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An existence theorem on Hamiltonian (g, f)-factors in networks

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Abstract

Let a, b, and r be nonnegative integers with $\max\{3, r+1\} \leqslant a < b-r$, let G be a graph of order n, and let g and f be two integer-valued functions defined on $V(G)$ with $\max\{3, r+1\} \leqslant a \leqslant g(x) < f(x) - r \leqslant b - r$ for any $x \in V(G)$. In this article, it is proved that if $n \geqslant \frac{(a+b-3)(a+b-5)+1}{a-1+r}$ and $\text{bind}(G) \geqslant \frac{(a+b-3)(n-1)}{(a-1+r)n-(a+b-3)}$, then G admits a Hamiltonian (g, f)-factor.

Keywords: Network, graph, binding number, (g, f)-factor, Hamiltonian (g, f)-factor.

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1. Introduction

Many real-world networks can conveniently be modeled by graphs or networks. Examples include a railroad network with nodes presenting railroad stations, and links corresponding to railways between two stations, or a communication network with nodes and links modeling cities and communication channels, or the world wide web with nodes presenting web pages, and links corresponding to hyperlinks between web pages, respectively. In our daily life, many problems on network design and optimization, e.g., scheduling problems, the file transfer problems on computer networks, building blocks, coding design and so on, are related to the factors and factorizations in graphs [1]. For example, file transfer problem in computer networks can be converted into factorizations of graphs. The problem on telephone network design can be converted into 1-factors (or P2-factors, or K2- factors) in graphs. Many other applications in this field can be found in a current survey [1]. It is well-known that a graph can represent a network. Vertices and edges of the graph model nodes and links between the nodes in the network. Henceforth we use the term graph instead of network.

In this article, we consider finite undirected graphs which have neither multiple edges nor loops. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For a vertex x of G, we use $d_G(x)$ to denote the degree of x in G. For any set S of vertices of G, we use $N_G(S)$ to denote the set of vertices in G which are adjacent to vertices in S; we denote by $G[S]$ the subgraph of G induced by S and write $G - S = G[V(G) \setminus S]$. The minimum degree of G is denoted by $\delta(G)$. The binding

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number $\mathrm{bind}(G)$ of G is denoted by

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

Let g and f be two integer-valued functions defined on $V(G)$ satisfying $0 \leqslant g(x) \leqslant f(x)$ for any $x \in V(G)$. We define a (g, f)-factor of G a spanning subgraph F of G such that $g(x) \leqslant d_F(x) \leqslant f(x)$ for any $x \in V(G)$. A (g, f)-factor F of a graph G is said to be a Hamiltonian (g, f)-factor if F includes a Hamiltonian cycle. A Hamiltonian (g, f)-factor is called a Hamiltonian $[a, b]$-factor if $g(x) = a$ and $f(x) = b$ hold for any $x \in V(G)$. For convenience, we write $f(S) = \sum_{x \in S} f(x)$ for any function f defined on $V(G)$ and $f(\emptyset) = 0$.

Factors and factorizations in graphs have been investigated by many authors [3–7, 9, 11, 13, 14, 16–18]. Matsuda [10] showed a degree condition for a Hamiltonian graph G to have a Hamiltonian $[a, b]$-factor. Zhou [15] studied the existence of Hamiltonian $[a, b]$-factors in Hamiltonian graphs depending on toughness. Cai and Liu [2] gave a binding number condition for a graph to have a Hamiltonian (g, f)-factor. The following results on Hamiltonian factors are known.

Theorem 1.1 ([12])**.** *Let G be a graph with $\mathrm{bind}(G) \geqslant \frac{3}{2}$. Then G has a Hamiltonian cycle (or a Hamiltonian 2-factor).*

Theorem 1.2 ([2])**.** *Let G be a connected graph of order n, and a, b be integers with $4 \leqslant a < b$. Let g and f be integer-valued functions defined on $V(G)$ satisfying $a \leqslant g(x) < f(x) \leqslant b$ for every $x \in V(G)$. If $n \geqslant \frac{(a+b-5)^2}{a-2}$, $\mathrm{bind}(G) \geqslant \frac{(a+b-5)(n-1)}{(a-2)n-3(a+b-5)}$, and for any non-empty independent subset X of $V(G)$, $|N_G(X)| \geqslant \frac{(b-3)n+(2a+2b-9)|X|}{a+b-5}$, then G contains a Hamiltonian (g, f)-factor.*

In the following, we give our main result in this article, which is an extension of Theorem 1.1 and an improvement of Theorem 1.2.

