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A new iterative scheme in CAT(0) spaces with convergence analysis

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Abstract

In this paper, we establish strong and Δ-convergence theorems in CAT(0) spaces for two total asymptotically nonexpansive non-self mappings via a new two-step iterative scheme for non-self-mappings. Our results extend and generalize several results from the current existing literature.

Keywords: Total asymptotically nonexpansive non-self-mapping, new two-step iterative scheme, common fixed point, strong convergence, Δ-convergence, CAT(0) space.

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1. Introduction

Fixed point theory in CAT(0) spaces was first studied by Kirk [20, 21] wherein it is shown that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leqslant 0$ since any CAT(k) space is a CAT(m) space for every $m \geqslant k$ (see [5], "Metric spaces of non-positive curvature").

The concept of Δ-convergence in a general metric space was introduced by Lim [24]. In 2008, Kirk and Panyanak [22] used the notion of Δ-convergence introduced by Lim [24] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [11] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space.

The notion of total asymptotically nonexpansive mapping was first introduced in Banach spaces by Alber et al. [3] in 2006. It is generalization of the asymptotically nonexpansive mappings introduced by Goebel and Kirk [12] in 1972 as well as the nearly asymptotically nonexpansive mappings introduced by Sahu [27] in 2005.

In 2012, Chang et al. [7] studied the demiclosed principle and convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Since then the convergence theorems of several it-

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eration procedures for this class of mappings have been rapidly developed (see e.g., [4, 7, 17, 26, 28–31, 33, 36–38]).

2. Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say that X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [5]).

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) inequality.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leqslant d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

Complete CAT(0) spaces are often called *Hadamard spaces* (see [18]). If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leqslant \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \tag{2.1}$$

Inequality (2.1) is the (CN) inequality of Bruhat and Tits [6]. The above inequality was extended in [11] as

$$d^2(z, \alpha x \oplus (1-\alpha)y) \leqslant \alpha d^2(z, x) + (1-\alpha) d^2(z, y) - \alpha(1-\alpha) d^2(x, y)$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [5, p.163]). Moreover, if X is a CAT(0) space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leqslant \alpha d(z, x) + (1-\alpha) d(z, y)$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$.

A subset K of a CAT(0) space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$.

The following are some elementary facts about CAT(0) spaces given in [11].

Lemma 2.1 ([11]). *Let X be a CAT(0) space. Then*

(i) *(X, d) is uniquely geodesic.*
(ii) *Let p, x, y be points of X and $\alpha \in [0, 1]$. Let n_1 and n_2 be the points in $[p, x]$ and $[p, y]$, respectively, satisfying $d(p, n_1) = \alpha\, d(p, x)$ and $d(p, n_2) = \alpha\, d(p, y)$. Then*

$$d(n_1, n_2) \leqslant \alpha d(x, y).$$

(iii) *Let* $x, y \in X$, $x \neq y$ *and* $z, w \in [x, y]$ *such that* $d(x, z) = d(x, w)$. *Then* $z = w$.

(iv) *Let* $x, y \in X$. *For each* $t \in [0, 1]$, *there exists a unique point* $z \in [x, y]$ *such that*

$$d(x,z) = t\, d(x,y) \quad \textit{and} \quad d(y,z) = (1-t)\, d(x,y). \tag{A}$$

We use the notation $(1-t)x \oplus ty$ *for the unique point* z *satisfying* (A).

(v) *For* $x, y, z \in X$ *and* $t \in [0, 1]$, *we have*

$$d((1-t)x \oplus ty, z) \leqslant (1-t)\, d(x,z) + t\, d(y,z).$$

Now, we give some definitions needed in the sequel.

Definition 2.2. Let T be a self-mapping on a nonempty subset C of X. Denote the set of fixed points of T by $F(T) = \{x \in C : T(x) = x\}$. We say that T is:

(1) nonexpansive if $d(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in C$;

(2) asymptotically nonexpansive ([12]) if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $d(T^n x, T^n y) \leqslant (1 + r_n) d(x, y)$ for all $x, y \in C$ and $n \geqslant 1$;

(3) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leqslant L\, d(x, y)$ for all $x, y \in C$ and $n \geqslant 1$;

(4) semi-compact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$;

(5) total asymptotically nonexpansive mapping ([7]) if there exist non-negative real sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\psi\colon [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$d(T^n x, T^n y) \leqslant d(x, y) + \nu_n \psi(d(x, y)) + \mu_n$$

for all $x, y \in C$ and $n \geqslant 1$.

Remark 2.3. In view of Definition 2.2, it is clear that each nonexpansive mapping is an asymptotically nonexpansive mapping with the constant sequence $\{k_n\} = \{1\}$ for all $n \geqslant 1$ and each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\mu_n = 0$, $\nu_n = k_n - 1$ for all $n \geqslant 1$, $\psi(t) = t$, $t \geqslant 0$.

Let C be a nonempty closed subset of a metric space X. Recall that C is said to be a retract of X if there exists a continuous mapping $P\colon X \to C$ such that $P(x) = x$ for $x \in C$. A mapping $P\colon X \to C$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $P(y) = y$ for all y in the range of P.

