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Lyapunov-type inequalities for Laplacian systems and applications to boundary value problems

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Abstract

In this paper, we establish some new Lyapunov-type inequalities for a class of Laplacian systems. With these, sufficient conditions for the non-existence of nontrivial solutions to certain boundary value problems are obtained. A lower bound for the eigenvalues is also deduced.

Keywords: Lyapunov type inequality, boundary value problem, Laplacian systems.

2010 MSC: 26D15, 26D10, 34B15.

1. Introduction

For the second-order linear differential equation

$$x''(t) + q(t)x(t) = 0, \tag{1.1}$$

where $q \in C([a, b], \mathbb{R})$, the following result is known:

Theorem 1.1. *Assume $x(t)$ is a solution of Eq. (1.1) such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$. Then*

$$(b-a)\int_a^b |q(s)|ds > 4.$$

Equation (1.1) is known as the Lyapunov inequality. Since this inequality has found extensive applications in the study of various properties of solutions of many differential and difference equations, including oscillation theory, disconjugacy, and eigenvalue problems, much efforts have been devoted to the improvement and extensions of the inequality. These include, for example, works on delay differential equations, higher order differential equations, discrete differential equations, and Hamiltonian systems. See, for example, [1–4, 6, 8–10], and the references therein.

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In 2003, Yang [11] considered the following second order half linear differential equation

$$\left(r(t)\left|x'(t)\right|^{\gamma-2}x'(t)\right)' + q(t)\left|x(t)\right|^{\gamma-2}x(t) = 0, \tag{1.2}$$

and obtained the following inequality

$$\int_a^b q^+(t)dt\left(\int_a^b [r(t)]^{-1/(\gamma-1)}dt\right)^{\gamma-1} > 2^\gamma,$$

provided that Eq. (1.2) has a nontrivial solution $x(t)$ which satisfies $x(a) = x(b) = 0$ and $x(t) \neq 0$, $t \in (a, b)$. Here, $q^+(t) := \max\{q(t),\ 0\}$, $r > 0$, $\gamma > 1$.

In 2006, Napoli and Pinasco in [5] established Lyapunov-type inequalities for the quasilinear system of resonant type

$$\begin{cases} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = f(t)\left|u(t)\right|^{\mu-2}\left|v(t)\right|^{\gamma}u(t), \\ -\left(\left|v'(t)\right|^{q-2}v'(t)\right)' = g(t)\left|u(t)\right|^{\mu}\left|v(t)\right|^{\gamma-2}v(t), \end{cases}$$

on the interval (a, b), with Dirichlet boundary conditions

$$u(a) = u(b) = v(a) = v(b) = 0.$$

In 2017, Jleli and Samet [7] obtained Lyapunov type inequalities for the following system involving one-dimensional $(p_i,\ q_i)$-Laplacian operators $(i = 1, 2)$:

$$\begin{cases} -\left(\left|u'(t)\right|^{p_1-2}u'(t)\right)' - \left(\left|u'(t)\right|^{q_1-2}u'(t)\right)' = f(t)\left|u(t)\right|^{\alpha-2}\left|v(t)\right|^{\beta}u(t), \\ -\left(\left|v'(t)\right|^{p_2-2}v'(t)\right)' - \left(\left|v'(t)\right|^{q_2-2}v'(t)\right)' = g(t)\left|u(t)\right|^{\alpha}\left|v(t)\right|^{\beta-2}v(t), \end{cases}$$

on the interval $(a,\ b)$, under the Dirichlet boundary conditions

$$u(a) = u(b) = v(a) = v(b) = 0.$$

In this paper, we shall establish some new Lyapunov-type inequalities which are interesting in their own right, and as applications, we obtain sufficient conditions for the non-existence of nontrivial solutions to certain boundary value problems, and a lower bound for the eigenvalues.