Theorem 1.3. *Let a, b and r be nonnegative integers with $\max\{3, r+1\} \leqslant a < b - r$, and let G be a graph of order n with $n \geqslant \frac{(a+b-3)(a+b-5)+1}{a-1+r}$. Let g and f be integer-valued functions defined on $V(G)$ such that $a \leqslant g(x) < f(x) - r \leqslant b - r$ for any $x \in V(G)$. If $\mathrm{bind}(G) \geqslant \frac{(a+b-3)(n-1)}{(a-1+r)n-(a+b-3)}$, then G admits a Hamiltonian (g, f)-factor.*

Since a Hamiltonian graph is 2-edge-connected, we obtain the following corollary by Theorem 1.3.

Corollary 1.4. *Let a, b, and r be nonnegative integers with $\max\{3, r+1\} \leqslant a < b - r$, and let G be a graph of order n with $n \geqslant \frac{(a+b-3)(a+b-5)+1}{a-1+r}$. Let g and f be integer-valued functions defined on $V(G)$ such that $a \leqslant g(x) < f(x) - r \leqslant b - r$ for any $x \in V(G)$. If $\mathrm{bind}(G) \geqslant \frac{(a+b-3)(n-1)}{(a-1+r)n-(a+b-3)}$, then G contains a 2-edge-connected (g, f)-factor.*

If $r = 0$ in Theorem 1.3, then we get the following corollary.

Corollary 1.5. *Let a, b be nonnegative integers with $3 \leqslant a < b$, and let G be a graph of order n with $n \geqslant \frac{(a+b-3)(a+b-5)+1}{a-1}$. Let g and f be integer-valued functions defined on $V(G)$ such that $a \leqslant g(x) < f(x) \leqslant b$ for any $x \in V(G)$. If $\mathrm{bind}(G) \geqslant \frac{(a+b-3)(n-1)}{(a-1)n-(a+b-3)}$, then G includes a Hamiltonian (g, f)-factor.*

According to $3 \leqslant a < b$, we may verify that $\frac{(a+b-5)^2}{a-2} \geqslant \frac{(a+b-3)(a+b-5)}{a-1}$ and $\frac{(a+b-5)(n-1)}{(a-2)n-3(a+b-5)} \geqslant \frac{(a+b-3)(n-1)}{(a-1)n-(a+b-3)}$, and so, the lower bounds for n and $\mathrm{bind}(G)$ required by Corollary 1.5 are weaker than that of Theorem 1.2. Moreover, Theorem 1.2 has an extra hypothesis on the neighborhoods of independent sets. Hence, the result of Theorem 1.2 is an immediate consequence of Corollary 1.5. Thus, Theorem 1.2 is a special case of Theorem 1.3. Furthermore, the author poses the following problem.

Problem 1.6. Is it possible to weaken the binding number condition in Theorem 1.3?

2. The proof of Theorem 1.3

The proof of Theorem 1.3 relies heavily on the following lemma, which is a special case of Lovász (g, f)-factor theorem.

Lemma 2.1 ([8]). *Let G be a graph, and let g and f be two positive integer-valued functions defined on $V(G)$ satisfying $g(x) < f(x)$ for any $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geqslant 0$$

for all disjoint vertex subsets S and T of G.

Lemma 2.2 ([12]). *Let c be a positive real, and let G be a graph of order n with $\mathrm{bind}(G) \geqslant c$. Then $\delta(G) \geqslant n - \frac{n-1}{c}$.*

Proof of Theorem 1.3. We easily prove that $\mathrm{bind}(G) \geqslant \frac{(a+b-3)(n-1)}{(a-1+r)n-(a+b-3)} \geqslant \frac{3}{2}$ by $\max\{3, r+1\} \leqslant a < b-r$. Then by Theorem 1.1, G admits a Hamiltonian cycle C. Let $G' = G - E(C)$. Clearly, G has a Hamiltonian (g, f)-factor if and only if G' has a $(g-2, f-2)$-factor. Hence, we need only to verify that G' admits a $(g-2, f-2)$-factor. For convenience, set $g'(x) = g(x) - 2$ and $f'(x) = f(x) - 2$. It is easy to see that $a - 2 \leqslant g'(x) < f'(x) - r \leqslant b - 2 - r$ for any $x \in V(G')$. By contradiction, we assume that G' has no (g', f')-factor. Then by Lemma 2.1, there exist two disjoint vertex subsets S and T of G' satisfying

$$\delta_{G'}(S, T) = f'(S) + d_{G'-S}(T) - g'(T) \leqslant -1. \tag{2.1}$$

We choose such vertex subsets S and T so that $|T|$ is minimum. Obviously, $T \neq \emptyset$ by (2.1). We now verify the following claims.