Definition 2.4 ([8])**.** Let C be a nonempty closed subset of a metric space X. Let $P\colon X \to C$ be a nonexpansive retraction of X onto C. A mapping $T\colon C \to X$ is said to be:

(i) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leqslant (1 + r_n) d(x, y)$ for all $x, y \in C$ and $n \geqslant 1$;

(ii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leqslant L\, d(x, y)$ for all $x, y \in C$ and $n \geqslant 1$;

(iii) total asymptotically nonexpansive mapping ([38]) if there exist non-negative real sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\psi\colon [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leqslant d(x, y) + \nu_n \psi(d(x, y)) + \mu_n$$

for all $x, y \in C$ and $n \geqslant 1$.

We now define the following.

Definition 2.5. A non-self-mapping $T\colon C \to X$ is said to be total asymptotically nonexpansive mapping ([38]) if there exist non-negative real sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$d((PT)^n x, (PT)^n y) \leqslant d(x,y) + \nu_n \psi(d(x,y)) + \mu_n$$

for all $x, y \in C$ and $n \geqslant 1$.

Remark 2.6. If $T\colon C \to X$ is total asymptotically nonexpansive in light of Definition 2.4 (iii) and $P\colon X \to C$ is a nonexpansive retraction, then $PT\colon C \to C$ is total asymptotically nonexpansive in light of $d(T^n x, T^n y) \leqslant d(x,y) + \nu_n \psi(d(x,y)) + \mu_n$ for all $x, y \in C$ and $n \geqslant 1$. Indeed by Definition 2.4 (iii), we have

$$\begin{aligned} d((PT)^n x, (PT)^n y) &= d(PT(PT)^{n-1}x, PT(PT)^{n-1}y) \\ &\leqslant d(T(PT)^{n-1}x, T(PT)^{n-1}y) \\ &\leqslant d(x,y) + \nu_n \psi(d(x,y)) + \mu_n \end{aligned}$$

for all $x, y \in C$ and $n \geqslant 1$. Conversely, it may not be true.

Let C be a nonempty closed subset of a CAT(0) space X and let $T_i\colon C \to X$ be total $(\nu_n^{(i)}, \mu_n^{(i)}, \psi^{(i)})$-total asymptotically nonexpansive non-self-mappings for each $i = 1, 2$. We consider the following iteration scheme:

$$\begin{aligned} x_1 &\in C, \\ y_n &= P[(1-\beta_n)x_n \oplus \beta_n (PT_1)^n x_n], \\ x_{n+1} &= P[(1-\alpha_n)(PT_1)^n x_n \oplus \alpha_n (PT_2)^n y_n],\ n \geqslant 1, \end{aligned} \tag{2.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0,1). This scheme is called modified Agarwal et al. [2] scheme for two non-self-mappings, where P is same as in Definition 2.4.

Remark 2.7.

(i) If we take $T_1 = T_2 = T$ and $P = I$, the identity mapping, then iteration scheme (2.2) reduces to the modified Agarwal et al. [2] iteration scheme for total asymptotically nonexpansive self-mapping.

(ii) By taking $T_1^n = T_2^n = T$ for all $n \geqslant 1$ and $P = I$, the identity mapping, then iteration scheme (2.2) reduces to S-iteration scheme ([2]) for nonexpansive self-mapping.

Let $\{x_n\}$ be a bounded sequence in a closed convex subset C of a CAT(0) space X. For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \big\{x \in X : r(\{x_n\}) = r(x, \{x_n\})\big\}.$$

It is known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [10, Proposition 7].

We now recall the definition of Δ-convergence and weak convergence ($\rightharpoonup$) in CAT(0) space.

Definition 2.8 ([22]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In the Banach space it is known that, every bounded sequence has a regular subsequence [13, Lemma 15.2].

Since in a CAT(0) space every regular sequence Δ-converges, we see that every bounded sequence in X has a Δ-convergent subsequence, also it is noticed that [22, p.3690].

Lemma 2.9 ([1]). *Given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$, then*

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y).$$

In a Banach space the above condition is known as the Opial property.

Now, recall the definition of weak convergence in a CAT(0) space.

Definition 2.10 ([15]). Let C be a closed convex subset of a CAT(0) space X. A bounded sequence $\{x_n\}$ in C is said to converge weakly to $q \in C$ if and only if $\Phi(q) = \inf_{x \in C} \Phi(x)$, where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

Note that $\{x_n\} \rightharpoonup q$ if and only if $A_C\{x_n\} = \{q\}$.

Nanjaras and Panyanak [25] established the following relation between Δ-convergence and weak convergence in a CAT(0) space.

Lemma 2.11 ([25, Proposition 3.12]). *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let C be a closed convex subset of X which contains $\{x_n\}$. Then*

(i) $\Delta\text{-}\lim_n x_n = x$ *implies* $x_n \rightharpoonup x$.