2. Lyapunov type inequalities

Theorem 2.1. *If a nontrivial continuous solution to the boundary value problem*

$$\begin{cases} -\sum_{i=1}^{m}\left(\left|u'(t)\right|^{p_i-2}u'(t)\right)' = r(t)\left|u(t)\right|^{\alpha-2}u(t), \\ u(a) + u(b) = 0, \\ u'(a) + u'(b) = 0, \end{cases} \tag{2.1}$$

exists, where r *is a positive continuous function on* $[a, b]$, $p_i > 1$, *and* $\alpha = \sum_{i=1}^{m} p_i/m$, *then*

$$\min_{1\leqslant i\leqslant m}\left\{\frac{m2^{p_i}}{(b-a)^{p_i-1}}\right\} \leqslant \int_a^b r(t)dt.$$

Proof. Let u be a nontrivial solution to (2.1). Multiplying equation (2.1) by u and integrating over (a, b), we obtain

$$\sum_{i=1}^{m}\int_a^b \left|u'(t)\right|^{p_i}dt = \int_a^b r(t)\left|u(t)\right|^{\alpha}dt.$$

For any $t_0 \in [a,b]$, we have

$$2u(t_0) = \int_a^{t_0} u'(t)dt - \int_{t_0}^b u'(t)dt.$$

So

$$|2u(t_0)| \leqslant \int_a^b |u'(t)|dt.$$

By using Hölder inequality with parameters p_i and $p_i' = \frac{p_i}{p_i - 1}$, we have

$$2|u(t_0)| \leqslant (b-a)^{\frac{1}{p_i'}} \left(\int_a^b |u'(t)|^{p_i} dt \right)^{\frac{1}{p_i}},$$

hence

$$\frac{2^{p_i}}{(b-a)^{p_i-1}} |u(t_0)|^{p_i} \leqslant \int_a^b |u'(t)|^{p_i} dt, \ i = 1, 2, \dots, m.$$

Now choosing t_0 as the point where $|u(t)|$ attains its maximum, we have

$$|u(t_0)|^\alpha \int_a^b r(t)dt > \int_a^b r(t)|u(t)|^\alpha dt \geqslant \sum_{i=1}^m \frac{2^{p_i}}{(b-a)^{p_i-1}} |u(t_0)|^{p_i} \geqslant \min_{1 \leqslant i \leqslant m} \left\{ \frac{2^{p_i}}{(b-a)^{p_i-1}} \right\} \sum_{i=1}^m |u(t_0)|^{p_i}.$$

Using the inequality

$$\sum_{i=1}^m a_i \geqslant m \sqrt[m]{a_1 a_2 \cdots a_m},$$

we get

$$\min_{1 \leqslant i \leqslant m} \left\{ \frac{m 2^{p_i}}{(b-a)^{p_i-1}} \right\} \leqslant \int_a^b r(t)dt.$$

The proof is complete. □

Theorem 2.2. *If a nontrivial continuous solution* (u_1, u_2) *to the Laplacian system*

$$\begin{cases} -\sum\limits_{i=1}^m \left(|u_1'(t)|^{p_i-2} u_1'(t) \right)' = r_1(t) |u_1(t)|^{\alpha_1 - 2} |u_2(t)|^{\alpha_2} u_1(t), \\ -\sum\limits_{i=1}^m \left(|u_2'(t)|^{q_i-2} u_2'(t) \right)' = r_2(t) |u_1(t)|^{\beta_1} |u_2(t)|^{\beta_2 - 2} u_2(t), \\ u_j(a) + u_j(b) = 0, \ \ j = 1, 2, \\ u_j'(a) + u_j'(b) = 0, \ \ j = 1, 2, \end{cases} \tag{2.2}$$

exists, where r_1, r_2 *are nonnegative real-valued functions,* p_i, $q_i > 1$, $i = 1, 2, \dots, m$, α_j, $\beta_j > 1$, $j = 1, 2$, *and*

$$\frac{m\alpha_1}{\sum\limits_{i=1}^m p_i} + \frac{m\alpha_2}{\sum\limits_{i=1}^m q_i} = 1, \qquad \frac{m\beta_1}{\sum\limits_{i=1}^m p_i} + \frac{m\beta_2}{\sum\limits_{i=1}^m q_i} = 1, \tag{2.3}$$

then

$$\left[\min_{1 \leqslant i \leqslant m} \left\{ \frac{2^{p_i}}{(b-a)^{p_i-1}} \right\} \right]^{\frac{m\beta_1}{\sum_{i=1}^m p_i}} \cdot \left[\min_{1 \leqslant i \leqslant m} \left\{ \frac{2^{q_i}}{(b-a)^{q_i-1}} \right\} \right]^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}} \leqslant \left(\frac{1}{m} \int_a^b r_1(t)dt \right)^{\frac{m\beta_1}{\sum_{i=1}^m p_i}} \cdot \left(\frac{1}{m} \int_a^b r_2(t)dt \right)^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}.$$

