1,\dots,n}$ be n for finite $n\in\{1,2,\dots\}$ positive continuous functions decreasing on the interval $[a,b]$. Let $a<t\leqslant b$, $\delta>0$, $\xi\geqslant\gamma_p>0$ for any fixed $p\in\{1,\dots,n\}$. Then for left-FCI operator ${}_a^{\beta}\mathfrak{J}^{\alpha}$, the following inequality holds true*

$$ \frac{{}_a^{\beta}\mathfrak{J}^{\alpha}\left[\prod_{j=1}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\right]}{{}_a^{\beta}\mathfrak{J}^{\alpha}\left[\prod_{j=1}^{n}f_j^{\gamma_j}(x)\right]}\geqslant\frac{{}_a^{\beta}\mathfrak{J}^{\alpha}\left[(x-a)^{\delta}\prod_{j=1}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\right]}{{}_a^{\beta}\mathfrak{J}^{\alpha}\left[(x-a)^{\delta}\prod_{j=1}^{n}f_j^{\gamma_j}(x)\right]}. \tag{2.1} $$

Proof. It is clear that for decreasing positive and continuous functions $(f_j)_{j=1,\dots,n}$ and $a<t\leqslant b$, $\delta>0$, $\xi\geqslant\gamma_p>0$ for any fixed $p\in\{1,\dots,n\}$, we have

$$ \left((\rho-a)^{\delta}-(\tau-a)^{\delta}\right)\left(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\right)\geqslant 0. $$

Define a function

$$
\begin{aligned}
{}^{\beta}_{a}\mathcal{T}^{\alpha}(x,\rho,\tau) = {} & \frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\
& \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\tau)}{(\tau-a)^{1-\alpha}}\Big((\rho-a)^{\delta}-(\tau-a)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big).
\end{aligned} \tag{2.2}
$$

With the assumptions in Theorem 2.1, the function ${}^{\beta}_{a}\mathcal{T}^{\alpha}(x,\rho,\tau)$ is positive for all $\tau \in (a,b]$. Integrating both sides of (2.2), with respect to τ from a to x, we have

$$
\begin{aligned}
0 \leqslant \int_a^x {}^{\beta}_{a}\mathcal{T}^{\alpha}(x,\rho,\tau)\,d\tau = {} & \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\
& \times \prod_{j=1}^{n} f_j^{\gamma_j}(\tau)\Big((\rho-a)^{\delta}-(\tau-a)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big)\frac{d\tau}{(\tau-a)^{1-\alpha}} \\
= {} & \Big[(\rho-a)^{\delta}\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] + f_p^{\xi-\gamma_p}(\tau)\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[(\tau-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big] \\
& -(\rho-a)^{\delta} f_p^{\xi-\gamma_p}(\rho)\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big] - {}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[(\tau-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big].
\end{aligned} \tag{2.3}
$$

Multiplying (2.3) by $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\right)^{\beta-1} \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\rho)}{(\rho-a)^{1-\alpha}}$ and integrating with respect to ρ from a to x implies

$$
0 \leqslant {}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big] - {}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[(x-a)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big].
$$

Dividing both sides by ${}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}\Big]\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]$, we get (2.1). □

Theorem 2.2. *Let* $(f_j)_{j=1,\dots,n}$ *be* n *for finite* $n \in \{1,2,\dots\}$ *positive continuous functions decreasing on the interval* $[a,b]$. *Let* $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ *for any fixed* $p \in \{1,\dots,n\}$. *Then for left-FCI operator* ${}^{\beta}_{a}\mathcal{J}^{\alpha}$, *the following inequality holds true*

$$
\frac{{}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] + {}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]}{{}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[(x-a)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]\,{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] + {}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]\,{}^{\beta}_{a}\mathcal{J}^{\beta}\Big[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]} \geqslant 1. \tag{2.4}
$$

Proof. It is clear that for decreasing positive and continuous functions $(f_j)_{j=1,\dots,n}$ and $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$, we have

$$
\Big((\rho-a)^{\delta}-(\tau-a)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big) \geqslant 0.
$$

