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Invariance analysis and exact solutions of some sixth-order difference equations

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Abstract

We perform a full Lie point symmetry analysis of difference equations of the form

$$u_{n+6} = \frac{u_n u_{n+4}}{u_{n+2}(A_n + B_n u_n u_{n+4})},$$

where the initial conditions are non-zero real numbers. Consequently, we obtain four non-trivial symmetries. Eventually, we get solutions of the difference equation for random sequences (A_n) and (B_n). This work is a generalization of a recent result by Khaliq and Elsayed [A. Khaliq, E. M. Elsayed, J. Nonlinear Sci. Appl., **9** (2016), 1052–1063].

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1. Introduction

In the past years, after the work of Lie [12] on differential equations, several researchers became interested in symmetries. Lie studied the group of mappings which leaves the differential equations invariant. The notion of symmetry is linked to conservation laws and the link between them has sparked great interest in researchers following the work of Noether [14]. It is a clear fact that as long as the symmetries and first integrals are related through the condition of invariance, one can execute the double reduction of the differential equations [13, 16]. The symmetry method has been used to find traveling wave solutions. For more on traveling waves, refer to [2, 17, 18] and [19]. The idea of symmetry method has been extended to difference equations and we refer the reader to [3–6, 11, 15] and references therein. Hydon [6] came up with a symmetry based procedure which makes it possible for one to find solutions of difference equations without trial and error. Much as Hydon, in his book [6], put emphasis on difference equations of second order, his algorithm works for any order. Unfortunately in such higher-order cases, computations are usually cumbersome and so certain assumptions are made to ease the computation. When this method is employed, we expect single solutions with fewer constraints on initial conditions.

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Our work is inspired by the results of Khaliq and Elsayed [9] where the following difference equation was studied

$$x_{n+1} = \frac{x_{n-1}x_{n-5}}{x_{n-3}(\pm 1 \pm x_{n-1}x_{n-5})}, \tag{1.1}$$

where x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0 are initial conditions and are positive real numbers. It is clear that (1.1) is specific case of a general form

$$x_{n+1} = \frac{x_{n-1}x_{n-5}}{x_{n-3}(a_n + b_n x_{n-1}x_{n-5})}, \tag{1.2}$$

where (a_n) and (b_n) are real sequences. Our goal is to use a symmetry based method to solve the more general difference equation (1.2). For definiteness, we study the equation

$$u_{n+6} = \frac{u_n u_{n+4}}{u_{n+2}(A_n + B_n u_n u_{n+4})},$$

instead.

For related work, refer to [1, 7, 8] and [10].

2. Preliminaries

In this section, we follow definitions and notation from [6].

Definition 2.1. A parameterized set of point transformations,

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x;\varepsilon),$$

where $x = x_i$, $i = 1, \cdots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$.
2. $\Gamma_a\Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0.
3. Each $\hat{x_i}$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore

$$\hat{x_i}(x:\varepsilon) = x_i + \varepsilon\xi_i(x) + O(\varepsilon^2), \quad i = 1, \cdots, p.$$

Given the p-th-order difference equation

$$u_{n+p} = \omega(u_n, \cdots, u_{n+p-1}) \tag{2.1}$$

for some function ω. We confine ourselves to symmetries where $\hat{u}_n$ depends on n and u_n only. Suppose that the point transformations take the following shape:

$$\hat{n} = n, \qquad \hat{u}_n = u_n + \epsilon Q(n, u_n). \tag{2.2}$$

Thus the symmetry condition is defined as

$$\hat{u}_{n+p} = \omega(n, \hat{u}_n, \hat{u}_{n+1}, \cdots, \hat{u}_{n+p-1}), \tag{2.3}$$

whenever (2.1) is true. Performing a substitution of Lie point symmetries (2.2) into the condition (2.3) results in the symmetry condition

$$S^{(p)}Q - X\omega = 0, \tag{2.4}$$

whenever (2.1) holds, in which S is the shift operator, that is, $S : n \mapsto n + 1$. With the following known infinitesimal symmetry generator

$$X = Q(n, u_n)\frac{\partial}{\partial u_n} + SQ(n, u_n)\frac{\partial}{\partial u_{n+1}} + \cdots + S^{n+p-1}Q(n, u_n)\frac{\partial}{\partial u_{n+p-1}},$$

it is important to take the canonical coordinate into consideration

$$s_n = \int \frac{du_n}{Q(n, u_n)}.$$

3. Symmetries and exact solutions

Considering the sixth-order difference equation of the form

$$u_{n+6} = \frac{u_n u_{n+4}}{u_{n+2}(A_n + B_n u_n u_{n+4})}. \tag{3.1}$$

Imposing the symmetry condition (2.4) and with a bit of simplification, we get

$$\begin{aligned} &Q(n+6, u_{n+6}) - \frac{A_n u_n}{u_{n+2}(B_n u_n u_{n+4} + A_n)^2} Q(n+4, u_{n+4}) \\ &+ \frac{u_n u_{n+4}}{{u_{n+2}}^2 (B_n u_n u_{n+4} + A_n)} Q(n+2, u_{n+2}) - \frac{A_n u_{n+4}}{u_{n+2}(B_n u_n u_{n+4} + A_n)^2} Q(n, u_n) = 0. \end{aligned} \tag{3.2}$$