Proof. Let (u_1, u_2) be a nontrivial solution to (2.2). Multiplying the first equation of (2.2) by u_1 and integrating over (a, b), we obtain

$$\sum_{i=1}^{m} \int_a^b |u_1'(t)|^{p_i} dt = \int_a^b r_1(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2} dt. \tag{2.4}$$

Multiplying the second equation of (2.2) by u_2 and integrating over (a, b), we obtain

$$\sum_{i=1}^{m} \int_a^b |u_2'(t)|^{q_i} dt = \int_a^b r_2(t)|u_1(t)|^{\beta_1}|u_2(t)|^{\beta_2} dt. \tag{2.5}$$

From the proof of Theorem 2.1, we get for any $t_0, x_0 \in [a, b]$,

$$\frac{2^{p_i}}{(b-a)^{p_i-1}}|u_1(t_0)|^{p_i} \leqslant \int_a^b |u_1'(t)|^{p_i} dt, \quad i = 1, 2, \dots, m, \tag{2.6}$$

$$\frac{2^{q_i}}{(b-a)^{q_i-1}}|u_2(x_0)|^{q_i} \leqslant \int_a^b |u_2'(t)|^{q_i} dt, \quad i = 1, 2, \dots, m. \tag{2.7}$$

From (2.4) and (2.6), we obtain

$$\sum_{i=1}^{m} \frac{2^{p_i}}{(b-a)^{p_i-1}}|u_1(t_0)|^{p_i} \leqslant \int_a^b r_1(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2} dt. \tag{2.8}$$

From (2.5) and (2.7), we have

$$\sum_{i=1}^{m} \frac{2^{q_i}}{(b-a)^{q_i-1}}|u_2(x_0)|^{q_i} \leqslant \int_a^b r_2(t)|u_1(t)|^{\beta_1}|u_2(t)|^{\beta_2} dt. \tag{2.9}$$

Let $t_0, x_0 \in [a, b]$ be such that

$$|u_1(t_0)| = \max\{|u_1(t)| : a \leqslant t \leqslant b\}, \quad |u_2(x_0)| = \max\{|u_2(t)| : a \leqslant t \leqslant b\}.$$

From (2.8), we have

$$\sum_{i=1}^{m} \frac{2^{p_i}}{(b-a)^{p_i-1}}|u_1(t_0)|^{p_i} \leqslant |u_1(t_0)|^{\alpha_1}|u_2(x_0)|^{\alpha_2} \int_a^b r_1(t) dt$$

and so

$$\min_{1\leqslant i\leqslant m} \left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\} \sum_{i=1}^{m} |u_1(t_0)|^{p_i} \leqslant |u_1(t_0)|^{\alpha_1}|u_2(x_0)|^{\alpha_2} \int_a^b r_1(t) dt.$$

By using the inequality

$$\sum_{i=1}^{m} a_i \geqslant m \sqrt[m]{a_1 a_2 \cdots a_m},$$

we get

$$\min_{1\leqslant i\leqslant m} \left\{\frac{m 2^{p_i}}{(b-a)^{p_i-1}}\right\} \leqslant |u_1(t_0)|^{\alpha_1 - \frac{\sum_{i=1}^{m} p_i}{m}} \cdot |u_2(x_0)|^{\alpha_2} \int_a^b r_1(t) dt. \tag{2.10}$$

Similarly, from (2.9) we obtain

$$\min_{1\leqslant i\leqslant m} \left\{\frac{m 2^{q_i}}{(b-a)^{q_i-1}}\right\} \leqslant |u_1(t_0)|^{\beta_1} \cdot |u_2(x_0)|^{\beta_2 - \frac{\sum_{i=1}^{m} q_i}{m}} \int_a^b r_2(t) dt. \tag{2.11}$$