(ii) *The converse of* (i) *is true if $\{x_n\}$ is regular.*

Lemma 2.12 ([11, Lemma 2.8]). *If $\{x_n\}$ is a bounded sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.13 ([9, Proposition 2.1]). *If C is a closed convex subset of a CAT(0) space X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.14 ([7, Theorem 3.8]). *Let C be closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = p$. Then $Tp = p$.*

Lemma 2.15 ([35]). *Suppose that $\{a_n\}$, $\{b_n\}$, and $\{r_n\}$ are sequences of nonnegative numbers such that $a_{n+1} \leqslant (1 + b_n)a_n + r_n$ for all $n \geqslant 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.16 ([7]). *Let X be a CAT(0) space and $x \in X$ a given point. Suppose that $\{t_n\}$ is a sequence in $[a, b]$ with $a, b \in (0, 1)$ and $0 < a(1 - b) \leqslant \frac{1}{2}$. If $\{x_n\}$ and $\{y_n\}$ are any sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \leqslant r, \ \limsup_{n\to\infty} d(y_n, x) \leqslant r,$$

and

$$\lim_{n\to\infty} d((1 - t_n)x_n \oplus t_n y_n, x) = r$$

for some $r \geqslant 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

In 2013, Yang and Zhao [38] established the following existence theorem besides results on convexity and closedness of a fixed point set in a CAT(0) spaces in respect of total asymptotically nonexpansive non-self-mappings.

Theorem 2.17 ([38]). *Let X be a complete CAT(0) space and C be a nonempty bounded closed and convex subset of X. If $T\colon C \to X$ is a uniformly L-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \psi)$-total asymptotically nonexpansive non-self-mapping, then T has a fixed point in C and the set of fixed points is closed and convex.*

Recently, Imdad and Dashputre established the following result.

Theorem 2.18 ([16, Theorem 2.17]). *Let C be a nonempty closed and convex subset of a complete CAT(0) space X, $T\colon C \to X$ a uniformly L-Lipschitzian and $(\{\nu_n\}, \{\mu_n\}, \psi)$-total asymptotically nonexpansive non-self-mapping. If $\{x_n\}$ is a bounded sequence in C which Δ-converges to x and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in C$ and $x = Tx$.*

3. Main results

In this section, we establish Δ-convergence and strong convergence theorems of newly defined iteration scheme (2.2) for $(\nu_n^{(i)}, \mu_n^{(i)}, \psi^{(i)})$-total asymptotically nonexpansive non-self-mappings for each $i = 1, 2$ in the setting of CAT(0) spaces. Let $F = F(T_1) \cap F(T_2)$ be the set of common fixed points of the mappings T_1 and T_2.

Theorem 3.1. *Let C be a nonempty closed convex and bounded subset of a complete CAT(0) space X. Let $T_i \colon C \to X$ be uniformly L_i-Lipschitzian and $(\nu_n^{(i)}, \mu_n^{(i)}, \psi^{(i)})$-total asymptotically nonexpansive non-self-mappings for each $i = 1, 2$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$, $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$, $i = 1, 2$;
(ii) *there exist constants $M_i > 0$, $M_i^* > 0$ such that $\psi^{(i)}(\lambda) \leqslant M_i^* \lambda$ for all $\lambda \geqslant M$, $i = 1, 2$;*
(iii) *there exist constants $a_1, a_2 \in (0, 1)$ with $0 < a_1(1 - a_2) \leqslant \frac{1}{2}$ such that $0 < a_1 \leqslant \alpha_n, \beta_n \leqslant a_2 < 1$.*

Suppose that $\{x_n\}$ is defined by (2.2). If $F = F(T_1) \cap F(T_2) \neq \emptyset$, then the following hold:

(1) $\lim_{n\to\infty} d(x_n, q)$ *exists for each $q \in F$;*
(2) $\lim_{n\to\infty} d(x_n, T_1 x_n) = 0$ *and* $\lim_{n\to\infty} d(x_n, T_2 x_n) = 0$;
(3) *the sequence $\{x_n\}$ Δ-converges to a common fixed point of T_1 and T_2.*

Proof.

(1) Using (ii) and strictly increasing function $\psi^{(i)}$ for $i = 1, 2$, we obtain

$$\psi^{(i)}(\lambda) \leqslant \psi^{(i)}(M_i) + \lambda M_i^* \text{ for } i = 1, 2.$$

Set $\nu_n = \max\{\nu_n^{(1)}, \nu_n^{(2)}\}$, $\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}\}$, $L = \max\{L_1, L_2\}$, $M = \max\{M_1, M_2\}$, $M^* = \max\{M_1^*, M_2^*\}$, and $\psi = \max\{\psi^{(1)}, \psi^{(2)}\}$. Since $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$, $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$, $i = 1, 2$, we know that $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$. Since T_i $(i = 1, 2)$ are total asymptotically nonexpansive non-self-mappings, by Lemma 2.1 (v) and (2.2), for each $q \in F$, we have