In solving for the characteristic function, differentiate (3.2) with respect to u_n (keeping ω fixed) and consider u_{n+2} as a function of u_n, u_{n+4} and ω. This yields

$$\begin{aligned} &B_n u_{n+4} u_n u_{n+2} \frac{\partial}{\partial u_{n+2}} Q\left(n+2, u_{n+2}\right) - B_n u_{n+4} u_n u_{n+2} \frac{\partial}{\partial u_n} Q\left(n, u_n\right) \\ &- B_n u_{n+4} u_n Q\left(n+2, u_{n+2}\right) + B_n u_n u_{n+2} Q\left(n+4, u_{n+4}\right) \\ &+ 2B_n u_{n+4} u_{n+2} Q\left(n, u_n\right) + A_n u_{n+2} \frac{\partial}{\partial u_{n+2}} Q\left(n+2, u_{n+2}\right) \\ &- A_n u_{n+2} \frac{\partial}{\partial u_n} Q\left(n, u_n\right) - A_n Q\left(n+2, u_{n+2}\right) + \frac{A_n u_{n+2}}{u_n} Q\left(n, u_n\right) = 0. \end{aligned} \tag{3.3}$$

We now proceed by differentiating (3.3) with respect to u_n twice (keeping u_{n+2} fixed) and obtain

$$\begin{aligned} &- B_n u_{n+4} u_n \frac{\partial^3}{\partial {u_n}^3} Q\left(n, u_n\right) - A_n \frac{\partial^3}{\partial {u_n}^3} Q\left(n, u_n\right) + \frac{A_n}{u_n} \frac{\partial^2}{\partial {u_n}^2} Q\left(n, u_n\right) \\ &- \frac{2A_n}{u_n^2} \frac{\partial}{\partial u_n} Q\left(n, u_n\right) + \frac{2A_n}{u_n^3} Q\left(n, u_n\right) = 0. \end{aligned}$$

The equation above is solved by separation of variables in powers of shifts of u_n. Thus we have

$$\begin{cases} u_{n+4} : -B_n u_n Q^{(3)} = 0, \\ 1 : -A_n Q^{(3)} + \dfrac{A_n}{u_n} Q^{(2)} - \dfrac{2A_n}{{u_n}^2} Q^{(1)} + \dfrac{2A_n}{{u_n}^3} Q = 0, \end{cases}$$

which has the solution

$$Q = Q\left(n, u_n\right) = \alpha_n {u_n}^2 + \beta_n u_n, \tag{3.4}$$

where α_n and β_n are some functions of n. We then substitute (3.4) in (3.2) and split the result so that the following is obtained:

$$\begin{cases} {u_{n+4}}^2 {u_n}^2 {u_{n+2}}^2 : \ B_n \alpha_{n+2} = 0, \\ {u_{n+4}}^2 {u_n}^2 u_{n+2} : \ B_n(\beta_{n+2} + \beta_{n+6}) = 0, \\ {u_{n+4}}^2 u_n u_{n+2} : \ A_n \alpha_{n+4} = 0, \\ u_{n+4} {u_n}^2 u_{n+2} : \ A_n \alpha_n = 0, \\ u_{n+4} u_n {u_{n+2}}^2 : \ A_n \alpha_{n+2} = 0, \\ {u_{n+4}}^2 {u_n}^2 : \ \alpha_{n+6} = 0, \\ u_{n+4} u_n u_{n+2} : \ A_n\left(-\beta_n + \beta_{n+2} - \beta_{n+4} + \beta_{n+6}\right) = 0. \end{cases} \tag{3.5}$$

The system above (3.5) has solutions

$$\begin{aligned}\alpha_n =&0, \qquad \text{and}\\ \beta_n =& \left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)^n c_1 + \left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^n c_2 + \left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)^n c_3 + \left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^n c_4\end{aligned}$$

for some arbitrary constants c_i, $i = 1, \cdots, 4$. So we obtain four characteristics and their corresponding generators are as follows:

$$\begin{aligned}X_1 =&(-1)^n\beta^n u_n\partial u_n - (-1)^n\beta^{n+1}u_{n+1}\partial_{u_{n+1}} + (-1)^n\beta^{n+2}u_{n+2}\partial_{u_{n+2}}\\ &-(-1)^n\beta^{n+3}u_{n+3}\partial_{u_{n+3}} + (-1)^n\beta^{n+4}u_{n+4}\partial_{u_{n+4}} - (-1)^n\beta^{n+5}u_{n+5}\partial_{u_{n+5}},\\ X_2 =&(-1)^n\bar\beta^n u_n\partial u_n - (-1)^n\bar\beta^{n+1}u_{n+1}\partial_{u_{n+1}} + (-1)^n\bar\beta^{n+2}u_{n+2}\partial_{u_{n+2}}\\ &-(-1)^n\bar\beta^{n+3}u_{n+3}\partial_{u_{n+3}} + (-1)^n\bar\beta^{n+4}u_{n+4}\partial_{u_{n+4}} - (-1)^n\bar\beta^{n+5}u_{n+5}\partial_{u_{n+5}},\\ X_3 =&\bar\beta^n u_n\partial u_n + \bar\beta^{n+1}u_{n+1}\partial_{u_{n+1}} + \bar\beta^{n+2}u_{n+2}\partial_{u_{n+2}} + \bar\beta^{n+3}u_{n+3}\partial_{u_{n+3}}\\ &+\bar\beta^{n+4}u_{n+4}\partial_{u_{n+4}} + \bar\beta^{n+5}u_{n+5}\partial_{u_{n+5}},\\ X_4 =&\beta^n u_n\partial u_n + \beta^{n+1}u_{n+1}\partial_{u_{n+1}} + \beta^{n+2}u_{n+2}\partial_{u_{n+2}} + \beta^{n+3}u_{n+3}\partial_{u_{n+3}}\\ &+\beta^{n+4}u_{n+4}\partial_{u_{n+4}} + \beta^{n+5}u_{n+5}\partial_{u_{n+5}},\end{aligned}$$

in which $\beta = \exp\left(\frac{\pi i}{4}\right)$.