Raising inequality (2.10) to a power $e_1 > 0$, inequality (2.11) to a power $e_2 > 0$, and multiplying the resulting inequalities, we obtain

$$\begin{aligned}&\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^{e_1}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^{e_2}\cdot m^{e_1+e_2}\\&\leqslant\left|u_1(t_0)\right|^{\left(\alpha_1-\frac{\sum_{i=1}^m p_i}{m}\right)e_1+\beta_1e_2}\cdot\left|u_2(x_0)\right|^{\alpha_2e_1+\left(\beta_2-\frac{\sum_{i=1}^m q_i}{m}\right)e_2}\left(\int_a^b r_1(t)dt\right)^{e_1}\left(\int_a^b r_2(t)dt\right)^{e_2}.\end{aligned}$$

Choose e_1 and e_2 such that $|u_1(t_0)|$, $|u_2(x_0)|$ cancel out, i.e., e_1, e_2 solve the homogeneous linear system

$$\begin{cases}\left(\alpha_1-\frac{\sum_{i=1}^m p_i}{m}\right)e_1+\beta_1e_2=0,\\ \alpha_2e_1+\left(\beta_2-\frac{\sum_{i=1}^m q_i}{m}\right)e_2=0.\end{cases}$$

Using (2.3), we may take

$$\begin{cases}e_1=\beta_1,\\ e_2=\frac{\sum_{i=1}^m p_i\alpha_2}{\sum_{i=1}^m q_i}.\end{cases}$$

Then we have

$$\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^{\beta_1}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^{\frac{\sum_{i=1}^m p_i\alpha_2}{\sum_{i=1}^m q_i}}\leqslant\left(\frac{1}{m}\int_a^b r_1(t)dt\right)^{\beta_1}\left(\frac{1}{m}\int_a^b r_2(t)dt\right)^{\frac{\sum_{i=1}^m p_i\alpha_2}{\sum_{i=1}^m q_i}},$$

and so

$$\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^{\frac{m\beta_1}{\sum_{i=1}^m p_i}}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}\leqslant\left(\frac{1}{m}\int_a^b r_1(t)dt\right)^{\frac{m\beta_1}{\sum_{i=1}^m p_i}}\left(\frac{1}{m}\int_a^b r_2(t)dt\right)^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}.$$

The proof is complete. □

Theorem 2.3. *If a nontrivial continuous solution* (u_1, u_2, u_3) *to the Laplacian system*

$$\begin{cases}-\sum_{i=1}^m(|u_1'(t)|^{p_i-2}u_1'(t))'=r_1(t)|u_1(t)|^{\alpha_1-2}|u_2(t)|^{\alpha_2}|u_3(t)|^{\alpha_3}u_1(t),\\ -\sum_{i=1}^m(|u_2'(t)|^{q_i-2}u_2'(t))'=r_2(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2-2}|u_3(t)|^{\alpha_3}u_2(t),\\ -\sum_{i=1}^m(|u_3'(t)|^{\gamma_i-2}u_3'(t))'=r_3(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2}|u_3(t)|^{\alpha_3-2}u_3(t),\\ u_j(a)+u_j(b)=0,\quad j=1,2,3,\\ u_j'(a)+u_j'(b)=0,\quad j=1,2,3,\end{cases}\tag{2.12}$$

exists, where r_1, r_2, r_3 *are nonnegative real-valued functions,* p_i ,q_i, $\gamma_i > 1$, $i = 1,2,\dots,m$, $\alpha_j > 1$, $j = 1,2,3$ *and*

$$\frac{m\alpha_1}{\sum_{i=1}^m p_i}+\frac{m\alpha_2}{\sum_{i=1}^m q_i}+\frac{m\alpha_3}{\sum_{i=1}^m \gamma_i}=1,\tag{2.13}$$

then

$$\begin{aligned}&m\cdot\Big[\min_{1\leqslant i\leqslant m}\Big\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\Big\}\Big]^{\frac{m\alpha_1}{\sum_{i=1}^m p_i}}\cdot\Big[\min_{1\leqslant i\leqslant m}\Big\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\Big\}\Big]^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}\cdot\Big[\min_{1\leqslant i\leqslant m}\Big\{\frac{2^{\gamma_i}}{(b-a)^{\gamma_i-1}}\Big\}\Big]^{\frac{m\alpha_3}{\sum_{i=1}^m \gamma_i}}\\&\leqslant\Big(\int_a^b r_1(t)dt\Big)^{\frac{m\alpha_1}{\sum_{i=1}^m p_i}}\cdot\Big(\int_a^b r_2(t)dt\Big)^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}\cdot\Big(\int_a^b r_3(t)dt\Big)^{\frac{m\alpha_3}{\sum_{i=1}^m \gamma_i}}.\end{aligned}\tag{2.14}$$