Define a function

$$
\begin{aligned}
{}^{\beta}_{a}\mathcal{T}^{\alpha}(x,\rho,\tau) = {} & \frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\
& \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\tau)}{(\tau-a)^{1-\alpha}}\Big((\rho-a)^{\delta}-(\tau-a)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big).
\end{aligned} \tag{2.5}
$$

With the assumptions in Theorem 2.2, the function ${}_{a}^{\beta}\mathcal{T}^{\alpha}(x,\rho,\tau)$ is positive for all $\tau \in (a,b]$. Integrating both sides of (2.5), with respect to τ from a to x, we have

$$
\begin{aligned}
0 \leqslant \int_a^x {}_{a}^{\beta}\mathcal{T}^{\alpha}(x,\rho,\tau)d\tau &= \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\
&\quad \times \prod_{j=1}^{n} f_j^{\gamma_j}(\tau)\Big((\rho-a)^{\delta}-(\tau-a)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big)\frac{d\tau}{(\tau-a)^{1-\alpha}} \\
&= \left[(\rho-a)^{\delta}\,{}_{a}^{\beta}\mathcal{J}^{\alpha}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + f_p^{\xi-\gamma_p}(\tau)\,{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[(\tau-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] \\
&\quad -(\rho-a)^{\delta} f_p^{\xi-\gamma_p}(\rho)\,{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] - {}_{a}^{\beta}\mathcal{J}^{\alpha}\left[(\tau-a)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right].
\end{aligned}
\tag{2.6}
$$

Multiplying (2.6) by $\frac{1}{\Gamma(\theta)}\left(\frac{(x-a)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\right)^{\theta-1} \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\rho)}{(\rho-a)^{1-\alpha}}$ and integrating with respect to ρ from a to x implies

$$
\begin{aligned}
0 \leqslant {}_{a}^{\theta}\mathcal{J}^{\alpha}&\left[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + {}_{a}^{\theta}\mathcal{J}^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] \\
-\,{}_{a}^{\theta}\mathcal{J}^{\alpha}&\left[(x-a)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] - {}_{a}^{\theta}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]{}_{a}^{\beta}\mathcal{J}^{\beta}\left[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right].
\end{aligned}
\tag{2.7}
$$

Dividing both sides of (2.7) by ${}_{a}^{\theta}\mathcal{J}^{\alpha}\left[(x-a)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + {}_{a}^{\theta}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]{}_{a}^{\beta}\mathcal{J}^{\beta}$ $\left[(x-a)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]$, we get (2.4). □

Theorem 2.3. *Let $(f_j)_{j=1,\dots,n}$ be n for finite $n \in \{1,2,\dots\}$ positive continuous functions decreasing on the interval $[a,b]$. Let $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$. Then for left-FCI operator ${}_{a}^{\beta}\mathcal{J}^{\alpha}$, the following inequality holds true*

$$
\frac{{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]}{{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]} \geqslant \frac{{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]}{{}_{a}^{\beta}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]}. \tag{2.8}
$$

Proof. It is clear that for decreasing positive and continuous functions $(f_j)_{j=1,\dots,n}$ and $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$, we have

$$
\Big(h^{\delta}(\rho)-h^{\delta}(\tau)\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big) \geqslant 0.
$$

Define a function

$$
\begin{aligned}
{}_{a}^{\beta}\mathcal{T}^{\alpha}(x,\rho,\tau) &= \frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\
&\quad \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\tau)}{(\tau-a)^{\alpha}}\Big(h^{\delta}(\rho)-h^{\delta}(\tau)\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big).
\end{aligned}
\tag{2.9}
$$

With the assumptions in Theorem 2.3, the function ${}_{a}^{\beta}\mathcal{T}^{\alpha}(x,\rho,\tau)$ is positive for all $\tau \in (a,b]$. Integrating both sides of (2.9), with respect to τ from a to x, we have

$$\begin{aligned} 0 \leqslant \int_a^x {}^{\beta}_{a}\mathcal{T}^{\alpha}(x,\rho,\tau)d\tau &= \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\ &\quad \times \prod_{j=1}^{n} f_j^{\gamma_j}(\tau)\Big(h^{\delta}(\rho)-h^{\delta}(\tau)\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big)\frac{d\tau}{(\tau-a)^{1-\alpha}} \\ &= \left[h^{\delta}(\rho){}^{\beta}_{a}\mathcal{J}^{\alpha}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + f_p^{\xi-\gamma_p}(\tau){}^{\beta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(\tau)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] \\ &\quad - h^{\delta}(\rho)f_p^{\xi-\gamma_p}(\rho){}^{\beta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] - {}^{\beta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]. \end{aligned} \tag{2.10}$$