Now, utilizing X_4, we introduce the canonical coordinate

$$s_n = \int \frac{du_n}{\beta^n u_n} = \bar\beta^n \ln|u_n|.$$

Set

$$\tilde V_n = \beta^{n+4}s_{n+4} + \beta^n s_n,$$

and

$$|V_n| = \exp\{-\tilde V_n\},$$

i.e., $V_n = \pm 1/(u_n u_{n+4})$ but we choose $V_n = 1/(u_n u_{n+4})$. Using (3.1), one can check that

$$V_{n+2} = A_n V_n + B_n,$$

and thus

$$V_{2n+k} = V_k\left(\prod_{k_1=0}^{n-1} A_{2k_1+k}\right) + \sum_{l=0}^{n-1}\left(B_{2l+k}\prod_{k_2=l+1}^{n-1} A_{2k_2+k}\right), \quad k = 0,1. \tag{3.6}$$

We have that

$$\begin{aligned}|u_n| =&\exp\{\beta^n s_n\}\\ =&\exp \beta^n\Bigg(c_6 + i^n c_7 + (-1)^n c_8 + (-i)^n c_9 + \frac{1}{4}\Bigg[\sum_{k_1=0}^{n-1} r_{k_1} + i^n\sum_{k_2=0}^{n-1}(-i)^{k_2} r_{k_2}\\ &+(-1)^n\sum_{k_3=0}^{n-1}(-1)^{k_3}r_{k_3} + (-i)^n\sum_{k_4=0}^{n-1}(i)^{k_4}r_{k_4} + \Bigg]\Bigg)\\ =&\exp \beta^n\Bigg(c_6 + i^n c_7 + (-1)^n c_8 + (-i)^n c_9 + \frac{1}{4}\Bigg[\sum_{k_1=0}^{n-1}\bar\beta^{k_1}\ln|V_{k_1}| + i^n\sum_{k_2=0}^{n-1}(-i)^{k_2}\bar\beta^{k_2}\ln|V_{k_2}|\\ &+(-1)^n\sum_{k_3=0}^{n-1}(-1)^{k_3}\bar\beta^{k_3}\ln|V_{k_3}| + (-i)^n\sum_{k_4=0}^{n-1}(i)^{k_4}\bar\beta^{k_4}\ln|V_{k_4}|\Bigg]\Bigg),\end{aligned} \tag{3.7}$$

$$|u_n| = \exp\left(\frac{1}{2}\sum_{k_1=0}^{n-1}[1+(-1)^{n+k_1}]\mathrm{Re}(\gamma(n,k_1))\ln|V_{k_1}| + \beta^n c_6 + (-\bar{\beta})^n c_7 + (-1)^n\beta^n c_8 + \bar{\beta}^n c_9\right), \tag{3.8}$$

where V_k is given in (3.6) with $\gamma(n,k) = \beta^n\bar{\beta}^k$. Observe that c_6, c_7, c_8 and c_9 satisfy

$$c_6 + c_7 + c_8 + c_9 = \ln|u_0|, \tag{3.9}$$
$$\beta c_6 - \bar{\beta}c_7 - \beta c_8 + \bar{\beta}c_9 = \ln|u_1|, \tag{3.10}$$
$$\beta^2 c_6 + \bar{\beta}^2 c_7 + \beta^2 c_8 + \bar{\beta}^2 c_9 = \ln|u_2|, \tag{3.11}$$
$$\beta^3 c_6 - \bar{\beta}^3 c_7 - \beta^3 c_8 + \bar{\beta}^3 c_9 = \ln|u_3|. \tag{3.12}$$

Equations (3.7), (3.8), (3.9), (3.10), (3.11), (3.12) give the solutions of (3.1).

The function $\gamma(n,k) = \beta^n\bar{\beta}^k$ is such that

$$\begin{aligned}&\gamma(0,1) = \bar{\beta}, \quad \gamma(0,2) = -i, \quad \gamma(0,3) = -\beta, \quad \gamma(1,0) = \beta, \quad \gamma(1,2) = \bar{\beta}, \quad \gamma(1,3) = -i, \quad \gamma(2,0) = i,\\ &\gamma(2,1) = \beta, \quad \gamma(2,3) = \bar{\beta}, \quad \gamma(3,0) = -\bar{\beta}, \quad \gamma(3,1) = i, \quad \gamma(3,2) = \beta, \quad \gamma(n,n) = 1,\\ &\gamma(n+4,k) = -\gamma(n,k), \quad \gamma(n,k+4) = -\gamma(n,k), \quad \gamma(8n,k) = \gamma(0,k), \quad \gamma(n,8k) = \gamma(n,0).\end{aligned} \tag{3.13}$$

From (3.13), we note that

$$|u_{8n+j}| = \exp\left(H_j + \frac{1}{2}\sum_{k_1=0}^{8n+j-1}[1+(-1)^{n+k_1}]\mathrm{Re}(\gamma(n,k_1))\ln|V_{k_1}|\right), \tag{3.14}$$

where

$$H_j = \beta^j c_6 + (-1)^j\bar{\beta}^j c_7 + (-1)^j\beta^j c_8 + \bar{\beta}^j c_9.$$

For $j = 0$, we have

$$|u_{8n}| = \exp(H_0 + \ln|V_0| - \ln|V_4| + \ln|V_8| - \ln|V_{12}| + \cdots + \ln|V_{8n-8}| - \ln|V_{8n-4}|) = \exp(H_0)\prod_{s=0}^{n-1}\left|\frac{V_{8s}}{V_{8s+4}}\right|.$$