$$\begin{aligned} d(y_n, q) &= d(P((1-\beta_n)x_n \oplus \beta_n (PT_1)^n x_n), P(q)) \\ &\leqslant d((1-\beta_n)x_n \oplus \beta_n (PT_1)^n x_n, q) \\ &\leqslant (1-\beta_n) d(x_n, q) + \beta_n d((PT_1)^n x_n, q) \\ &\leqslant (1-\beta_n) d(x_n, q) + \beta_n [d(x_n, q) + \nu_n \psi(d(x_n, q)) + \mu_n] \\ &\leqslant (1-\beta_n) d(x_n, q) + \beta_n [d(x_n, q) + \nu_n \psi(M) + \nu_n M^* d(x_n, q) + \mu_n] \\ &\leqslant (1 + \nu_n M^*) d(x_n, q) + \beta_n \nu_n \psi(M) + \beta_n \mu_n \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} d(x_{n+1}, q) &= d(P((1-\alpha_n)(PT_1)^n x_n \oplus \alpha_n (PT_2)^n y_n), P(q)) \\ &\leqslant d((1-\alpha_n)(PT_1)^n x_n \oplus \alpha_n (PT_2)^n y_n, q) \\ &\leqslant (1-\alpha_n) d((PT_1)^n x_n, q) + \alpha_n d((PT_2)^n y_n, q) \\ &\leqslant (1-\alpha_n)[d(x_n, q) + \nu_n \psi(d(x_n, q)) + \mu_n] \\ &\quad + \alpha_n [d(y_n, q) + \nu_n \psi(d(y_n, q)) + \mu_n] \\ &\leqslant (1-\alpha_n)[d(x_n, q) + \nu_n \psi(M) + \nu_n M^* d(x_n, q) + \mu_n] \\ &\quad + \alpha_n [d(y_n, q) + \nu_n \psi(M) + \nu_n M^* d(y_n, q) + \mu_n] \\ &= (1-\alpha_n)[(1 + \nu_n M^*) d(x_n, q) + \nu_n \psi(M) + \mu_n] \\ &\quad + \alpha_n [(1 + \nu_n M^*) d(y_n, q) + \nu_n \psi(M) + \mu_n] \\ &= (1-\alpha_n)(1 + \nu_n M^*) d(x_n, q) + \nu_n \psi(M) + \mu_n + \alpha_n (1 + \nu_n M^*) d(y_n, q). \end{aligned} \tag{3.2}$$

Inserting (3.1) into (3.2), we get

$$\begin{aligned} d(x_{n+1},q) &\leqslant (1-\alpha_n)(1+\nu_n M^*)d(x_n,q)+\nu_n\psi(M)+\mu_n \\ &\quad +\alpha_n(1+\nu_n M^*)[(1+\nu_n M^*)d(x_n,q)+\beta_n\nu_n\psi(M)+\beta_n\mu_n] \\ &\leqslant (1-\alpha_n)(1+\nu_n M^*)^2 d(x_n,q)+\nu_n\psi(M)+\mu_n \\ &\quad +\alpha_n(1+\nu_n M^*)^2 d(x_n,q)+\alpha_n\beta_n\nu_n(1+\nu_n M^*)\psi(M)+\alpha_n\beta_n\mu_n(1+\nu_n M^*) \\ &= (1+\nu_n M^*)^2 d(x_n,q)+(\nu_n\psi(M)+\mu_n)\times[1+\alpha_n\beta_n(1+\nu_n M^*)] \\ &\leqslant (1+\nu_n R^*)d(x_n,q)+\theta_n, \end{aligned} \tag{3.3}$$

where $\theta_n = (\nu_n\psi(M)+\mu_n)[1+\alpha_n\beta_n(1+\nu_n M^*)]$ and for some $R^* > 0$. Since $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$, it follows that $\sum_{n=1}^{\infty}\theta_n < \infty$. Hence by Lemma 2.15, $\lim_{n\to\infty} d(x_n,q)$ exists for each $q \in F$. This completes the proof of part (1).

(2) We show that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0,\ i = 1,2.$$

From the proof of part (1), we obtain $\lim_{n\to\infty} d(x_n,q)$ exists. We may assume that

$$\lim_{n\to\infty} d(x_n,q) = l \geqslant 0. \tag{3.4}$$

Taking the limsup on both sides in (3.1), we have

$$\limsup_{n\to\infty} d(y_n,q) \leqslant l. \tag{3.5}$$

Since

$$\begin{aligned} d((PT_1)^n x_n,q) &\leqslant d(x_n,q)+\nu_n\psi(d(x_n,q))+\mu_n \\ &\leqslant d(x_n,q)+\nu_n\psi(M)+\nu_n M^* d(x_n,q)+\mu_n \\ &= (1+\nu_n M^*)d(x_n,q)+\nu_n\psi(M)+\mu_n, \end{aligned}$$

we have that

$$\limsup_{n\to\infty} d((PT_1)^n x_n,q) \leqslant l. \tag{3.6}$$

Hence

$$\limsup_{n\to\infty} d((PT_2)^n y_n,q) \leqslant \limsup_{n\to\infty}[(1+\nu_n M^*)d(y_n,q)+\nu_n\psi(M)+\mu_n] \leqslant l.$$

Since

$$\begin{aligned} \lim_{n\to\infty} d(x_{n+1},q) &= \lim_{n\to\infty} d(P((1-\alpha_n)(PT_1)^n x_n \oplus \alpha_n(PT_2)^n y_n),q) \\ &\leqslant \lim_{n\to\infty}[(1+\nu_n R^*)d(x_n,q)+\theta_n] \leqslant l. \end{aligned}$$