Multiplying (2.10) by $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\right)^{\beta-1} \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\rho)}{(\rho-a)^{1-\alpha}}$ and integrating with respect to ρ from a to x implies

$$0 \leqslant {}^{\beta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}^{\beta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] - {}^{\beta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}^{\beta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right].$$

Dividing both sides by ${}^{\beta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}\right]{}^{\beta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]$, we get (2.8). □

Theorem 2.4. *Let $(f_j)_{j=1,\dots,n}$ be n for finite $n \in \{1,2,\dots\}$ positive continuous functions decreasing on the interval $[a,b]$. Let $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$. Then for left-FCI operator ${}^{\beta}_{a}\mathcal{J}^{\alpha}$, the following inequality holds true*

$$\frac{{}^{\theta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]{}^{\beta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + {}^{\theta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}^{\beta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]}{{}^{\theta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}^{\beta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + {}^{\theta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]{}^{\beta}_{a}\mathcal{J}^{\beta}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]} \geqslant 1. \tag{2.11}$$

Proof. It is clear that for decreasing positive and continuous functions $(f_j)_{j=1,\dots,n}$ and $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$, we have

$$\Big(h^{\delta}(\rho)-h^{\delta}(\tau)\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big) \geqslant 0.$$

Define a function

$$\begin{aligned} {}^{\beta}_{a}\mathcal{T}^{\alpha}(x,\rho,\tau) &= \frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\ &\quad \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\tau)}{(\tau-a)^{1-\alpha}}\Big(h^{\delta}(\rho)-h^{\delta}(\tau)\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big). \end{aligned} \tag{2.12}$$

With the assumptions in Theorem 2.4, the function ${}^{\beta}_{a}\mathcal{T}^{\alpha}(x,\rho,\tau)$ is positive for all $\tau \in (a,b]$. Integrating both sides of (2.12), with respect to τ from a to x, we have

$$\begin{aligned} 0 \leqslant \int_a^x {}^{\beta}_{a}\mathcal{T}^{\alpha}(x,\rho,\tau)d\tau &= \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\ &\quad \times \prod_{j=1}^{n} f_j^{\gamma_j}(\tau)\Big(h^{\delta}(\rho)-h^{\delta}(\tau)\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big)\frac{d\tau}{(\tau-a)^{1-\alpha}} \\ &= \left[h^{\delta}(\rho){}^{\beta}_{a}\mathcal{J}^{\alpha}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + f_p^{\xi-\gamma_p}(\tau){}^{\beta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(\tau)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] \\ &\quad - h^{\delta}(\rho)f_p^{\xi-\gamma_p}(\rho){}^{\beta}_{a}\mathcal{J}^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] - {}^{\beta}_{a}\mathcal{J}^{\alpha}\left[h^{\delta}(x)\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]. \end{aligned} \tag{2.13}$$

Multiplying (2.13) by $\frac{1}{\Gamma(\theta)}\left(\frac{(x-a)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\right)^{\theta-1}\times\frac{\prod_{j=1}^{n}f_j^{\gamma_j}(\rho)}{(\rho-a)^{1-\alpha}}$ and integrating with respect to ρ from a to x implies

$$\begin{aligned}0\leqslant{}& {}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[h^{\delta}(x)\prod_{j=1}^{n}f_j^{\gamma_j}(x)\Big]{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j\neq p}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]+{}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j\neq p}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[h^{\delta}(x)\prod_{j=1}^{n}f_j^{\gamma_j}(x)\Big]\\&-{}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[h^{\delta}(x)\prod_{j\neq p}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j\neq p}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]-{}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j=1}^{n}f_j^{\gamma_j}(x)\Big]{}^{\beta}_{a}\mathcal{J}^{\beta}\Big[h^{\delta}(x)\prod_{j=1}^{n}f_j^{\gamma_j}(x)\Big].\end{aligned}\tag{2.14}$$

