



Finite difference method for Riesz space fractional diffusion equations with delay and a nonlinear source term



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Abstract

In this paper, we propose a finite difference method for the Riesz space fractional diffusion equations with delay and a nonlinear source term on a finite domain. The proposed method combines a time scheme based on the predictor-corrector method and the Crank-Nicolson scheme for the spatial discretization. The corresponding theoretical results including stability and convergence are provided. Some numerical examples are presented to validate the proposed method. ©2017 All rights reserved.

Keywords: Riesz fractional derivative, fractional diffusion equations, Crank-Nicolson scheme, stability, convergence.

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1. Introduction

In this paper, we consider the following Riesz space fractional diffusion equations with delay and a nonlinear source term

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = K_\alpha(x,t) \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t,u,u(x,t-s)), 1 < \alpha \leq 2, (x,t) \in (0,L) \times [0,T], \\ u(0,t) = u(L,t) = 0, t \in [0,T], \\ u(x,t) = \varphi(x,t), (x,t) \in (0,L) \times [-s,0], \end{cases} \quad (1.1)$$

where $1 < \alpha \leq 2$, $s > 0$, and $f : D = [0,L] \times [0,T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous mapping and satisfies the following Lipschitz condition

$$|f(x,t,u_1,u_2) - f(x,t,\bar{u}_1,\bar{u}_2)| \leq \beta_1|u_1 - \bar{u}_1| + \beta_2|u_2 - \bar{u}_2|, \quad \forall u_1, \bar{u}_1, u_2, \bar{u}_2 \in \mathbb{R}, \quad (1.2)$$

where $\beta_1 > 0, \beta_2 > 0$, and $K_\alpha \geq 0$ represents the dispersion coefficient. The Riesz space fractional

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operators $\frac{\partial^\alpha u(x,t)}{\partial |\alpha|^\alpha}$ on a finite domain $[0, L]$ can be defined as

$$\frac{\partial^\alpha u(x,t)}{\partial |\alpha|^\alpha} = -c_\alpha [{}_0D_x^\alpha u(x,t) + {}_x D_L^\alpha u(x,t)],$$

where

$$c_\alpha = \frac{1}{2\cos(\frac{\pi\alpha}{2})},$$

and

$${}_0D_x^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_0^x (x-\tau)^{1-\alpha} u(\tau,t) d\tau, \quad {}_x D_L^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^L (\tau-x)^{1-\alpha} u(\tau,t) d\tau,$$

where $\Gamma(\cdot)$ represents the Euler gamma function.

In the past decades, fractional calculus has been a hot spot of researches in science and engineering [1, 6, 8, 9, 15, 18, 20–22]. The numerical approach to fractional differential equations (FDEs) has been discussed by numerous researchers [2–5, 7, 10–14, 17, 19, 26]. Recently, in [23–25], some new finite memory fractional differences have been proposed. And these definitions can be considered for discrete fractional modeling. Consideration numerical methods for solving the fractional advection-dispersion equations (FADE) have been proposed. In 2004, Meerschaert and Tadjeran [16] developed practical numerical methods to solve the one-dimensional space FADE with variable coefficients on a finite domain. Liu et al. [10] proposed a class of method of lines for solving the space fractional Fokker-Planck equation in 2004. In 2009, Zhuang et al. [27] discussed a variable order fractional advection diffusion equation with a nonlinear source term on a finite domain. In addition, some numerical methods including finite difference method, finite element method, spectral method are also employed to solve the FADE [12]. Recently, these numerical methods for Riesz space fractional advection-dispersion equations (RFDEs) have also been a subject of many investigations [12].

As we all know, functional differential equations including delay differential equations have been widely used in physics, biology, medicine, chemistry, economics, and so on. In my opinion, it is valuable to study numerical methods for Riesz space fractional diffusion equations with delay. In this paper, our aim is to construct the numerical method for the Riesz space fractional diffusion equations with delay and obtain the corresponding theoretical results.

This paper is organized as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, we present the finite difference method for the Riesz space fractional diffusion equations with delay. The proofs of the stability and convergence of the method are provided in Section 4. Finally we carry out some numerical tests to verify the theoretical results in Section 5.

2. Preliminaries

In this section, we first review the basic definitions and some useful lemmas.