It follows from Lemma 2.16 that

$$\lim_{n\to\infty} d((PT_1)^n x_n,(PT_2)^n y_n) = 0. \tag{3.7}$$

From (2.2) and (3.7), we have

$$\begin{aligned} d(x_{n+1},(PT_1)^n x_n) &= d(P((1-\alpha_n)(PT_1)^n x_n \oplus \alpha_n(PT_2)^n y_n),(PT_1)^n x_n) \\ &= d((1-\alpha_n)(PT_1)^n x_n \oplus \alpha_n(PT_2)^n y_n,(PT_1)^n x_n) \\ &\leqslant \alpha_n d((PT_1)^n x_n,(PT_2)^n y_n) \\ &\leqslant a_2 d((PT_1)^n x_n,(PT_2)^n y_n) \\ &\to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.8}$$

Now

$$d(x_{n+1},(PT_2)^n y_n) \leqslant d(x_{n+1},(PT_1)^n x_n) + d((PT_1)^n x_n,(PT_2)^n y_n),$$

so that from (3.7) and (3.8), we have

$$\lim_{n\to\infty} d(x_{n+1},(PT_2)^n y_n) = 0. \tag{3.9}$$

Also,

$$\begin{aligned} d(x_{n+1},q) &\leqslant d(x_{n+1},(PT_2)^n y_n) + d((PT_2)^n y_n,q) \\ &\leqslant d(x_{n+1},(PT_2)^n y_n) + (1+\nu_n M^*)d(y_n,q) + \nu_n\psi(M) + \mu_n, \end{aligned}$$

using (3.9), which gives that

$$l \leqslant \liminf_{n\to\infty} d(y_n,q). \tag{3.10}$$

Using (3.5) and (3.10), we obtain

$$l = \lim_{n\to\infty} d(y_n,q) = \lim_{n\to\infty} d(P((1-\beta_n)x_n \oplus \beta_n(PT_1)^n x_n),q). \tag{3.11}$$

From (3.4), (3.6), (3.11), and Lemma 2.16, we obtain

$$\lim_{n\to\infty} d(x_n,(PT_1)^n x_n) = 0. \tag{3.12}$$

Observe that

$$\begin{aligned} d(x_{n+1},(PT_2)^n x_n) &\leqslant d(x_{n+1},(PT_2)^n y_n) + d((PT_2)^n y_n,(PT_2)^n x_n) \\ &\leqslant d(x_{n+1},(PT_2)^n y_n) + d(x_n,y_n) + \nu_n\psi(d(x_n,y_n)) + \mu_n \\ &\leqslant d(x_{n+1},(PT_2)^n y_n) + d(x_n,y_n) + \nu_n\psi(M) + \nu_n M^* d(x_n,y_n) + \mu_n \\ &= d(x_{n+1},(PT_2)^n y_n) + (1+\nu_n M^*)d(x_n,y_n) + \nu_n\psi(M) + \mu_n, \end{aligned} \tag{3.13}$$

where

$$d(x_n,y_n) = d(x_n,P((1-\beta_n)x_n \oplus \beta_n(PT_1)^n x_n)) \leqslant \beta_n d(x_n,(PT_1)^n x_n) \leqslant a_2 d(x_n,(PT_1)^n x_n).$$

It follows from (3.12) that

$$\lim_{n\to\infty} d(x_n,y_n) = 0. \tag{3.14}$$

Thus, from (3.9), (3.13), and (3.14), we have

$$\lim_{n\to\infty} d(x_{n+1},(PT_2)^n x_n) = 0. \tag{3.15}$$

In addition, since

$$d(x_{n+1},x_n) \leqslant d(x_{n+1},(PT_1)^n x_n) + d((PT_1)^n x_n,x_n), \tag{3.16}$$

using (3.8) and (3.12) in (3.16), we obtain

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0. \tag{3.17}$$

Again note that

$$d(x_n,(PT_2)^n x_n) \leqslant d(x_n,x_{n+1}) + d(x_{n+1},(PT_2)^n x_n).$$

Using (3.15) and (3.17) in the above inequality, we obtain

$$\lim_{n\to\infty} d(x_n,(PT_2)^n x_n) = 0. \tag{3.18}$$

Finally, since T_i $(i = 1, 2)$ is uniformly L_i-Lipschitzian, we have

$$\begin{aligned} d(x_n, T_1x_n) &\leqslant d(x_{n+1}, x_n) + d(x_{n+1},(PT_1)^{n+1}x_{n+1}) + d((PT_1)^{n+1}x_{n+1},(PT_1)^{n+1}x_n) \\ &\quad + d((PT_1)^{n+1}x_n, T_1x_n) \\ &\leqslant (1 + L)d(x_{n+1}, x_n) + d(x_{n+1},(PT_1)^{n+1}x_{n+1}) + Ld((PT_1)^n x_n, x_n). \end{aligned} \tag{3.19}$$

Using (3.12) and (3.17) in (3.19), we obtain

$$\lim_{n\to\infty} d(x_n, T_1x_n) = 0. \tag{3.20}$$

Similarly, since

$$\begin{aligned} d(x_n, T_2x_n) &\leqslant d(x_{n+1}, x_n) + d(x_{n+1},(PT_2)^{n+1}x_{n+1}) \\ &\quad + d((PT_2)^{n+1}x_{n+1},(PT_2)^{n+1}x_n) + d((PT_2)^{n+1}x_n, T_2x_n) \\ &\leqslant (1 + L)d(x_{n+1}, x_n) + d(x_{n+1},(PT_2)^{n+1}x_{n+1}) + Ld((PT_2)^n x_n, x_n). \end{aligned}$$

Using (3.17) and (3.18) in (3.20), we obtain

$$\lim_{n\to\infty} d(x_n, T_2x_n) = 0. \tag{3.21}$$

This completes the proof of part (2).