Proof. Let (u_1, u_2, u_3) be a nontrivial solution to (2.12). Multiplying the first equation of (2.12) by u_1 and integrating over (a, b), we obtain

$$\sum_{i=1}^m\int_a^b|u_1'(t)|^{p_i}dt=\int_a^b r_1(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2}|u_3(t)|^{\alpha_3}dt.\tag{2.15}$$

Let t_0, x_0, $y_0 \in [a, b]$ be such that

$$\begin{aligned}|u_1(t_0)|&=\max\{|u_1(t)|:a\leqslant t\leqslant b\},\\|u_2(x_0)|&=\max\{|u_2(t)|:a\leqslant t\leqslant b\},\\|u_3(y_0)|&=\max\{|u_3(t)|:a\leqslant t\leqslant b\}.\end{aligned}$$

From the proof of Theorem 2.1, we get

$$\frac{2^{p_i}}{(b-a)^{p_i-1}}|u_1(t_0)|^{p_i}\leqslant\int_a^b|u_1'(t)|^{p_i}dt\quad i=1,2,\dots,m.\tag{2.16}$$

From (2.15) and (2.16), we obtain

$$\sum_{i=1}^m\frac{2^{p_i}}{(b-a)^{p_i-1}}|u_1(t_0)|^{p_i}\leqslant|u_1(t_0)|^{\alpha_1}|u_2(x_0)|^{\alpha_2}|u_3(y_0)|^{\alpha_3}\int_a^b r_1(t)dt,$$

which yields

$$\min_{1\leqslant i\leqslant m}\Big\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\Big\}\sum_{i=1}^m|u_1(t_0)|^{p_i}\leqslant|u_1(t_0)|^{\alpha_1}|u_2(x_0)|^{\alpha_2}|u_3(y_0)|^{\alpha_3}\int_a^b r_1(t)dt.$$

By using the inequality

$$\sum_{i=1}^m a_i\geqslant m\sqrt[m]{a_1a_2\cdots a_m},$$

we get

$$\min_{1\leqslant i\leqslant m}\Big\{\frac{m2^{p_i}}{(b-a)^{p_i-1}}\Big\}\leqslant|u_1(t_0)|^{\alpha_1-\frac{\sum_{i=1}^m p_i}{m}}|u_2(x_0)|^{\alpha_2}|u_3(y_0)|^{\alpha_3}\int_a^b r_1(t)dt.\tag{2.17}$$

Similarly, we obtain

$$\min_{1\leqslant i\leqslant m}\Big\{\frac{m2^{q_i}}{(b-a)^{q_i-1}}\Big\}\leqslant|u_1(t_0)|^{\alpha_1}|u_2(x_0)|^{\alpha_2-\frac{\sum_{i=1}^m q_i}{m}}|u_3(y_0)|^{\alpha_3}\int_a^b r_2(t)dt,\tag{2.18}$$

$$\min_{1\leqslant i\leqslant m}\Big\{\frac{m2^{\gamma_i}}{(b-a)^{\gamma_i-1}}\Big\}\leqslant|u_1(t_0)|^{\alpha_1}|u_2(x_0)|^{\alpha_2}|u_3(y_0)|^{\alpha_3-\frac{\sum_{i=1}^m \gamma_i}{m}}\int_a^b r_3(t)dt.\tag{2.19}$$

Raising inequality (2.17) to a power $e_1 > 0$, inequality (2.18) to a power $e_2 > 0$, inequality (2.19) to a power $e_3 > 0$, and multiplying the resulting inequalities, we obtain

$$\begin{aligned}&\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^{e_1}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^{e_2}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{\gamma_i}}{(b-a)^{\gamma_i-1}}\right\}\right]^{e_3}m^{e_1+e_2+e_3}\\&\leqslant\left|u_1(t_0)\right|^{\left(\alpha_1-\frac{\sum_{i=1}^m p_i}{m}\right)e_1+\alpha_1e_2+\alpha_1e_3}\cdot\left|u_2(x_0)\right|^{\alpha_2e_1+\left(\alpha_2-\frac{\sum_{i=1}^m q_i}{m}\right)e_2+\alpha_2e_3}\\&\quad\cdot\left|u_3(y_0)\right|^{\alpha_3e_1+\alpha_3e_2+\left(\alpha_3-\frac{\sum_{i=1}^m \gamma_i}{m}\right)e_3}\left(\int_a^b r_1(t)dt\right)^{e_1}\left(\int_a^b r_2(t)dt\right)^{e_2}\left(\int_a^b r_3(t)dt\right)^{e_3}.\end{aligned}$$