Definition 2.1 ([17]). The left and right Riemann-Liouville fractional derivatives of order γ ($n-1 < \gamma < n$) of the function $v(x)$ on $[a, b]$, are given by

$${}_a^R D_x^\gamma v(x) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_a^x (x-\tau)^{n-\gamma-1} v(\tau) d\tau,$$

and

$${}_x^R D_b^\gamma v(x) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_x^b (t-\tau)^{n-\gamma-1} v(\tau) d\tau,$$

respectively.

In the interval $[0, L]$, we take the mesh points $x_i = ih, i = 1, 2, \dots, M$ and $t_n = n\tau, n = 1, 2, \dots, N$, where $h = \frac{L}{M}, \tau = \frac{T}{N}$, i.e., h and τ are the uniform spatial step size and temporal step size, $s = m\tau$. Now, we give the following lemmas.

Lemma 2.2 ([12]). *Supposing $1 < \gamma < 2$, let $v(x) \in L^1(\mathbb{R})$, then*

$${}^R_0D_x^\gamma v(x_i) = \frac{1}{h^\gamma} \sum_{k=0}^{i+1} \omega_k^\gamma v(x_{i-k+1}) + O(h^2), \quad {}^R_xD_x L^\gamma v(x_i) = \frac{1}{h^\gamma} \sum_{k=0}^{m-i+1} \omega_k^\gamma v(x_{i+k-1}) + O(h^2),$$

where

$$\omega_0^{(\gamma)} = \frac{\gamma}{2} g_0^{(\gamma)}, \quad \omega_k^{(\gamma)} = \frac{\gamma}{2} g_k^{(\gamma)} + \frac{2-\gamma}{2} g_{k-1}^{(\gamma)}, \quad g_0^{(\gamma)} = 1, \quad g_k^{(\gamma)} = \left(1 - \frac{\gamma+1}{k}\right) g_{k-1}^{(\gamma)}, \quad k = 1, 2, \dots$$

In fact the coefficients $g_k^{(\gamma)}$ are the coefficients of the power series of the function $(1-z)^\gamma$ for all $|z| \leq 1$.

Lemma 2.3 ([12]). *Supposing that $1 < \gamma \leq 2$, then the coefficient $g_k^{(\gamma)}$ satisfies*

$$\begin{cases} g_0^{(\gamma)} = 1, \quad g_1^{(\gamma)} = -\gamma, \quad g_2^{(\gamma)} = \frac{\gamma(\gamma-1)}{2} > 0, \\ 1 \geq g_2^{(\gamma)} \geq g_3^{(\gamma)} \geq \dots \geq 0, \quad \sum_{k=0}^{\infty} g_k^{(\gamma)} = 0, \quad \sum_{k=0}^m g_k^{(\gamma)} < 0, \quad m \geq 1. \end{cases}$$

Lemma 2.4 ([12]). *Supposing that $1 < \gamma \leq 2$, then the coefficient $\omega_k^{(\gamma)}$ satisfies*

$$\begin{cases} \omega_0^{(\gamma)} = \frac{\gamma}{2}, \quad \omega_1^{(\gamma)} = \frac{2-\gamma-\gamma^2}{2} < 0, \quad g_2^{(\gamma)} = \frac{\gamma(\gamma^2+\gamma-4)}{4}, \\ 1 \geq \omega_2^{(\gamma)} \geq \omega_3^{(\gamma)} \geq \dots \geq 0, \quad \sum_{k=0}^{\infty} \omega_k^{(\gamma)} = 0, \quad \sum_{k=0}^m \omega_k^{(\gamma)} < 0, \quad m \geq 2. \end{cases}$$

3. The finite difference method for Riesz space fractional diffusion equations with delay and a non-linear source term

In this section, we discuss the numerical method of (1.1). First, we define $t_n = n\tau, n = 0, 1, \dots, N$, let $\Omega = [0, L]$ be a finite domain, setting S_h be a uniform partition of Ω , which is given by $x_i = ih$ for $i = 0, 1, \dots, M, s = m\tau$, where $\tau = \frac{T}{N}$ and $h = \frac{L}{M}$ are the time and space stepsize, respectively.