One can show that there is no need for the absolute values by using the fact that

$$V_i = \frac{1}{u_i u_{i+4}}. \tag{3.15}$$

To find $\exp(H_0)$, we set $n = 0$ in (3.14) and observe that $|u_0| = \exp(H_0)$. Thus

$$u_{8n} = u_0\prod_{s=0}^{n-1}\frac{V_{8s}}{V_{8s+4}}.$$

Clearly, $8s = 2(4s) + 0$ so that $i = 0$ and $n = 4s$ in (3.6). From (3.6), we obtain

$$\begin{aligned}V_{8s} &= V_0\left(\prod_{k_1=0}^{4s-1}A_{2k_1} + \frac{1}{V_0}\sum_{l=0}^{4s-1}B_{2l}\prod_{k_2=l+1}^{4s-1}A_{2k_2}\right)\\ &= \frac{1}{u_0u_4}\left(\prod_{k_1=0}^{4s-1}A_{2k_1} + u_0u_4\sum_{l=0}^{4s-1}B_{2l}\prod_{k_2=l+1}^{4s-1}A_{2k_2}\right),\end{aligned}$$

using (3.15). Similarly, we have

$$V_{8s+4} = \frac{1}{u_0u_4}\left(\prod_{k_1=0}^{4s+1}A_{2k_1} + u_0u_4\sum_{l=0}^{4s+1}B_{2l}\prod_{k_2=l+1}^{4s-1}A_{2k_2}\right),$$

so that

$$u_{8n} = u_0 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s-1} A_{2k_1} + u_0u_4 \sum_{l=0}^{4s-1} B_{2l} \prod_{k_2=l+1}^{4s-1} A_{2k_2}}{\prod_{k_1=0}^{4s+1} A_{2k_1} + u_0u_4 \sum_{l=0}^{4s+1} B_{2l} \prod_{k_2=l+1}^{4s+1} A_{2k_2}}.$$

Hence

$$x_{8n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s-1} A_{2k_1} + x_{-5}x_{-1} \sum_{l=0}^{4s-1} B_{2l} \prod_{k_2=l+1}^{4s-1} A_{2k_2}}{\prod_{k_1=0}^{4s+1} A_{2k_1} + x_{-5}x_{-1} \sum_{l=0}^{4s+1} B_{2l} \prod_{k_2=l+1}^{4s+1} A_{2k_2}}. \tag{3.16}$$

For $j = 1$, (3.14) becomes

$$|u_{8n+1}| = \exp(H_1)\exp(\ln|V_1| - \ln|V_5| + \ln|V_9| - \ln|V_{13}| + \cdots + \ln|V_{8n-7}| - \ln|V_{8n-3}|).$$

However, using (3.14) with $j = 1, n = 0$, $|u_1| = \exp(H_1)$. Thus

$$u_{8n+1} = u_1 \prod_{s=0}^{n-1} \frac{V_{8s+1}}{V_{8s+5}}.$$

By (3.6), we have

$$\begin{aligned} V_{8s+1} &= V_1 \left(\prod_{k_1=0}^{4s-1} A_{2k_1+1} + \frac{1}{V_1} \sum_{l=0}^{4s-1} B_{2l+1} \prod_{k_2=l+1}^{4s-1} A_{2k_2} \right) \\ &= \frac{1}{u_1u_5} \left(\prod_{k_1=0}^{4s-1} A_{2k_1+1} + u_1u_5 \sum_{l=0}^{4s-1} B_{2l+1} \prod_{k_2=l+1}^{4s-1} A_{2k_2+1} \right), \end{aligned}$$

and similarly,

$$V_{8s+5} = \frac{1}{u_1u_5} \left(\prod_{k_1=0}^{4s+1} A_{2k_1+1} + u_1u_5 \sum_{l=0}^{4s+1} B_{2l+1} \prod_{k_2=l+1}^{4s+1} A_{2k_2+1} \right).$$

Now

$$u_{8n+1} = u_1 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s-1} A_{2k_1+1} + u_1u_5 \sum_{l=0}^{4s-1} B_{2l+1} \prod_{k_2=l+1}^{4s-1} A_{2k_2+1}}{\prod_{k_1=0}^{4s+1} A_{2k_1+1} + u_1u_5 \sum_{l=0}^{4s+1} B_{2l+1} \prod_{k_2=l+1}^{4s+1} A_{2k_2+1}},$$

which implies that

$$x_{8n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s-1} A_{2k_1+1} + x_{-4}x_0 \sum_{l=0}^{4s-1} B_{2l+1} \prod_{k_2=l+1}^{4s-1} A_{2k_2+1}}{\prod_{k_1=0}^{4s+1} A_{2k_1+1} + x_{-4}x_0 \sum_{l=0}^{4s+1} B_{2l+1} \prod_{k_2=l+1}^{4s+1} A_{2k_2+1}}. \tag{3.17}$$

For $j = 2$, we find that (3.14) becomes

$$|u_{8n+2}| = \exp(H_2)\exp(\ln|V_2| - \ln|V_6| + \ln|V_{10}| - \cdots + \ln|V_{8n-6}| - \ln|V_{8n-2}|).$$

Similar to the earlier cases, setting $n = 0$ and $j = 2$ yields the equation $|u_2| = \exp(H_2)$. So we have

$$u_{8n+2} = u_2 \prod_{s=0}^{n-1} \frac{V_{8s+2}}{V_{8s+6}}.$$

The expressions for V_{8s+2} and V_{8s+6} are obtained from (3.6), by setting $n = 4s+1,\ i = 0$ and $n = 4s+3,\ i = 0$, respectively. They are as follows:

$$V_{8s+2} = V_0 \left(\prod_{k_1=0}^{4s} A_{2k_1} + \frac{1}{V_0} \sum_{l=0}^{4s} B_{2l} \prod_{k_2=l+1}^{4s} A_{2k_2} \right)$$

$$= \frac{1}{u_0u_4}\left(\prod_{k_1=0}^{4s} A_{2k_1} + u_0u_4\sum_{l=0}^{4s} B_{2l}\prod_{k_2=l+1}^{4s} A_{2k_2}\right),$$

and

$$V_{8s+6} = V_0\left(\prod_{k_1=0}^{4s+2} A_{2k_1} + \frac{1}{V_0}\sum_{l=0}^{4s+2} B_{2l}\prod_{k_2=l+1}^{4s} A_{2k_2}\right)$$
$$= \frac{1}{u_0u_4}\left(\prod_{k_1=0}^{4s+2} A_{2k_1} + u_0u_4\sum_{l=0}^{4s+2} B_{2l}\prod_{k_2=l+1}^{4s+2} A_{2k_2}\right).$$