Choose e_1, e_2, and e_3 such that $|u_1(t_0)|$, $|u_2(x_0)|$, $|u_3(y_0)|$ cancel out, i.e., e_1, e_2, e_3 solve the homogeneous linear system

$$\begin{cases}\left(\alpha_1-\frac{\sum_{i=1}^m p_i}{m}\right)e_1+\alpha_1e_2+\alpha_1e_3=0,\\ \alpha_2e_1+\left(\alpha_2-\frac{\sum_{i=1}^m q_i}{m}\right)e_2+\alpha_2e_3=0,\\ \alpha_3e_1+\alpha_3e_2+\left(\alpha_3-\frac{\sum_{i=1}^m \gamma_i}{m}\right)e_3=0.\end{cases}$$

From (2.13), we may take

$$\begin{cases}e_1=\alpha_1,\\ e_2=\dfrac{\sum_{i=1}^m p_i\alpha_2}{\sum_{i=1}^m q_i},\\ e_3=\dfrac{\sum_{i=1}^m p_i\alpha_3}{\sum_{i=1}^m \gamma_i},\end{cases}$$

and get

$$\begin{aligned}&\left[\min_{1\leqslant i\leqslant m}\left\{\frac{m2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^{\alpha_1}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{m2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^{\frac{\sum_{i=1}^m p_i\alpha_2}{\sum_{i=1}^m q_i}}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{m2^{\gamma_i}}{(b-a)^{\gamma_i-1}}\right\}\right]^{\frac{\sum_{i=1}^m p_i\alpha_3}{\sum_{i=1}^m \gamma_i}}\\&\leqslant\left(\int_a^b r_1(t)dt\right)^{\alpha_1}\cdot\left(\int_a^b r_2(t)dt\right)^{\frac{\sum_{i=1}^m p_i\alpha_2}{\sum_{i=1}^m q_i}}\cdot\left(\int_a^b r_3(t)dt\right)^{\frac{\sum_{i=1}^m p_i\alpha_3}{\sum_{i=1}^m \gamma_i}}.\end{aligned}$$

From (2.13), we get

$$\begin{aligned}&m\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^{\frac{m\alpha_1}{\sum_{i=1}^m p_i}}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{\gamma_i}}{(b-a)^{\gamma_i-1}}\right\}\right]^{\frac{m\alpha_3}{\sum_{i=1}^m \gamma_i}}\\&\leqslant\left(\int_a^b r_1(t)dt\right)^{\frac{m\alpha_1}{\sum_{i=1}^m p_i}}\cdot\left(\int_a^b r_2(t)dt\right)^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}\cdot\left(\int_a^b r_3(t)dt\right)^{\frac{m\alpha_3}{\sum_{i=1}^m \gamma_i}}.\end{aligned}$$

The proof is then complete. □

As immediate consequences of Theorems 2.1-2.3, the following corollary gives sufficient conditions for the non-existence of nontrivial solutions to the above boundary value problems.

Corollary 2.4.

(a) *If*

$$\int_a^b r(t)dt < \min_{1\leqslant i\leqslant m}\left\{\frac{m2^{p_i}}{(b-a)^{p_i-1}}\right\},$$

then BVP (2.1) *has no nontrivial solution.*

(b) *If*

$$\left(\frac{1}{m}\int_a^b r_1(t)dt\right)^{\frac{m\beta_1}{\sum_{i=1}^m p_i}}\cdot\left(\frac{1}{m}\int_a^b r_2(t)dt\right)^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}$$
$$< \left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^{\frac{m\beta_1}{\sum_{i=1}^m p_i}}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}},$$