Now, we present the semi-discrete form of (1.1),

$$\frac{\partial u(x_i, t)}{\partial t} = K_\alpha \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} \Big|_{x=x_i} + f(x_i, t, u(x_i, t), u(x_i, t-s)). \tag{3.1}$$

Applying the predictor-corrector method and the Crank-Nicolson scheme to solve (3.1), we have

$$\begin{cases} \bar{u}_i^n = u_i^{n-1} - \frac{\tau K_\alpha c_\alpha}{h^\alpha} \left[\sum_{k=0}^{i+1} \omega_k^{(\alpha)} u_{i-k+1}^{n-1} + \sum_{k=0}^{M-i+1} \omega_k^{(\alpha)} u_{i+k-1}^{n-1} \right] + \tau f(x_i, t_n, \bar{u}_i^{n-1}, \bar{u}_i^{n-m}), \\ u_i^n = u_i^{n-1} - \frac{\tau K_\alpha c_\alpha}{2h^\alpha} \left[\left(\sum_{k=0}^{i+1} \omega_k^{(\alpha)} u_{i-k+1}^{n-1} + \sum_{k=0}^{M-i+1} \omega_k^{(\alpha)} u_{i+k-1}^{n-1} \right) + \left(\sum_{k=0}^{i+1} \omega_k^{(\alpha)} \bar{u}_{i-k+1}^{n-1} \right. \right. \\ \left. \left. + \sum_{k=0}^{M-i+1} \omega_k^{(\alpha)} \bar{u}_{i+k-1}^{n-1} \right) \right] + \frac{\tau}{2} [f(x_i, t_n, u_i^{n-1}, u_i^{n-m}) + f(x_i, t_n, \bar{u}_i^{n-1}, \bar{u}_i^{n-m})], \end{cases} \tag{3.2}$$

where u_i^n is approximate solution of $u(x_i, t_n)$. Denote

$$A = \begin{bmatrix} \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & \cdots & 0 & 0 \\ \omega_3^{(\alpha)} & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_{M-2}^{(\alpha)} & \omega_{M-3}^{(\alpha)} & \omega_{M-4}^{(\alpha)} & \cdots & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \\ \omega_{M-1}^{(\alpha)} & \omega_{M-2}^{(\alpha)} & \omega_{M-3}^{(\alpha)} & \cdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} \end{bmatrix}, \quad F(U) = \begin{bmatrix} f(x_1, t_n, U_1, u_1^{n-m}) \\ f(x_2, t_n, U_2, u_2^{n-m}) \\ \vdots \\ f(x_{M-1}, t_n, U_{M-1}, u_{M-1}^{n-m}) \end{bmatrix},$$

and $D = -\frac{\tau K_{\alpha} c_{\alpha}}{h^{\alpha}}(A + A^T)$, $U^n = [u_1^n, u_2^n, \dots, u_{M-1}^n]$, $\bar{U}^n = [\bar{u}_1^n, \bar{u}_2^n, \dots, \bar{u}_{M-1}^n]$. Thus, (3.2) can be simplified as

$$\begin{cases} \bar{U}^n = U^{n-1} + DU^{n-1} + \tau F(U^{n-1}), \\ U^n = U^{n-1} + \frac{1}{2}D(U^{n-1} + \bar{U}^n) + \frac{\tau}{2}(F(U^{n-1}) + F(\bar{U}^n)). \end{cases} \tag{3.3}$$

Obviously, the above scheme can be written as the following scheme

$$U^n = U^{n-1} + \frac{1}{2}D(2U^{n-1} + DU^{n-1} + \tau F(U^{n-1})) + \frac{\tau}{2}(F(U^{n-1}) + F(U^{n-1} + DU^{n-1} + \tau F(U^{n-1}))).$$

The boundary and initial conditions are discretized as

$$u_i^0 = \varphi(ih, 0), \quad U^0 = (u_1^0, u_2^0, \dots, u_{M-1}^0), i = 1, 2, \dots, M-1, \quad \text{and} \quad u_i^k = \varphi(ih, k\tau), k < 0.$$

4. Theoretical analysis of the finite difference method

In this section, we present the proofs of stability and convergence of the proposed method.