(3) Now, we show that the sequence $\{x_n\}$ Δ-converges to a common fixed point of T_1 and T_2.

Let $W_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $W_w(x_n) \subseteq F$ and $W_w(x_n)$ consists of exactly one point. Let $u \in W_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.13, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. In view of (3.20) and (3.21), $\lim_{n\to\infty} d(x_n, T_1x_n) = 0$ and $\lim_{n\to\infty} d(x_n, T_2x_n) = 0$. It follows from Theorem 2.18 that $v \in F = F(T_1) \cap F(T_2)$, so by part (1) of Theorem 3.1, the $\lim_{n\to\infty} d(x_n, v)$ exists. By Lemma 2.12, $u = v \in F$. This implies that $W_w(x_n) \subseteq F$.

To show that $\{x_n\}$ Δ-converges to a point in F, it is sufficient to show that $W_w(x_n)$ consists of exactly one point.

Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. Since $w \in W_w(x_n) \subseteq F$ and by part (1) of Theorem 3.1, $\lim_{n\to\infty} d(x_n, w)$ exists. Again by Lemma 2.12, we have $x = w \in F$. This implies that $W_w(x_n)$ consists of exactly one point. This shows that $\{x_n\}$ Δ-converges to a common fixed point of T_1 and T_2. This completes the proof of part (3). □

Remark 3.2. Theorem 3.1 extends the corresponding results of Chang et al. [7], Khan and Abbas [19], and Yang and Zhao [38] to the case of modified Agarwal et al. iteration scheme [2] and two total asymptotically nonexpansive non-self-mappings.

We can get the following corollary for the self-mappings.

Corollary 3.3. *Let* C *and* X *be the same as in Theorem* 3.1 *and let* $T_i\colon C \to C$ *be uniformly* L_i*-Lipschitzian and* $(\nu_n^{(i)}, \mu_n^{(i)}, \psi^{(i)})$*-total asymptotically nonexpansive mappings for each* $i = 1, 2$*. If* $F = F(T_1) \cap F(T_2) \neq \emptyset$ *and the conditions* (i)-(iii) *in Theorem* 3.1 *are satisfied, then the sequence* $\{x_n\}$ *defined by*

$$\begin{aligned} x_1 &\in C, \\ y_n &= (1 - \beta_n)x_n \oplus \beta_n T_1^n x_n, \\ x_{n+1} &= (1 - \alpha_n)T_1^n x_n \oplus \alpha_n T_2^n y_n,\ n \geqslant 1, \end{aligned} \tag{3.22}$$

is Δ*-convergent to a common fixed point of* T_1 *and* T_2*.*

Proof. Since T_i, $i = 1,2$ is a self-mapping from C into C, take $P = I$ (the identity mapping on C), then $(PT_i)^n = T_i^n$ for $i = 1,2$. The conclusion of Corollary 3.3 will be obtained from part (3) of Theorem 3.1. This completes the proof. □

Next, we prove some strong convergence theorems via newly proposed iteration scheme (2.2) for $(\nu_n^{(i)}, \mu_n^{(i)}, \psi^{(i)})$-total asymptotically nonexpansive non-self-mappings for each $i = 1,2$.

Theorem 3.4. *Let C and X be the same as in Theorem 3.1 and let $T_i : C \to C$ be uniformly L_i-Lipschitzian and $(\nu_n^{(i)}, \mu_n^{(i)}, \psi^{(i)})$-total asymptotically nonexpansive non-self-mappings for each $i = 1,2$. If $F = F(T_1) \cap F(T_2) \neq \emptyset$ and the conditions (i)-(iii) in Theorem 3.1 are satisfied, then the sequence $\{x_n\}$ defined by (2.2) converges strongly to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. From (3.3), we have

$$d(x_{n+1}, F) \leqslant (1 + \nu_n R^*) d(x_n, F) + \theta_n,\ n \geqslant 1.$$

By Lemma 2.16, $\lim_{n\to\infty} d(x_n, F)$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F) = 0$.

It is well-known that $(1 + t) \leqslant e^t$ for all $t \geqslant 0$, from (3.3), we obtain

$$\begin{aligned}
d(x_{n+m}, q) &\leqslant (1 + R^*\nu_{n+m-1}) d(x_{n+m-1}, q) + \theta_{n+m-1} \\
&\leqslant e^{R^*\nu_{n+m-1}} d(x_{n+m-1}, q) + \theta_{n+m-1} \\
&\leqslant e^{R^*\nu_{n+m-1}} [e^{R^*\nu_{n+m-2}} d(x_{n+m-2}, q) + \theta_{n+m-2}] + \theta_{n+m-1} \\
&\leqslant e^{[R^*\nu_{n+m-1} + R^*\nu_{n+m-2}]} d(x_{n+m-2}, q) + e^{R^*\nu_{n+m-1}} [\theta_{n+m-1} + \theta_{n+m-2}] \\
&\ \vdots \\
&\leqslant \left(e^{R^* \sum_{j=n}^{n+m-1} \nu_j}\right) d(x_n, q) + \left(e^{R^* \sum_{j=n}^{n+m-1} \nu_j}\right) \sum_{j=n}^{n+m-1} \theta_j \\
&\leqslant Q\Big[d(x_n, q) + \sum_{j=n}^{n+m-1} \theta_j\Big]
\end{aligned}$$