Hence

$$u_{8n+2} = u_2\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s} A_{2k_1} + u_0u_4\sum_{l=0}^{4s} B_{2l}\prod_{k_2=l+1}^{4s} A_{2k_2}}{\prod_{k_1=0}^{4s+2} A_{2k_1} + u_0u_4\sum_{l=0}^{4s+2} B_{2l}\prod_{k_2=l+1}^{4s+2} A_{2k_2}},$$

which gives

$$x_{8n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s} A_{2k_1} + x_{-5}x_{-1}\sum_{l=0}^{4s} B_{2l}\prod_{k_2=l+1}^{4s} A_{2k_2}}{\prod_{k_1=0}^{4s+2} A_{2k_1} + x_{-5}x_{-1}\sum_{l=0}^{4s+2} B_{2l}\prod_{k_2=l+1}^{4s+2} A_{2k_2}}. \tag{3.18}$$

For $j = 3$, we find that (3.14) becomes

$$|u_{8n+3}| = \exp(H_3)\exp(\ln|V_3| - \ln|V_7| + \ln|V_{11}| - \cdots + \ln|V_{8n-5}| - \ln|V_{8n-1}|).$$

Setting $n = 0$ and $j = 3$, we find that $|u_3| = \exp(H_3)$, hence

$$u_{8n+3} = u_3\prod_{s=1}^{n-1}\frac{V_{8s+3}}{V_{8s+7}}.$$

Following a similar approach as was done in the earlier cases ($j = 0, 1, 2$), the reader can verify that

$$u_{8n+3} = u_3\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s} A_{2k_1+1} + u_1u_5\sum_{l=0}^{4s} B_{2l+1}\prod_{k_2=l+1}^{4s} A_{2k_2+1}}{\prod_{k_1=0}^{4s+2} A_{2k_1+1} + u_1u_5\sum_{l=0}^{4s+2} B_{2l+1}\prod_{k_2=l+1}^{4s+2} A_{2k_2+1}}.$$

Thus

$$x_{8n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s} A_{2k_1+1} + x_{-4}x_0\sum_{l=0}^{4s} B_{2l+1}\prod_{k_2=l+1}^{4s} A_{2k_2+1}}{\prod_{k_1=0}^{4s+2} A_{2k_1+1} + x_{-4}x_0\sum_{l=0}^{4s+2} B_{2l+1}\prod_{k_2=l+1}^{4s+2} A_{2k_2+1}}. \tag{3.19}$$

For $j = 4$, (3.14) becomes

$$|u_{8n+4}| = \exp(H_4)\exp(-\ln|V_0| + \ln|V_4| - \ln|V_8| + \cdots + \ln|V_{8n-4}| - \ln|V_{8n}|).$$

Setting $j = 0$ and $n = 0$ in (3.14) yields $|u_4| = \exp(H_4)\exp(-\ln|V_0|)$, so that $\exp(H_4) = |u_4||V_0|$. Thus

$$u_{8n+4} = u_4\prod_{s=1}^{n-1}\frac{V_{8s+4}}{V_{8s+8}}.$$

As before, similar steps can be carried out and one obtains

$$u_{8n+4} = u_4\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s+1} A_{2k_1} + u_0u_4\sum_{l=0}^{4s+1} B_{2l}\prod_{k_2=l+1}^{4s+1} A_{2k_2}}{\prod_{k_1=0}^{4s+3} A_{2k_1} + u_0u_4\sum_{l=0}^{4s+3} B_{2l}\prod_{k_2=l+1}^{4s+3} A_{2k_2}},$$

which yields

$$x_{8n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} A_{2k_1} + x_{-5}x_{-1} \sum_{l=0}^{4s+1} B_{2l} \prod_{k_2=l+1}^{4s+1} A_{2k_2}}{\prod_{k_1=0}^{4s+3} A_{2k_1} + x_{-5}x_{-1} \sum_{l=0}^{4s+3} B_{2l} \prod_{k_2=l+1}^{4s+3} A_{2k_2}}. \tag{3.20}$$

For $j = 5$, (3.14) becomes

$$|u_{8n+5}| = \exp(H_5) \exp(-\ln|V_1| + \ln|V_5| - \ln|V_9| + \cdots + \ln|V_{8n-3}| - \ln|V_{8n+1}|).$$

But setting $n = 0$ and $j = 5$ in the same equation (3.14), we get $|u_5| = \exp(-\ln|V_1|)\exp(H_5)$ so that $\exp(H_5) = |u_5||V_1|$. Thus

$$u_{8n+5} = u_5 \prod_{s=0}^{n-1} \frac{V_{8s+5}}{V_{8s+9}}.$$

After performing similar substitutions as before, we get

$$u_{8n+5} = u_5 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} A_{2k_1+1} + u_1u_5 \sum_{l=0}^{4s+1} B_{2l+1} \prod_{k_2=l+1}^{4s+1} A_{2k_2+1}}{\prod_{k_1=0}^{4s+3} A_{2k_1+1} + u_1u_5 \sum_{l=0}^{4s+3} B_{2l+1} \prod_{k_2=l+1}^{4s+3} A_{2k_2+1}},$$

which leads to

$$x_{8n} = x_0 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} A_{2k_1+1} + x_{-4}x_0 \sum_{l=0}^{4s+1} B_{2l+1} \prod_{k_2=l+1}^{4s+1} A_{2k_2+1}}{\prod_{k_1=0}^{4s+3} A_{2k_1+1} + x_{-4}x_0 \sum_{l=0}^{4s+3} B_{2l+1} \prod_{k_2=l+1}^{4s+3} A_{2k_2+1}}. \tag{3.21}$$