Lemma 4.1 ([12]). *The matrix D is symmetric positive definite.*

Theorem 4.2 (Stability). *If $\|I + D\|_{\infty} \leq 1$, the method (3.3) is stable.*

Proof. Let U_i^n and \tilde{u}_i^n be the numerical and approximate solution vectors, respectively, and

$$\tilde{u}^n = [\tilde{u}(x_1, t_n), \tilde{u}(x_2, t_n), \dots, \tilde{u}(x_{M-1}, t_n)]^T, \quad \text{and} \quad \epsilon^n = U^n - u^n.$$

Then,

$$\begin{aligned} \epsilon^n &= \epsilon^{n-1} + \frac{1}{2}D(2\epsilon^{n-1} + D\epsilon^{n-1} + \tau(F(U^{n-1}) - F(u^{n-1}))) \\ &\quad + \frac{\tau}{2}[(F(U^{n-1}) + F(U^{n-1} + DU^{n-1} + \tau F(U^{n-1}))) - (F(u^{n-1}) + F(u^{n-1} + Du^{n-1} + \tau F(u^{n-1})))] \\ &= (I + D + \frac{D^2}{2})\epsilon^{n-1} + \frac{\tau}{2}D(F(U^{n-1}) - F(u^{n-1})) \\ &\quad + \frac{\tau}{2}[F(U^{n-1}) - F(u^{n-1}) + F((I + D)U^{n-1} + \tau F(U^{n-1})) - F((I + D)u^{n-1} + \tau F(u^{n-1}))]. \end{aligned}$$

Since $\|I + D\|_{\infty} \leq 1$, then

$$\|I + D + \frac{1}{2}D^2\|_{\infty} = \|\frac{1}{2}(I + (I + D)^2)\|_{\infty} \leq \frac{1}{2}(1 + \|I + D\|_{\infty}^2) \leq 1.$$

By using the condition (1.2), we have

$$\|\epsilon^n\|_{\infty} \leq \|I + D + \frac{1}{2}D^2\|_{\infty} \|\epsilon^{n-1}\|_{\infty} + \frac{\tau}{2}\|D\|_{\infty}(\beta_1\|\epsilon^{n-1}\|_{\infty} + \beta_2\|\epsilon^{n-m}\|_{\infty}) + \frac{\tau}{2}[(\beta_1\|\epsilon^{n-1}\|_{\infty}$$

$$+ \beta_2 \|\epsilon^{n-m}\|_\infty) + \beta_1 (\|I + D\|_\infty \|\epsilon^{n-1}\|_\infty + \tau(\beta_1 \|\epsilon^{n-1}\|_\infty + \beta_2 \|\epsilon^{n-m}\|_\infty)) + \beta_2 \|\epsilon^{n-m}\|_\infty].$$

Denoting $e^n = \max_{0 \leq i \leq n} \{\|\epsilon^i\|_\infty\}$, we have

$$\begin{aligned} e^n &\leq e^{n-1} + \frac{\tau}{2} \|D\|_\infty (\beta_1 e^{n-1} + \beta_2 e^{n-1}) \\ &\quad + \frac{\tau}{2} [(\beta_1 e^{n-1} + \beta_2 e^{n-1}) + \beta_1 (e^{n-1} + \tau(\beta_1 e^{n-1} + \beta_2 e^{n-1})) + \beta_2 e^{n-1}] \\ &\leq [1 + \tau(\beta_1 + \beta_2)(1 + \frac{1}{2}(\|D\|_\infty + \beta_1 \tau))] e^{n-1} \leq (1 + \tau L) e^{n-1}, \end{aligned}$$

where $L = (\beta_1 + \beta_2)(1 + \frac{1}{2}(\|D\|_\infty + \beta_1 \tau))$. Obviously, because of $\|I + D\|_\infty \leq 1$, $\|D\|_\infty \leq C_0 \ll +\infty$. Thus,

$$e^n \leq (1 + \tau L)^n e^0 \leq e^{n\tau L} e^0 = e^{\tau L} e^0. \quad \square$$

Theorem 4.3 (Convergence). *If $\|I + D\|_\infty \leq 1$, the method (3.3) is convergent.*

Proof. Let U^n and u^n be the numerical and exact solution vectors, respectively, and

$$u^n = [u(x_1, t_n), u(x_2, t_n), \dots, u(x_{M-1}, t_n)]^T, \quad \epsilon^n = U^n - u(x_i, t_n).$$