for each $q \in F$ and $m, n \geqslant 1$, where $Q = e^{R^* \sum_{j=n}^{n+m-1} \nu_j} > 0$. As, $R^* > 0$ and $\sum_{n=1}^{\infty} \nu_n < \infty$, so $Q^* = e^{R^* \sum_{n=1}^{\infty} \nu_n} \geqslant Q = e^{R^* \sum_{j=n}^{n+m-1} \nu_j}$. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} \theta_n < \infty$, there exists a positive integer N_0 such that

$$d(x_n, F) < \frac{\varepsilon}{4Q^*} \text{ and } \sum_{j=N_0}^{n+m+1} \theta_j < \frac{\varepsilon}{6Q^*} \text{ for } n \geqslant N_0.$$

In particular, $\inf\{d(x_{N_0}, q) : q \in F\} < \frac{\varepsilon}{4Q^*}$. Thus there must exist $z \in F$ such that

$$d(x_{N_0}, z) < \frac{\varepsilon}{3Q^*}.$$

Hence for $n \geqslant N_0$, we have

$$d(x_{n+m}, x_n) \leqslant d(x_{n+m}, z) + d(z, x_n) \leqslant 2Q^* \Big[d(x_{N_0}, z) + \sum_{j=N_0}^{\infty} \theta_j\Big] < 2Q^* \Big(\frac{\varepsilon}{3Q^*} + \frac{\varepsilon}{6Q^*}\Big) = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in closed subset C of a complete CAT(0) space X, which implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = q^*$. Since C is closed, therefore $q^* \in C$. Next, we show that $q^* \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ we get $d(q^*, F) = 0$, closedness of F gives that $q^* \in F$. Thus $\{x_n\}$ converges strongly to a point in F. This completes the proof. □

Recall the following definition.

Definition 3.5 ([34]). A mapping T from a subset C of a metric space (X, d) into itself with $F(T) \neq \emptyset$ is said to satisfy condition (A) if there exists a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $f(d(x, F(T))) \leqslant d(x, Tx)$ for all $x \in C$.

Now, we generalize the above definition for two mappings T_1 and T_2.

Definition 3.6. Two mappings T_1 and T_2 from a subset C of a metric space (X, d) into itself with $F = F(T_1) \cap F(T_2) \neq \emptyset$ are said to satisfy condition (A^*) if there exists a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $f(d(x, F)) \leqslant [A_1 d(x, T_1x) + A_2 d(x, T_2x)]$ for all $x \in C$, where A_1 and A_2 are positive real numbers such that $A_1 + A_2 = 1$.

Remark 3.7. If $T_1 = T_2 = T$, then condition (A^*) reduces to condition (A) ([34]).

As an application of Theorem 3.4, we establish another strong convergence result employing condition (A^*).

Theorem 3.8. *Let C and X be the same as in Theorem 3.1 and let $T_i\colon C \to C$ be uniformly L_i-Lipschitzian and $(\nu_n^{(i)}, \mu_n^{(i)}, \psi^{(i)})$-total asymptotically nonexpansive non-self-mappings for each $i = 1, 2$. Suppose $F = F(T_1) \cap F(T_2) \neq \emptyset$ and the conditions (i)-(iii) in Theorem 3.1 are satisfied and T_1 and T_2 satisfy the condition (A^*). Then the sequence $\{x_n\}$ defined by (2.2) converges strongly to a common fixed point of the mappings T_1 and T_2.*

Proof. By part (2) of Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, T_ix_n) = 0$ for $i = 1, 2$. Further, by condition (A^*), $f(d(x_n, F)) \leqslant [A_1 d(x_n, T_1x_n) + A_2 d(x_n, T_2x_n)]$, so that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a non-decreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Theorem 3.4 implies that $\{x_n\}$ converges strongly to a point in F. This completes the proof. $\square$

For our next result, we need the following definition.

Definition 3.9. A mapping $T\colon C \to X$ is said to be *demi-compact* if for any sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) = 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some point $q \in C$.

Theorem 3.10. *Let C and X be the same as in Theorem 3.1 and let $T_i\colon C \to C$ be uniformly L_i-Lipschitzian and $(\nu_n^{(i)}, \mu_n^{(i)}, \psi^{(i)})$-total asymptotically nonexpansive non-self-mappings for each $i = 1, 2$. Let $F = F(T_1) \cap F(T_2) \neq \emptyset$ and the conditions (i)-(iii) in Theorem 3.1 are satisfied. If one of T_1 and T_2 is demi-compact, then the sequence $\{x_n\}$ defined by (2.2) converges strongly to a common fixed point of the mappings T_1 and T_2.*