For $j = 6$ and $n = 0$, (3.14) becomes $|u_6| = \exp(H_6)\exp(-\ln|V_2|)$, i.e., $\exp(H_6) = |u_6||V_2|$. However, setting $j = 6$ in the same equation, we get

$$|u_{8n+6}| = \exp(H_6) \exp(-\ln|V_2| + \ln|V_6| - \ln|V_{10}| + \cdots + \ln|V_{8n-2}| - \ln|V_{8n+2}|).$$

We have

$$u_{8n+6} = u_6 \prod_{s=0}^{n-1} \frac{V_{8s+6}}{V_{8s+10}}.$$

The reader can verify that

$$u_{8n+6} = u_6 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} A_{2k_1} + u_0u_4 \sum_{l=0}^{4s+2} B_{2l} \prod_{k_2=l+1}^{4s+2} A_{2k_2}}{\prod_{k_1=0}^{4s+4} A_{2k_1} + u_0u_4 \sum_{l=0}^{4s+4} B_{2l} \prod_{k_2=l+1}^{4s+4} A_{2k_2}},$$

which gives

$$x_{8n+1} = x_1 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} A_{2k_1} + x_{-5}x_{-1} \sum_{l=0}^{4s+2} B_{2l} \prod_{k_2=l+1}^{4s+2} A_{2k_2}}{\prod_{k_1=0}^{4s+4} A_{2k_1} + x_{-5}x_{-1} \sum_{l=0}^{4s+4} B_{2l} \prod_{k_2=l+1}^{4s+4} A_{2k_2}}.$$

Since $x_1 = \frac{x_{-1}x_{-5}}{x_{-3}(a_0+b_0x_{-1}x_{-5})}$, we have

$$x_{8n+1} = \frac{x_{-1}x_{-5}}{x_{-3}(a_0+b_0x_{-1}x_{-5})} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} A_{2k_1} + x_{-5}x_{-1} \sum_{l=0}^{4s+2} B_{2l} \prod_{k_2=l+1}^{4s+2} A_{2k_2}}{\prod_{k_1=0}^{4s+4} A_{2k_1} + x_{-5}x_{-1} \sum_{l=0}^{4s+4} B_{2l} \prod_{k_2=l+1}^{4s+4} A_{2k_2}}. \tag{3.22}$$

For $j = 7$, (3.14) becomes

$$|u_{8n+7}| = \exp(H_7) \exp(-\ln|V_3| + \ln|V_7| - \ln|V_{11}| + \cdots + \ln|V_{8n-1}| - \ln|V_{8n+3}|).$$

By setting $n = 0$ and $j = 7$ in (3.14), we get $|u_7| = \exp(-\ln|V_3|)\exp(H_7)$ so that $\exp(H_7) = |u_7||V_3|$. Hence

$$u_{8n+7} = u_7 \prod_{s=0}^{n-1} \frac{V_{8s+7}}{V_{8s+11}}.$$

After replacing V_i's with their appropriate expressions as done before, we get

$$u_{8n+7} = u_7 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} A_{2k_1+1} + u_1u_5 \sum_{l=0}^{4s+2} B_{2l+1} \prod_{k_2=l+1}^{4s+2} A_{2k_2+1}}{\prod_{k_1=0}^{4s+4} A_{2k_1+1} + u_1u_5 \sum_{l=0}^{4s+4} B_{2l+1} \prod_{k_2=l+1}^{4s+4} A_{2k_2+1}},$$

which implies that

$$x_{8n+2} = x_2 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} A_{2k_1+1} + x_{-4}x_0 \sum_{l=0}^{4s+2} B_{2l+1} \prod_{k_2=l+1}^{4s+2} A_{2k_2+1}}{\prod_{k_1=0}^{4s+4} A_{2k_1+1} + x_{-4}x_0 \sum_{l=0}^{4s+4} B_{2l+1} \prod_{k_2=l+1}^{4s+4} A_{2k_2+1}}.$$

Since $x_2 = \frac{x_0x_{-4}}{x_{-2}(a_1+b_2x_0x_{-4})}$, we have

$$x_{8n+2} = \frac{x_0x_{-4}}{x_{-2}(a_1+b_1x_0x_{-4})} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} A_{2k_1+1} + x_{-4}x_0 \sum_{l=0}^{4s+2} B_{2l+1} \prod_{k_2=l+1}^{4s+2} A_{2k_2+1}}{\prod_{k_1=0}^{4s+4} A_{2k_1+1} + x_{-4}x_0 \sum_{l=0}^{4s+4} B_{2l+1} \prod_{k_2=l+1}^{4s+4} A_{2k_2+1}}.$$

Remark 3.1. It is important to make sure that the denominators in the expressions for x_{8n+j} where $j = -5, -4, \cdots, 2$ are non-zero so that the solution is well-defined.