Claim 1. $d_{G'-S}(x) \leqslant g'(x) - 1 \leqslant b - 3 - r$ for any $x \in T$.

Proof. Assume that $d_{G'-S}(x) \geqslant g'(x)$ for some $x \in T$. Then the vertex subsets S and $T \setminus \{x\}$ satisfy (2.1), which contradicts the choice of S and T. Hence, we have

$$d_{G'-S}(x) \leqslant g'(x) - 1 \leqslant b - 3 - r$$

for any $x \in T$. This completes the proof of Claim 1. □

Claim 2. $d_{G-S}(x) \leqslant d_{G'-S}(x) + 2 \leqslant b - 1 - r$ for any $x \in T$.

Proof. Note that $G' = G - E(C)$. Thus, we have

$$d_{G-S}(x) \leqslant d_{G'-S}(x) + 2$$

for any $x \in T$. Combining this with Claim 1, we obtain

$$d_{G-S}(x) \leqslant d_{G'-S}(x) + 2 \leqslant b - 1 - r$$

for any $x \in T$. This completes the proof of Claim 2. □

Note that $T \neq \emptyset$. Define

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

In terms of Claim 2, we obtain

$$0 \leqslant h \leqslant b - 1 - r.$$

We use $\rho := \mathrm{bind}(G)$ to simplify the notation below, and choose $u \in T$ with $d_{G-S}(u) = h$. Thus, we obtain

$$\delta(G) \leqslant d_G(u) \leqslant d_{G-S}(u) + |S| = h + |S|,$$

that is,

$$|S| \geqslant \delta(G) - h. \tag{2.2}$$

Combining this with Lemma 2.2, we have

$$|S| \geqslant n - \frac{n-1}{\rho} - h. \tag{2.3}$$

In the following, we shall consider three cases by the value of h and derive a contradiction in each case.

Case 1. $h = 0$.

Claim 3. $\rho \leqslant a + b - 3$.

Proof. Let $\rho > a + b - 3$. Then by Lemma 2.2, we have

$$\delta(G) \geqslant n - \frac{n-1}{\rho} > \frac{(a+b-4)n+1}{a+b-3}. \tag{2.4}$$

On the other hand, we obtain by (2.1), (2.2), Claim 2, and $|S| + |T| \leqslant n$ that

$$\begin{aligned}
-1 \geqslant \delta_{G'}(S,T) &= f'(S) + d_{G'-S}(T) - g'(T) \\
&\geqslant (a-1+r)|S| + d_{G-S}(T) - 2|T| - (b-3-r)|T| \\
&\geqslant (a-1+r)|S| - (b-1-r)|T| \\
&\geqslant (a-1+r)|S| - (b-1-r)(n-|S|) \\
&= (a+b-2)|S| - (b-1-r)n \geqslant (a+b-2)\delta(G) - (b-1-r)n,
\end{aligned}$$

which implies

$$\delta(G) \leqslant \frac{(b-1-r)n-1}{a+b-2},$$

which contradicts (2.4) since $3 \leqslant a < b - r$. The proof of Claim 3 is complete. □

We write $X = \{x : x \in T, d_{G-S}(x) = 0\}$. Clearly, $X \neq \emptyset$ and $N_G(V(G) \setminus S) \cap X = \emptyset$. Thus, we have

$$|N_G(V(G) \setminus S)| \leqslant n - |X|.$$

Combining this with the definition of $\mathrm{bind}(G)$, we obtain

$$\rho \leqslant \frac{|N_G(V(G)\setminus S)|}{|V(G)\setminus S|} \leqslant \frac{n-|X|}{n-|S|},$$

which implies

$$|S| \geqslant \frac{(\rho-1)n}{\rho} + \frac{|X|}{\rho}. \tag{2.5}$$

In terms of (2.1), (2.5), Claim 2, Claim 3, $|S| + |T| \leqslant n$, and the assumptions of Theorem 1.3, we have

$$\begin{aligned}
0 > \delta_{G'}(S,T) &= f'(S) + d_{G'-S}(T) - g'(T) \\
&\geqslant (a-1+r)|S| + d_{G-S}(T) - 2|T| - (b-3-r)|T| \\
&\geqslant (a-1+r)|S| + |T| - |X| - (b-1-r)|T| \\
&= (a-1+r)|S| - (b-2-r)|T| - |X| \\
&\geqslant (a-1+r)|S| - (b-2-r)(n-|S|) - |X| \\
&= (a+b-3)|S| - (b-2-r)n - |X|
\end{aligned}$$

$$
\begin{aligned}
&\geqslant (a+b-3)\left(\frac{(\rho-1)n}{\rho}+\frac{|X|}{\rho}\right)-(b-2-r)n-|X| \\
&= \frac{((a-1+r)\rho-(a+b-3))n}{\rho}+\left(\frac{a+b-3}{\rho}-1\right)|X| \\
&\geqslant \frac{((a-1+r)\rho-(a+b-3))n}{\rho}+\frac{a+b-3}{\rho}-1 \\
&> (a-1+r)n-\frac{(a+b-3)(n-1)}{\rho}-(a+b-3)\geqslant 0,
\end{aligned}
$$

which is a contradiction.