Dividing both sides of (2.14) by ${}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[h^{\delta}(x)\prod_{j\neq p}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]{}^{\beta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j\neq p}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]+{}^{\theta}_{a}\mathcal{J}^{\alpha}\Big[\prod_{j=1}^{n}f_j^{\gamma_j}(x)\Big]{}^{\beta}_{a}\mathcal{J}^{\beta}\Big[h^{\delta}(x)\prod_{j=1}^{n}f_j^{\gamma_j}(x)\Big]$, we get (2.11). □

Now using the right-FCI operator we get following theorem.

Theorem 2.5. *Let* $(f_j)_{j=1,\dots,n}$ *be* n *for finite* $n\in\{1,2,\dots\}$ *positive continuous functions decreasing on the interval* $[a,b]$*. Let* $a<t\leqslant b$, $\delta>0$, $\xi\geqslant\gamma_p>0$ *for any fixed* $p\in\{1,\dots,n\}$*. Then for right-FCI operator* ${}^{\beta}\mathcal{J}^{\alpha}_{b}$*, the following inequality holds true*

$$\frac{{}^{\beta}\mathcal{J}^{\alpha}_{b}\Big[\prod_{j=1}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]}{{}^{\beta}\mathcal{J}^{\alpha}_{b}\Big[\Pi_{j=1}^{n}f_j^{\gamma_j}(x)\Big]}\geqslant\frac{{}^{\beta}\mathcal{J}^{\alpha}_{b}\Big[(b-x)^{\delta}\prod_{j=1}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]}{{}^{\beta}\mathcal{J}^{\alpha}_{b}\Big[(b-x)^{\delta}\Pi_{j=1}^{n}f_j^{\gamma_j}(x)\Big]}.\tag{2.15}$$

Proof. It is clear that for decreasing positive and continuous functions $(f_j)_{j=1,\dots,n}$ and $a<t\leqslant b$, $\delta>0$, $\xi\geqslant\gamma_p>0$ for any fixed $p\in\{1,\dots,n\}$, we have

$$\Big((b-\rho)^{\delta}-(b-\tau)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big)\geqslant 0.$$

Define a function

$$\begin{aligned}{}^{\beta}\mathcal{T}^{\alpha}_{b}(x,\rho,\tau)={}&\frac{1}{\Gamma(\beta)}\Big(\frac{(b-x)^{\alpha}-(b-\tau)^{\alpha}}{\alpha}\Big)^{\beta-1}\\&\times\frac{\prod_{j=1}^{n}f_j^{\gamma_j}(\tau)}{(b-\tau)^{1-\alpha}}\Big((b-\rho)^{\delta}-(b-\tau)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big).\end{aligned}\tag{2.16}$$

With the assumptions in Theorem 2.5, the function ${}^{\beta}\mathcal{T}^{\alpha}_{b}(x,\rho,\tau)$ is positive for all $\tau\in(a,b]$. Integrating both sides of (2.16), with respect to τ from x to b, we have

$$\begin{aligned}0\leqslant\int_x^b{}^{\beta}\mathcal{T}^{\alpha}_{b}(x,\rho,\tau)d\tau={}&\frac{1}{\Gamma(\beta)}\int_x^b\Big(\frac{(b-x)^{\alpha}-(b-\tau)^{\alpha}}{\alpha}\Big)^{\beta-1}\\&\times\prod_{j=1}^{n}f_j^{\gamma_j}(\tau)\Big((b-\rho)^{\delta}-(b-\tau)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau)-f_p^{\xi-\gamma_p}(\rho)\Big)\frac{d\tau}{(b-\tau)^{1-\alpha}}\\={}&\Big[(b-\rho)^{\delta}\,{}^{\beta}\mathcal{J}^{\alpha}_{b}\prod_{j\neq p}^{n}f_j^{\gamma_j}f_p^{\xi}(x)\Big]+f_p^{\xi-\gamma_p}(\tau)\,{}^{\beta}\mathcal{J}^{\alpha}_{b}\Big[(b-\tau)^{\delta}\prod_{j=1}^{n}f_j^{\gamma_j}(x)\Big]\\&-(b-\rho)^{\delta}f_p^{\xi-\gamma_p}(\rho)\,{}^{\beta}\mathcal{J}^{\alpha}_{b}\Big[\prod_{j=1}^{n}f_j^{\gamma_j}(\tau)\Big]-{}^{\beta}\mathcal{J}^{\alpha}_{b}\Big[(b-\tau)^{\delta}\prod_{j=1}^{n}f_j^{\gamma_j}(\tau)\Big].\end{aligned}\tag{2.17}$$