Proof. By part (2) of Theorem 3.1, we know that $\lim_{n\to\infty} d(x_n, T_ix_n) = 0$ for $i = 1, 2$, and one of T_1 and T_2 is demi-compact, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some point $p \in C$. Moreover, by the uniform continuity of T_1 and T_2, for each $i = 1, 2$, we have

$$d(T_i(p), p) \leqslant d(T_i(p), T_i(x_{n_k})) + d(T_i(x_{n_k}), x_{n_k}) + d(x_{n_k}, p) \to 0 \text{ as } k \to \infty.$$

This implies that $p \in F = F(T_1) \cap F(T_2)$. Again by part (1) of Theorem 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$, that is, $\{x_n\}$ converges strongly to a common fixed point of the mappings T_1 and T_2. This completes the proof. $\square$

Corollary 3.11. *Under the assumptions of Corollary 3.3, if one of T_1 and T_2 is demi-compact, then the sequence $\{x_n\}$ defined by (3.22) converges strongly to a common fixed point of the mappings T_1 and T_2.*

Remark 3.12.

(i) In view of Remark 2.3, Theorems 3.1, 3.4, and 3.8 extend the corresponding results of Chidume et al. [8] from Banach spaces to CAT(0) spaces. They also extend the corresponding results of Dhompongsa et al. [11] from the class of nonexpansive mappings to the class of total asymptotically nonexpansive non-self-mappings.

(ii) Theorems 3.1, 3.4, and 3.8 extend Lemmas 3.1, 3.2, and Theorems 3.1, 3.3, 3.4 of Saluja [30] from Banach spaces to CAT(0) spaces.

(iii) Theorems 3.1, 3.4, and 3.8 extend Lemma 3.1, 3.3, and Theorems 3.2, 3.4, 3.5, 3.9 of Saluja [28] for asymptotically nonexpansive non-self-mappings.

(iv) Our results also extend and improve the corresponding results contained in [4, 7, 16, 19, 23, 32, 36, 38] and many others from the existing literature.

Now, we give some examples in support of our result.

Example 3.13 ([14])**.** Let $X = \mathbb{R}$ be the real line with the usual metric $d(x, y) = |x - y|$, $C = [-1, 1]$ and P be the identity mapping. For each $x \in C$, define two mappings T_1, $T_2 \colon C \to C$ by

$$T_1(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0), \end{cases} \quad \text{and} \quad T_2(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

In [14], the authors proved that both T_1 and T_2 are asymptotically nonexpansive mappings with $k_n = 1$ for all $n \geqslant 1$. Therefore, they are total asymptotically nonexpansive mappings with $\mu_n = \nu_n = 0$. Additionally, they are uniformly L-Lipschitzian mappings with $L = \sup_{n\geqslant 1}(k_n)$. It is clear that $F(T_1) = 0$ and $F(T_2) = \{0 \leqslant x \leqslant 1\}$, that is, the unique common fixed point set $F = F(T_1) \cap F(T_2) = \{0\}$. Set $\alpha_n = \frac{n}{2n+1}$ and $\beta_n = \frac{n}{3n+1}$. Thus, the conditions of Theorem 3.4 are fulfilled. Hence the iterative sequence $\{x_n\}$ defined by (2.2) converges strongly to 0.

Example 3.14. Let $X = C = [0, 1]$ with the usual metric $d(x, y) = |x - y|$, $\{x_n\} = \{\frac{1}{n}\}$, $\{u_{n_k}\} = \{\frac{1}{k^n}\}$ for all $n, k \geqslant 1$ are sequences in C. Then $A(\{x_n\}) = \{0\}$ and $A(\{u_{n_k}\}) = \{0\}$. This shows that $\{x_n\}$ Δ-converges to 0, that is, $\Delta - \lim_{n\to\infty} x_n = 0$. The sequence $\{x_n\}$ also converges strongly to 0, that is, $|x_n - 0| \to 0$ as $n \to \infty$. Also it is weakly convergent to 0, that is, $x_n \rightharpoonup 0$ as $n \to \infty$, by Lemma 2.11. Thus, we conclude that

$$\text{strong convergence} \quad \Rightarrow \quad \Delta\text{-convergence} \quad \Rightarrow \quad \text{weak convergence},$$

but the converse is not true in general.

The following example shows that, if the sequence $\{x_n\}$ is weakly convergent, then it is not Δ-convergent.

Example 3.15 ([25])**.** Let $X = \mathbb{R}$, d be the usual metric on X, $C = [-1, 1]$, $\{x_n\} = \{1, -1, 1, -1, \dots\}$, $\{u_n\} = \{-1, -1, -1, \dots\}$, and $\{v_n\} = \{1, 1, 1, \dots\}$. Then $A(\{x_n\}) = A_C(\{x_n\}) = \{0\}$, $A(\{u_n\}) = \{-1\}$, and $A(\{v_n\}) = \{1\}$. This shows that $\{x_n\} \rightharpoonup 0$ as $n \to \infty$ ($\rightharpoonup$ means weakly) but it does not have a Δ-limit.

4. Conclusion

In this paper, we proposed and studied a new two-step iteration scheme for two total asymptotically nonexpansive non-self-mappings in CAT(0) spaces and established a Δ-convergence and some strong convergence results for said scheme and mappings in the setting of CAT(0) spaces. Our results extend, improve, and generalize the corresponding results of [4, 7, 8, 11, 16, 19, 23, 28, 30, 32, 36, 38] and many others.

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