Then,

$$\begin{aligned} \epsilon^n &= \epsilon^{n-1} + \frac{1}{2} D(2\epsilon^{n-1} + D\epsilon^{n-1} + \tau(F(U^{n-1}) - F(u^{n-1}))) + \frac{\tau}{2} [(F(U^{n-1}) \\ &= F(U^{n-1} + DU^{n-1} + \tau F(U^{n-1})) - (F(u^{n-1}) + F(u^{n-1} + Du^{n-1} + \tau F(u^{n-1}))) + O(\tau^3 + \tau h^2) \\ &= (I + D + \frac{D^2}{2})\epsilon^{n-1} + \frac{\tau}{2} D(F(U^{n-1}) - F(u^{n-1})) + \frac{\tau}{2} [F(U^{n-1}) - F(u^{n-1}) \\ &\quad + F((I + D)U^{n-1} + \tau F(U^{n-1})) - F((I + D)u^{n-1} + \tau F(u^{n-1}))] + O(\tau^3 + \tau h^2). \end{aligned}$$

By means of the condition (1.2), we have

$$\begin{aligned} \|\epsilon^n\|_\infty &\leq \|I + D + \frac{1}{2} D^2\|_\infty \|\epsilon^{n-1}\|_\infty + \frac{\tau}{2} \|D\|_\infty (\beta_1 \|\epsilon^{n-1}\|_\infty + \beta_2 \|\epsilon^{n-m}\|_\infty) + \frac{\tau}{2} [(\beta_1 \|\epsilon^{n-1}\|_\infty + \beta_2 \|\epsilon^{n-m}\|_\infty) \\ &\quad + \beta_1 (\|I + D\|_\infty \|\epsilon^{n-1}\|_\infty + \tau(\beta_1 \|\epsilon^{n-1}\|_\infty + \beta_2 \|\epsilon^{n-m}\|_\infty)) + \beta_2 \|\epsilon^{n-m}\|_\infty] + O(\tau^3 + \tau h^2). \end{aligned}$$

Denote $e^n = \max_{0 \leq i \leq n} \{\|\epsilon^i\|_\infty\}$. Note that $\|I + D + \frac{1}{2} D^2\|_\infty \leq 1$, we have

$$\begin{aligned} e^n &\leq e^{n-1} + \frac{\tau}{2} \|D\|_\infty (\beta_1 e^{n-1} + \beta_2 e^{n-1}) \\ &\quad + \frac{\tau}{2} [(\beta_1 e^{n-1} + \beta_2 e^{n-1}) + \beta_1 (e^{n-1} + \tau(\beta_1 e^{n-1} + \beta_2 e^{n-1})) + \beta_2 e^{n-1}] + O(\tau^3 + \tau h^2) \\ &\leq \left[1 + \tau(\beta_1 + \beta_2)(1 + \frac{1}{2}(\|D\|_\infty + \beta_1 \tau)) \right] e^{n-1} + O(\tau^3 + \tau h^2) \leq (1 + \tau L) e^{n-1} + O(\tau^3 + \tau h^2), \end{aligned}$$

where $L = (\beta_1 + \beta_2)(1 + \frac{1}{2}(\|D\|_\infty + \beta_1 \tau))$. Thus,

$$e^n \leq (1 + \tau L)^n e^0 \leq e^{n\tau L} e^0 + \frac{O(\tau^3 + \tau h^2)}{\tau L} [(1 + \tau L)^n - 1].$$

Note that $e^0 = 0$, then

$$e^n \leq \frac{O(\tau^3 + \tau h^2)}{\tau L} [(1 + \tau L)^n - 1] \leq \frac{O(\tau^2 + h^2)}{L} [e^{\tau L} - 1] = O(\tau^2 + h^2). \quad \square$$

Remark 4.4. Let $\mu_\alpha = -\frac{\tau K_\alpha c_\alpha}{h^\alpha}$ and $B = A + A^T$. Then, the matrix $I + D$ can be written as $I + D = I + \mu_\alpha B$. In the following, we discuss the range of value μ_α when $\|I + \mu_\alpha B\| \leq 1$.

If we take $M = 40, 200$ and investigate the range of value μ_α when α varies from 1.1 to 2, the numerical results can be shown in Fig. 1. From Fig. 1, we can see that the range of value μ_α is dependent on α , but it is almost independent of M . In this paper, we select μ_α as shown in Table 1.