then BVP (2.2) *has no nontrivial solution.*

(c) *If*

$$\left(\int_a^b r_1(t)dt\right)^{\frac{m\alpha_1}{\sum_{i=1}^m p_i}}\cdot\left(\int_a^b r_2(t)dt\right)^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}\cdot\left(\int_a^b r_3(t)dt\right)^{\frac{m\alpha_3}{\sum_{i=1}^m \gamma_i}}$$
$$< m\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^{\frac{m\alpha_1}{\sum_{i=1}^m p_i}}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^{\frac{m\alpha_2}{\sum_{i=1}^m q_i}}\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{\gamma_i}}{(b-a)^{\gamma_i-1}}\right\}\right]^{\frac{m\alpha_3}{\sum_{i=1}^m \gamma_i}},$$

then BVP (2.12) *has no nontrivial solution.*

Theorem 2.5. *Let* (λ,μ,γ) *be a generalized eigenvalue of the following system*

$$\begin{cases} -\sum_{i=1}^m (|u_1'(t)|^{p_i-2}u_1'(t))' = \lambda\alpha_1 v(t)|u_1(t)|^{\alpha_1-2}|u_2(t)|^{\alpha_2}|u_3(t)|^{\alpha_3}u_1(t),\\ -\sum_{i=1}^m (|u_2'(t)|^{q_i-2}u_2'(t))' = \mu\alpha_2 v(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2-2}|u_3(t)|^{\alpha_3}u_2(t),\\ -\sum_{i=1}^m (|u_3'(t)|^{\gamma_i-2}u_3'(t))' = \gamma\alpha_3 v(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2}|u_3(t)|^{\alpha_3-2}u_3(t),\\ u_j(a)+u_j(b)=0, \quad j=1,2,3,\\ u_j'(a)+u_j'(b)=0, \quad j=1,2,3, \end{cases}$$

where p_i, q_i, γ_i, α_j *satisfy condition* (2.13), $v \geqslant 0$, $v\in L^1(a,b)$. *Then*

$$\gamma \geqslant h(\lambda,\ \mu),$$

where $h:(0,\ \infty)\times(0,\ \infty)\to(0,\ \infty)$ *is the function defined by*

$$h(x,y)=\frac{1}{\alpha_3}\left(\frac{C}{x^\sigma y^\theta\int_a^b v(t)dt}\right)^{\eta^{-1}},$$

with

$$\sigma=\frac{m\alpha_1}{\sum_{i=1}^m p_i},\quad \theta=\frac{m\alpha_2}{\sum_{i=1}^m q_i},\quad \eta=\frac{m\alpha_3}{\sum_{i=1}^m \gamma_i},$$

and

$$C\alpha_1^\sigma\alpha_2^\theta = m\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\right\}\right]^\sigma\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\right\}\right]^\theta\cdot\left[\min_{1\leqslant i\leqslant m}\left\{\frac{2^{\gamma_i}}{(b-a)^{\gamma_i-1}}\right\}\right]^\eta.$$

Proof. Let (λ, μ, γ) be a generalized eigenvalue, with corresponding nontrivial solutions u_1, u_2, u_3. By substituting into (2.14) the functions

$$r_1(t) = \lambda\alpha_1 v(t), \quad r_2(t) = \mu\alpha_2 v(t), \quad r_3(t) = \gamma\alpha_3 v(t),$$

and using condition (2.13), we have

$$\begin{aligned} m\Big[\min_{1\leqslant i\leqslant m}\Big\{\frac{2^{p_i}}{(b-a)^{p_i-1}}\Big\}\Big]^{\sigma}\cdot\Big[\min_{1\leqslant i\leqslant m}\Big\{\frac{2^{q_i}}{(b-a)^{q_i-1}}\Big\}\Big]^{\theta}\cdot\Big[\min_{1\leqslant i\leqslant m}\Big\{\frac{2^{\gamma_i}}{(b-a)^{\gamma_i-1}}\Big\}\Big]^{\eta} \\ \leqslant \lambda^{\sigma}\alpha_1^{\sigma}\mu^{\theta}\alpha_2^{\theta}\gamma^{\eta}\alpha_3^{\eta}\int_a^b v(t)dt. \end{aligned}$$

Hence

$$\gamma^{\eta} \geqslant \frac{C}{\alpha_3^{\eta}\lambda^{\sigma}\mu^{\theta}\int_a^b v(t)dt}.$$

The proof is complete. □

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