Multiplying (2.3) by $\frac{1}{\Gamma(\beta)}\left(\frac{(b-x)^{\alpha}-(\rho-a)^{1-\alpha}}{\alpha}\right)^{\beta-1}\times\frac{\prod_{j=1}^{n}f_j^{\gamma_j}(\rho)}{(\rho-a)^{\alpha}}$ and integrating with respect to ρ from x to b implies

$$0 \leqslant {}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big] - {}^{\beta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(\rho)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(\tau)\Big].$$

Dividing both sides by ${}^{\beta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]$, we get (2.15). □

Theorem 2.6. *Let $(f_j)_{j=1,\dots,n}$ be n for finite $n \in \{1,2,\dots\}$ positive continuous functions decreasing on the interval $[a,b]$. Let $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$. Then for right-FCI operator ${}^{\beta}\mathcal{J}_b^{\alpha}$, the following inequality holds true*

$$\frac{{}^{\theta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] + {}^{\theta}\mathcal{J}_b^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]}{{}^{\theta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] + {}^{\theta}\mathcal{J}_b^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]{}^{\beta}\mathcal{J}_b^{\beta}\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]} \geqslant 1. \tag{2.18}$$

Proof. It is clear that for decreasing positive and continuous functions $(f_j)_{j=1,\dots,n}$ and $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$, we have

$$\Big((b-\rho)^{\delta} - (b-\tau)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\Big) \geqslant 0.$$

Define a function

$$\begin{aligned} {}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho,\tau) &= \frac{1}{\Gamma(\beta)}\Big(\frac{(b-x)^{\alpha} - (b-\tau)^{\alpha}}{\alpha}\Big)^{\beta-1} \\ &\quad \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\tau)}{(b-\tau)^{1-\alpha}}\Big((b-\rho)^{\delta} - (b-\tau)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\Big). \end{aligned} \tag{2.19}$$

With the assumptions in Theorem 2.6, the function ${}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho,\tau)$ is positive for all $\tau \in (a,b]$. Integrating both sides of (2.19), with respect to τ from x to b, we have

$$\begin{aligned} 0 \leqslant \int_x^b {}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho,\tau)d\tau &= \frac{1}{\Gamma(\beta)}\int_x^b \Big(\frac{(b-x)^{\alpha} - (b-\tau)^{\alpha}}{\alpha}\Big)^{\beta-1} \\ &\quad \times \prod_{j=1}^{n} f_j^{\gamma_j}(\tau)\Big((b-\rho)^{\delta} - (b-\tau)^{\delta}\Big)\Big(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\Big)\frac{d\tau}{(b-\tau)^{1-\alpha}} \\ &= \Big[(b-\rho)^{\delta}\,{}^{\beta}\mathcal{J}_b^{\alpha}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] + f_p^{\xi-\gamma_p}(\tau)\,{}^{\beta}\mathcal{J}_b^{\alpha}\Big[(b-\tau)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big] \\ &\quad - (b-\rho)^{\delta} f_p^{\xi-\gamma_p}(\rho)\,{}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big] - {}^{\beta}\mathcal{J}_b^{\alpha}\Big[(b-\tau)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]. \end{aligned} \tag{2.20}$$