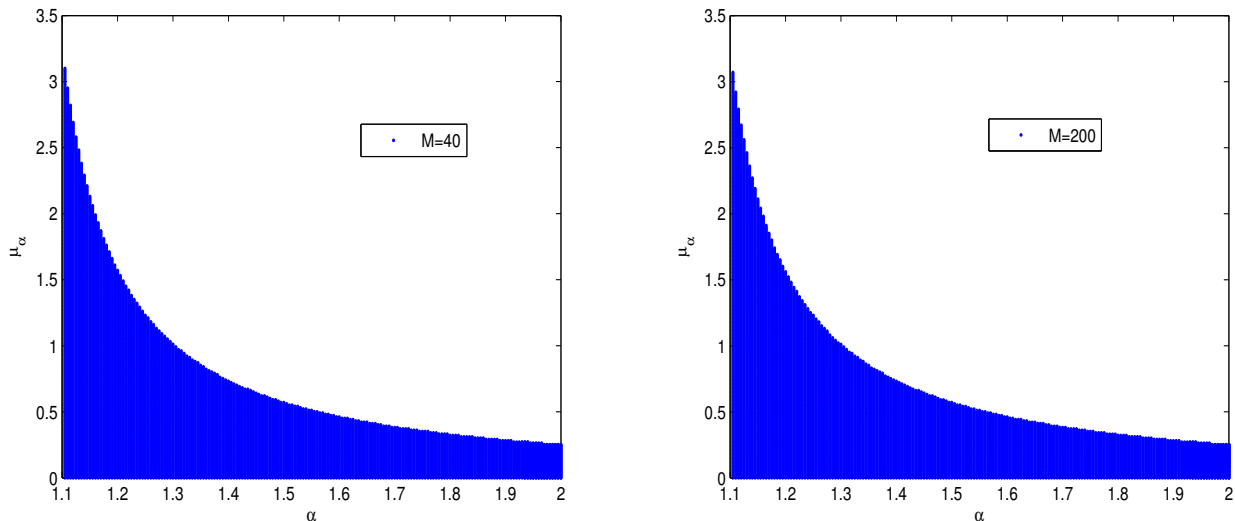


Figure 1: The range of value μ_α when α varies from 1.1 to 2.

Table 1: The range of the value μ_α .

α	(1.0, 1.1]	(1.1, 1.2]	(1.2, 1.3]	(1.3, 1.4]	(1.4, 1.5]
μ_α	(0, 3.23]	(0, 1.56]	(0, 1.01]	(0, 0.73]	(0, 0.57]
α	(1.5, 1.6]	(1.6, 1.7]	(1.7, 1.8]	(1.8, 1.9]	(1.9, 2.0]
μ_α	(0, 0.46]	(0, 0.38]	(0, 0.32]	(0, 0.28]	(0, 0.25]

5. Numerical examples

In order to demonstrate our theoretical results, we present some numerical examples in this section.

Example 5.1. Consider the following Riesz space fractional diffusion equations with delay:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(t, x, u(x, t), u(x, t-s)), & t \in [0, T], \\ u(x, t) = x^2(x-1)^2 e^{-t}, & t \in [-s, 0], \quad 0 \leq x \leq 1, \\ u(0, t) = u(1, t) = 0, & 0 \leq t \leq T, \end{cases} \quad (5.1)$$

where $T = 10, s = 1.0, K_\alpha = 1, 1 < \alpha \leq 2$,

$$\begin{aligned} f(x, t, u(x, t), u(x, t-s)) = & u(x, t)u(x, t-s) - x^2(1-x)^2 e^{-t} - x^4(1-x)^4 e^{-2t+\tau} \\ & + \frac{e^{-t}}{2\cos(\frac{\alpha\pi}{2})} \left\{ \frac{24}{\Gamma(5-\alpha)} [x^{4-\alpha} + (1-x)^{4-\alpha}] - \frac{12}{\Gamma(4-\alpha)} [x^{3-\alpha} + (1-x)^{3-\alpha}] \right. \\ & \left. + \frac{2}{\Gamma(3-\alpha)} [x^{2-\alpha} + (1-x)^{2-\alpha}] \right\}, \end{aligned}$$

and the exact solution is $u(x, t) = x^2(1-x)^2 e^{-t}$.

For different $\alpha \in (1, 2]$, the numerical solutions of (5.1) are obtained by using (3.3). When $\alpha = 1.1$, the errors and the computing orders of (3.3) are shown in Table 2. When $\alpha = 1.5$, we select $\mu_\alpha = 0.57$, the corresponding numerical results are shown in Table 2. All of the numerical results show that the errors between the numerical solutions and the exact solutions are small and (3.3) for solving (1.1) is convergent and robust.