Case 2. $h=1$.

It follows from (2.1), (2.3), $|S|+|T|\leqslant n$, and Claim 2 that

$$
\begin{aligned}
0>\delta_{G'}(S,T) &= f'(S)+d_{G'-S}(T)-g'(T) \\
&\geqslant (a-1+r)|S|+d_{G-S}(T)-2|T|-(b-3-r)|T| \\
&\geqslant (a-1+r)|S|+|T|-(b-1-r)|T| \\
&= (a-1+r)|S|-(b-2-r)|T| \\
&\geqslant (a-1+r)|S|-(b-2-r)(n-|S|) \\
&= (a+b-3)|S|-(b-2-r)n \\
&\geqslant (a+b-3)\left(n-\frac{n-1}{\rho}-1\right)-(b-2-r)n \\
&= (a-1+r)n-(a+b-3)-\frac{(a+b-3)(n-1)}{\rho},
\end{aligned}
$$

which implies

$$
\rho<\frac{(a+b-3)(n-1)}{(a-1+r)n-(a+b-3)},
$$

which contradicts $\mathrm{bind}(G)\geqslant \frac{(a+b-3)(n-1)}{(a-1+r)n-(a+b-3)}$.

Case 3. $2\leqslant h\leqslant b-1-r$.

In view of (2.1), (2.3), $|S|+|T|\leqslant n$, Claim 2, and the assumptions of Theorem 1.3, we obtain

$$
\begin{aligned}
-1\geqslant \delta_{G'}(S,T) &= f'(S)+d_{G'-S}(T)-g'(T) \\
&\geqslant (a-1+r)|S|+d_{G-S}(T)-2|T|-(b-3-r)|T| \\
&\geqslant (a-1+r)|S|+h|T|-(b-1-r)|T| \\
&= (a-1+r)|S|-(b-1-r-h)|T| \\
&\geqslant (a-1+r)|S|-(b-1-r-h)(n-|S|) \\
&= (a+b-2-h)|S|-(b-1-r-h)n \\
&\geqslant (a+b-2-h)\left(n-\frac{n-1}{\rho}-h\right)-(b-1-r-h)n \\
&\geqslant (a+b-2-h)\left(n-\frac{(a-1+r)n-(a+b-3)}{a+b-3}-h\right)-(b-1-r-h)n \\
&= (a+b-2-h)\left(\frac{(b-2-r)n}{a+b-3}+1-h\right)-(b-1-r-h)n,
\end{aligned}
$$

that is,

$$
-1\geqslant (a+b-2-h)\left(\frac{(b-2-r)n}{a+b-3}+1-h\right)-(b-1-r-h)n. \tag{2.6}
$$

Set $F(h) = (a+b-2-h)(\frac{(b-2-r)n}{a+b-3}+1-h)-(b-1-r-h)n$. Then by $n \geqslant \frac{(a+b-3)(a+b-5)+1}{a-1+r}$, we may calculate

$$\begin{aligned}\frac{dF(h)}{dh} &= -(\frac{(b-2-r)n}{a+b-3}+1-h)-(a+b-2-h)+n\\ &= \frac{(a-1+r)n}{a+b-3}+2h-(a+b-1)\\ &\geqslant \frac{(a+b-3)(a+b-5)+1}{a+b-3}+4-(a+b-1)=\frac{1}{a+b-3}>0,\end{aligned}$$

and so $F(h)$ attains its minimum value at $h=2$. Hence, we obtain

$$F(h) \geqslant F(2).$$

Combining this with (2.6), we have

$$-1 \geqslant F(h) \geqslant F(2) = (a+b-4)(\frac{(b-2-r)n}{a+b-3}-1)-(b-3-r)n = \frac{(a-1+r)n}{a+b-3}-(a+b-4),$$

which implies

$$n \leqslant \frac{(a+b-3)(a+b-5)}{a-1+r},$$

which contradicts that $n \geqslant \frac{(a+b-3)(a+b-5)+1}{a-1+r}$. This completes the proof of Theorem 1.3. □

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