Multiplying (2.20) by $\frac{1}{\Gamma(\theta)}\Big(\frac{(b-x)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\Big)^{\theta-1} \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\rho)}{(\rho-a)^{1-\alpha}}$ and integrating with respect to ρ from x to b implies

$$\begin{aligned} 0 \leqslant\ & {}^{\theta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] + {}^{\theta}\mathcal{J}_b^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big] \\ & - {}^{\theta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] - {}^{\theta}\mathcal{J}_b^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]{}^{\beta}\mathcal{J}_b^{\beta}\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]. \end{aligned} \tag{2.21}$$

Dividing both sides of (2.21) by ${}^{\theta}\mathcal{J}_b^{\alpha}\Big[(b-x)^{\delta}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big]{}^{\beta}\mathcal{J}_b^{\alpha}\Big[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\Big] + {}^{\theta}\mathcal{J}_b^{\alpha}\Big[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]{}^{\beta}\mathcal{J}_b^{\beta}$ $\Big[(b-x)^{\delta}\prod_{j=1}^{n} f_j^{\gamma_j}(x)\Big]$, we get (2.18). □

Theorem 2.7. *Let $(f_j)_{j=1,\dots,n}$ be n for finite $n \in \{1,2,\dots\}$ positive continuous functions decreasing on the interval $[a,b]$. Let $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$. Then for right-FCI operator ${}^{\beta}\mathcal{J}_b^{\alpha}$, the following inequality holds true*

$$\frac{{}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]}{{}^{\beta}\mathcal{J}_b^{\alpha}\left[\Pi_{j=1}^{n} f_j^{\gamma_j}(x)\right]} \geqslant \frac{{}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]}{{}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\Pi_{j=1}^{n} f_j^{\gamma_j}(x)\right]}. \tag{2.22}$$

Proof. It is clear that for decreasing positive and continuous functions $(f_j)_{j=1,\dots,n}$ and $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots,n\}$, we have

$$\left(h^{\delta}(\rho) - h^{\delta}(\tau)\right)\left(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\right) \geqslant 0.$$

Define a function

$$\begin{aligned}{}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho,\tau) &= \frac{1}{\Gamma(\beta)}\left(\frac{(b-x)^{\alpha} - (b-\tau)^{\alpha}}{\alpha}\right)^{\beta-1} \\ &\quad\times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\tau)}{(b-\tau)^{1-\alpha}}\left(h^{\delta}(\rho) - h^{\delta}(\tau)\right)\left(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\right).\end{aligned} \tag{2.23}$$

With the assumptions in Theorem 2.7, the function ${}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho,\tau)$ is positive for all $\tau \in (a,b]$. Integrating both sides of (2.23), with respect to τ from x to b, we have

$$\begin{aligned}0 \leqslant \int_x^b {}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho,\tau)d\tau &= \frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha} - (b-\tau)^{\alpha}}{\alpha}\right)^{\beta-1} \\ &\quad\times \prod_{j=1}^{n} f_j^{\gamma_j}(\tau)\left(h^{\delta}(\rho) - h^{\delta}(\tau)\right)\left(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\right)\frac{d\tau}{(b-\tau)^{1-\alpha}} \\ &= \left[h^{\delta}(\rho){}^{\beta}\mathcal{J}_b^{\alpha}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + f_p^{\xi-\gamma_p}(\tau){}_a^{\beta}\mathcal{J}^{\alpha}\left[h^{\delta}(\tau)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] \\ &\quad - h^{\delta}(\rho) f_p^{\xi-\gamma_p}(\rho){}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] - {}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(\tau)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right].\end{aligned} \tag{2.24}$$

Multiplying (2.24) by $\frac{1}{\Gamma(\beta)}\left(\frac{(b-x)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\right)^{\beta-1} \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\rho)}{(\rho-a)^{1-\alpha}}$ and integrating with respect to ρ from x to b implies

$$0 \leqslant {}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] - {}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]. \tag{2.25}$$

Dividing both sides of (2.25) by ${}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}\right]{}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]$, we get (2.22). □