Table 2: The error and convergence order of the method (3.3).

$\alpha = 1.1$				$\alpha = 1.5$			
τ	h	$\ e(h, \tau)\ _\infty$	Order	τ	h	$\ e(h, \tau)\ _\infty$	Order
$\frac{1}{8}$	$\frac{10}{200}$	1.5206E-03		$\frac{1}{8}$	$\frac{10}{280}$	2.2081E-03	
$\frac{1}{16}$	$\frac{10}{400}$	4.2502E-04	1.81	$\frac{1}{16}$	$\frac{10}{793}$	4.5625E-04	2.27
$\frac{1}{32}$	$\frac{10}{800}$	1.3587E-04	1.68	$\frac{1}{32}$	$\frac{10}{2245}$	1.1408E-04	2.00
$\frac{1}{64}$	$\frac{10}{1600}$	4.2558E-05	1.67	$\frac{1}{64}$	$\frac{10}{6351}$	3.2659E-05	1.80

Example 5.2. Consider the following Riesz space fractional diffusion equations with delay:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(t, x, u(x, t), u(x, t-s)), & t \in [0, T], \\ u(x, t) = x^2(x-1)^2 e^{-t}, & t \in [-s, 0], \quad 0 \leq x \leq 1, \\ u(0, t) = u(1, t) = 0, & 0 \leq t \leq T, \end{cases} \quad (5.2)$$

where $T = 10, s = 0.5, 1 < \alpha \leq 2, K_\alpha = 2$,

$$\begin{aligned} f(x, t, u(x, t), u(x, t-s)) &= (u(x, t)u(x, t-s))^2 - x^2(1-x)^2 e^{-t} - x^8(1-x)^8 e^{-4t+1} \\ &+ \frac{K_\alpha e^{-t}}{2\cos(\frac{\alpha\pi}{2})} \left\{ \frac{24}{\Gamma(5-\alpha)} [x^{4-\alpha} + (1-x)^{4-\alpha}] - \frac{12}{\Gamma(4-\alpha)} [x^{3-\alpha} + (1-x)^{3-\alpha}] \right. \\ &\left. + \frac{2}{\Gamma(3-\alpha)} [x^{12-\alpha} + (1-x)^{2-\alpha}] \right\}, \end{aligned}$$

and the exact solution is $u(x, t) = x^2(1-x)^2 e^{-t}$.

For different $\alpha \in (1, 2]$, the numerical solutions of (5.2) are obtained by using (3.3). When $\alpha = 1.2$ and $\alpha = 1.5$, we select $\mu_\alpha = 1.56$ and $\mu_\alpha = 0.57$, respectively, the errors and the computing orders of (3.3) are shown in Table 3. From the above computing results, we can find that the errors are also very small. Thus, we can draw the conclusion that (3.3) for solving (1.1) is stable and convergent.

Table 3: The error and convergence order of the method (3.3).

$\alpha = 1.2$				$\alpha = 1.5$			
τ	h	$\ e(h, \tau)\ _\infty$	Order	τ	h	$\ e(h, \tau)\ _\infty$	Order
$\frac{1}{8}$	$\frac{10}{251}$	2.0059E-03		$\frac{1}{8}$	$\frac{10}{561}$	2.7748E-03	
$\frac{1}{16}$	$\frac{10}{577}$	4.9531E-04	2.02	$\frac{1}{16}$	$\frac{10}{1587}$	6.2062E-04	2.23
$\frac{1}{32}$	$\frac{10}{1327}$	1.5728E-04	1.66	$\frac{1}{32}$	$\frac{10}{4491}$	1.3245E-04	2.23
$\frac{1}{64}$	$\frac{10}{3050}$	4.2916E-05	1.87	$\frac{1}{64}$	$\frac{10}{12703}$	3.4513E-05	1.94

6. Results and discussion

In this research, the finite difference method which combines a time scheme based on the predictor-corrector method and the Crank-Nicolson scheme for the spatial discretization has been employed to solve a class of the Riesz space fractional diffusion equations with delay and a nonlinear source term.

The corresponding theoretical results including stability and convergence are provided. Compared to the other implicit methods, the nonlinear system which is derived from discrete form of (1.1) does not have to be solved in this paper. The numerical results obtained by the proposed method confirm the robustness and the efficiency of it. And we hope that this work is a step in this direction.

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