4. The case when A_j and B_j are 2-periodic sequences

We assume that $\{A_j\}_{j\geqslant 0} = \{A_0, A_1, A_0, \cdots\}$ where $A_0 \neq A_1$, and $\{B\}_{j\geqslant 0} = \{B_0, B_1, B_0, B_1, \cdots\}$ with $B_0 \neq B_1$. Then after substitution, (3.16), (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22) become

$$x_{8n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{A_0^{4s} + x_{-5}x_{-1}B_0 \sum_{j=0}^{4s-1} A_0^j}{A_0^{4s+2} + x_{-5}x_{-1}B_0 \sum_{j=0}^{4s+1} A_0^j}, \qquad x_{8n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{A_1^{4s} + x_{-4}x_0B_1 \sum_{j=0}^{4s-1} A_1^j}{A_1^{4s+2} + x_{-4}x_0B_1 \sum_{j=0}^{4s+1} A_1^j},$$

$$x_{8n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{A_0^{4s+1} + x_{-5}x_{-1}B_0 \sum_{j=0}^{4s} A_0^j}{A_0^{4s+3} + x_{-5}x_{-1}B_0 \sum_{j=0}^{4s+2} A_0^j}, \qquad x_{8n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{A_1^{4s+1} + x_{-4}x_0B_1 \sum_{j=0}^{4s} A_1^j}{A_1^{4s+3} + x_{-4}x_0B_1 \sum_{j=0}^{4s+2} A_1^j},$$

$$x_{8n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{A_0^{4s+2} + x_{-5}x_{-1}B_0 \sum_{j=0}^{4s+1} A_0^j}{A_0^{4s+4} + x_{-5}x_{-1}B_0 \sum_{j=0}^{4s+3} A_0^j}, \qquad x_{8n} = x_0 \prod_{s=0}^{n-1} \frac{A_1^{4s+2} + x_{-4}x_0B_1 \sum_{j=0}^{4s+1} A_1^j}{A_1^{4s+4} + x_{-4}x_0B_1 \sum_{j=0}^{4s+3} A_1^j},$$

$$x_{8n+1} = x_1 \prod_{s=0}^{n-1} \frac{A_0^{4s+3} + x_{-5}x_{-1}B_0 \sum_{j=0}^{4s+2} A_0^j}{A_0^{4s+5} + x_{-5}x_{-1}B_0 \sum_{j=0}^{4s+4} A_0^j}, \qquad x_{8n+2} = x_2 \prod_{s=0}^{n-1} \frac{A_1^{4s+3} + x_{-4}x_0B_1 \sum_{j=0}^{4s+2} A_1^j}{A_1^{4s+5} + x_{-4}x_0B_1 \sum_{j=0}^{4s+4} A_1^j},$$

where $x_1 = \frac{x_1x_{-5}}{x_{-3}(A_0+B_0x_{-1}x_{-5})}$ and $x_2 = \frac{x_0x_{-4}}{x_{-2}(A_1+B_1x_0x_{-4})}$.

The solution is only valid when the denominator in each of the solution equations above is non-zero. To state this precisely, define $h(s, j, i)$, for $i = 0, 1$, as follows:

$$h(s,j,i) = \begin{cases} \frac{A_i^{4s+j-1}(1-A_i)}{B_i(1-A_i^{4s+j+1})}, & A_i \neq 1, \\ \frac{1}{B_i(4s+j+1)}, & A_i = 1. \end{cases}$$

Thus the solution is well-defined if

$$x_{-2}x_{-3}\left(x_{-4}x_0 + \frac{A_1}{B_1}\right)\left(x_{-1}x_{-5} + \frac{A_0}{B_0}\right)\prod_{j=1}^{4}(x_{-4}x_0 + h(s,j,1))(x_{-1}x_{-5} + h(s,j,0)) \neq 0,$$

for all $s = 1, 2, \cdots, n-1$.

5. The case where A_j and B_j are 1-periodic sequences

In this setting we assume that $A_j = A_0$, and $B_j = B_0$ for all j. The solution is found by replacing A_1 and B_1 with A_0 and B_0, respectively in the solution equations for Section 4. We find the following solution equations:

$$x_{8n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{A_0^{4s}+x_{-5}x_{-1}B_0\sum_{j=0}^{4s-1}A_0^j}{A_0^{4s+2}+x_{-5}x_{-1}B_0\sum_{j=0}^{4s+1}A_0^j}, \qquad x_{8n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{A_0^{4s}+x_{-4}x_0B_0\sum_{j=0}^{4s-1}A_0^j}{A_0^{4s+2}+x_{-4}x_0B_0\sum_{j=0}^{4s+1}A_0^j},$$

$$x_{8n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{A_0^{4s+1}+x_{-5}x_{-1}B_0\sum_{j=0}^{4s}A_0^j}{A_0^{4s+3}+x_{-5}x_{-1}B_0\sum_{j=0}^{4s+2}A_0^j}, \qquad x_{8n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{A_0^{4s+1}+x_{-4}x_0B_0\sum_{j=0}^{4s}A_0^j}{A_0^{4s+3}+x_{-4}x_0B_0\sum_{j=0}^{4s+2}A_0^j},$$

$$x_{8n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{A_0^{4s+2}+x_{-5}x_{-1}B_0\sum_{j=0}^{4s+1}A_0^j}{A_0^{4s+4}+x_{-5}x_{-1}B_0\sum_{j=0}^{4s+3}A_0^j}, \qquad x_{8n} = x_0\prod_{s=0}^{n-1}\frac{A_0^{4s+2}+x_{-4}x_0B_0\sum_{j=0}^{4s+1}A_0^j}{A_0^{4s+4}+x_{-4}x_0B_0\sum_{j=0}^{4s+3}A_0^j},$$

$$x_{8n+1} = x_1\prod_{s=0}^{n-1}\frac{A_0^{4s+3}+x_{-5}x_{-1}B_0\sum_{j=0}^{4s+2}A_0^j}{A_0^{4s+5}+x_{-5}x_{-1}B_0\sum_{j=0}^{4s+4}A_0^j}, \qquad x_{8n+2} = x_2\prod_{s=0}^{n-1}\frac{A_0^{4s+3}+x_{-4}x_0B_0\sum_{j=0}^{4s+2}A_0^j}{A_0^{4s+5}+x_{-4}x_0B_0\sum_{j=0}^{4s+4}A_0^j},$$

where $x_1 = \frac{x_1x_{-5}}{x_{-3}(A_0+B_0x_{-1}x_{-5})}$ and $x_2 = \frac{x_0x_{-4}}{x_{-2}(A_0+B_0x_0x_{-4})}$.