Theorem 2.8. *Let $(f_j)_{j=1,\dots,n}$ be n for finite $n \in \{1,2,\dots,n\}$ positive continuous functions decreasing on the interval $[a,b]$. Let $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1,\dots\}$. Then for right-FCI operator ${}^{\beta}\mathcal{J}_b^{\alpha}$, the following inequality holds true*

$$\frac{{}^{\theta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]{}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + {}^{\theta}\mathcal{J}_b^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]}{{}^{\theta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]{}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + {}^{\theta}\mathcal{J}_b^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]{}^{\beta}\mathcal{J}_b^{\beta}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]} \geqslant 1. \tag{2.26}$$

Proof. It is clear that for decreasing positive and continuous functions $(f_j)_{j=1,\dots,n}$ and $a < t \leqslant b$, $\delta > 0$, $\xi \geqslant \gamma_p > 0$ for any fixed $p \in \{1, \dots, n\}$, we have

$$\left(h^{\delta}(\rho) - h^{\delta}(\tau)\right)\left(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\right) \geqslant 0.$$

Define a function

$$\begin{aligned} {}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho,\tau) = \frac{1}{\Gamma(\beta)}&\left(\frac{(b-x)^{\alpha}-(b-\tau)^{\alpha}}{\alpha}\right)^{\beta-1} \\ &\times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\tau)}{(b-\tau)^{1-\alpha}}\left(h^{\delta}(\rho) - h^{\delta}(\tau)\right)\left(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\right). \end{aligned} \tag{2.27}$$

With the assumptions in Theorem 2.8, the function ${}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho,\tau)$ is positive for all $\tau \in (a, b]$. Integrating both sides of (2.27), with respect to τ from x to b, we have

$$\begin{aligned} 0 \leqslant \int_x^b {}^{\beta}\mathcal{T}_b^{\alpha}(x,\rho)d\tau &= \frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha}-(b-\tau)^{\alpha}}{\alpha}\right)^{\beta-1} \\ &\quad\times \prod_{j=1}^{n} f_j^{\gamma_j}(\tau)\left(h^{\delta}(\rho) - h^{\delta}(\tau)\right)\left(f_p^{\xi-\gamma_p}(\tau) - f_p^{\xi-\gamma_p}(\rho)\right)\frac{d\tau}{(b-\tau)^{1-\alpha}} \\ &= \left[h^{\delta}(\rho)\, {}^{\beta}\mathcal{J}_b^{\alpha}\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + f_p^{\xi-\gamma_p}(\tau)\, {}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(\tau)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] \\ &\quad - h^{\delta}(\rho) f_p^{\xi-\gamma_p}(\rho)\, {}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] - {}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right]. \end{aligned} \tag{2.28}$$

Multiplying (2.28) by $\frac{1}{\Gamma(\theta)}\left(\frac{(b-x)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\right)^{\theta-1} \times \frac{\prod_{j=1}^{n} f_j^{\gamma_j}(\rho)}{(\rho-a)^{1-\alpha}}$ and integrating with respect to ρ from x to b implies

$$\begin{aligned} 0 \leqslant\, &{}^{\theta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] {}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + {}^{\theta}\mathcal{J}_b^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] {}^{\beta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] \\ &- {}^{\theta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] {}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] - {}^{\theta}\mathcal{J}_b^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] {}^{\beta}\mathcal{J}_b^{\beta}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right]. \end{aligned} \tag{2.29}$$

Dividing both sides of (2.29) by

$${}^{\theta}\mathcal{J}_b^{\alpha}\left[h^{\delta}(x)\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] {}^{\beta}\mathcal{J}_b^{\alpha}\left[\prod_{j\neq p}^{n} f_j^{\gamma_j} f_p^{\xi}(x)\right] + {}^{\theta}\mathcal{J}_b^{\alpha}\left[\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right] {}^{\beta}\mathcal{J}_b^{\beta}\left[h^{\delta}(x)\prod_{j=1}^{n} f_j^{\gamma_j}(x)\right],$$

we get (2.26). □

3. Conclusion

In this paper we have considered the generalization of some integral inequalities for a general class of n positive non-increasing functions such that $n \in N$ by using the newly introduced left- and right-FCI operators. This work generalizes many results in the available literature. Several more special cases can be drawn from our results which will be equivalent to the pre-existing works. The results can be used in different directions. We suggest the readers for its applications to the existence of non-trivial solution of FDEs of different classes.

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