We also notice that the solution is well-defined if

$$x_{-2}x_{-3}\left(x_{-4}x_0+\frac{A_0}{B_0}\right)\left(x_{-1}x_{-5}+\frac{A_0}{B_0}\right)\prod_{j=1}^{4}(x_{-4}x_0+h(s,j,0))(x_{-1}x_{-5}+h(s,j,0)) \neq 0$$

for all $s = 1, 2, \cdots, n-1$, where the definition of $h(s, j, i)$ is given in the previous section.

5.1. *The case of* $A_j = B_j = -1$

In this case, set $A_0 = B_0 = -1$ in the above equations. This yields the following solution equations, which appear in [9, Theorem 5.1].

$$x_{8n-5} = x_{-5}, \qquad x_{8n-4} = x_{-4}, \qquad x_{8n-3} = x_{-3},$$
$$x_{8n-2} = x_{-2}, \qquad x_{8n-1} = x_{-1}, \qquad x_{8n} = x_0,$$
$$x_{8n+1} = \frac{x_1x_{-5}}{x_{-3}(-1-x_{-1}x_{-5})}, \qquad x_{8n+2} = \frac{x_0x_{-4}}{x_{-2}(-1-x_0x_{-4})}.$$

5.2. *The case of* $A_j = -1$ *and* $B_j = 1$

This case yields the following solution equations which appear in [9, Theorem 3.1].

$$x_{8n-5} = x_{-5}, \qquad x_{8n-4} = x_{-4}, \qquad x_{8n-3} = x_{-3},$$
$$x_{8n-2} = x_{-2}, \qquad x_{8n-1} = x_{-1}, \qquad x_{8n} = x_0,$$
$$x_{8n+1} = \frac{x_1x_{-5}}{x_{-3}(-1+x_{-1}x_{-5})}, \qquad x_{8n+2} = \frac{x_0x_{-4}}{x_{-2}(-1+x_0x_{-4})}.$$

5.3. *The case of* $A_j = 1$ *and* $B_j = -1$

Our general solution yields the following solution equations which appear in [9, Theorem 4.1].

$$x_{8n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{1-x_{-5}x_{-1}4s}{1-x_{-5}x_{-1}(4s+2)}, \qquad x_{8n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{1-x_{-4}x_0 4s}{1-x_{-4}x_0(4s+2)},$$

$$x_{8n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1-x_{-5}x_{-1}(4s+1)}{1-x_{-5}x_{-1}(4s+3)}, \qquad x_{8n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1-x_{-4}x_0(4s+1)}{1-x_{-4}x_0(4s+3)},$$

$$x_{8n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{1 - x_{-5}x_{-1}(4s+2)}{1 - x_{-5}x_{-1}(4s+4)}, \qquad x_{8n} = x_0 \prod_{s=0}^{n-1} \frac{1 - x_{-4}x_0(4s+2)}{1 - x_{-4}x_0(4s+4)},$$

$$x_{8n+1} = x_1 \prod_{s=0}^{n-1} \frac{1 - x_{-5}x_{-1}(4s+3)}{1 - x_{-5}x_{-1}(4s+5)}, \qquad x_{8n+2} = x_2 \prod_{s=0}^{n-1} \frac{1 - x_{-4}x_0(4s+3)}{1 - x_{-4}x_0(4s+5)},$$

where $x_1 = \frac{x_1 x_{-5}}{x_{-3}(1 - x_{-1}x_{-5})}$ and $x_2 = \frac{x_0 x_{-4}}{x_{-2}(1 - x_0 x_{-4})}$.

5.4. *The case of* $A_j = B_j = 1$

This case yields the following solution equations which appear in [9, Theorem 2.1].

$$x_{8n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{1 + x_{-5}x_{-1}4s}{1 + x_{-4}x_0(4s+2)}, \qquad x_{8n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{1 + x_{-4}x_0 4s}{1 + x_{-4}x_0(4s+2)},$$

$$x_{8n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{1 + x_{-5}x_{-1}(4s+1)}{1 + x_{-5}x_{-1}(4s+3)}, \qquad x_{8n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{1 + x_{-4}x_0(4s+1)}{1 + x_{-4}x_0(4s+3)},$$

$$x_{8n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{1 + x_{-5}x_{-1}(4s+2)}{1 + x_{-5}x_{-1}(4s+4)}, \qquad x_{8n} = x_0 \prod_{s=0}^{n-1} \frac{1 + x_{-4}x_0(4s+2)}{1 + x_{-4}x_0(4s+4)},$$

$$x_{8n+1} = x_1 \prod_{s=0}^{n-1} \frac{1 + x_{-5}x_{-1}(4s+3)}{1 + x_{-5}x_{-1}(4s+5)}, \qquad x_{8n+2} = x_2 \prod_{s=0}^{n-1} \frac{1 + x_{-4}x_0(4s+3)}{1 + x_{-4}x_0(4s+5)},$$

where $x_1 = \frac{x_1 x_{-5}}{x_{-3}(1 + x_{-1}x_{-5})}$ and $x_2 = \frac{x_0 x_{-4}}{x_{-2}(1 + x_0 x_{-4})}$.

6. Conclusion

In this paper, by using a symmetry based method, we have obtained a solution of a more general form of the difference equations considered in [9]. Thus we studied difference equations of the form

$$x_{n+1} = \frac{x_{-1}x_{n-5}}{x_{n-3}(A_n + B_n x_{n-1} x_{n-5})},$$

and obtained their solutions. It should be noted that Khaliq and Elsayed [9] considered the cases with all possible combinations of $A_n \in \{-1, 1\}$ and $B_n \in \{-1, 1\}$, and our case goes beyond the ± 1 setting by generalizing the situation to (B_n) and (A_n) being non-zero sequences of real numbers.

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