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On the rational difference equation $y_{n+1} = \frac{\alpha_0 y_n+\alpha_1 y_{n-p}+\alpha_2 y_{n-q}+\alpha_3 y_{n-r}+\alpha_4 y_{n-s}}{\beta_0 y_n+\beta_1 y_{n-p}+\beta_2 y_{n-q}+\beta_3 y_{n-r}+\beta_4 y_{n-s}}$

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Abstract

In this paper, we examine and explore the boundedness, periodicity, and global stability of the positive solutions of the rational difference equation

$$y_{n+1} = \frac{\alpha_0 y_n+\alpha_1 y_{n-p}+\alpha_2 y_{n-q}+\alpha_3 y_{n-r}+\alpha_4 y_{n-s}}{\beta_0 y_n+\beta_1 y_{n-p}+\beta_2 y_{n-q}+\beta_3 y_{n-r}+\beta_4 y_{n-s}},$$

where the coefficients $\alpha_i, \beta_i \in (0,\infty)$, $i = 0,1,2,3,4$, and p, q, r, and s are positive integers. The initial conditions $y_{-s},\ldots,y_{-r},\ldots, y_{-q},\ldots, y_{-p},\ldots,y_{-1},y_0$ are arbitrary positive real numbers such that $p < q < r < s$. Some numerical examples will be given to illustrate our result.

Keywords: Difference equation, boundedness, prime period two solution, global stability.

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1. Introduction

Difference equations occur as mathematical models in various real world applications such as problems in physiology, engineering, ecology, and many more. In this direction, many such problems are nonlinear in nature and thus the research focus is on nonlinear difference equations. One such class of equations which have attracted attentions of researchers is the rational difference equations.

This paper is inspired and extends the work on rational difference equation in [23]. Specifically, the core of our work here is to study qualitative properties such as the local and global stability, boundedness, periodicity of the positive solutions of the rational difference equation

$$y_{n+1} = \frac{\alpha_0 y_n+\alpha_1 y_{n-p}+\alpha_2 y_{n-q}+\alpha_3 y_{n-r}+\alpha_4 y_{n-s}}{\beta_0 y_n+\beta_1 y_{n-p}+\beta_2 y_{n-q}+\beta_3 y_{n-r}+\beta_4 y_{n-s}}, \tag{1.1}$$

where the coefficients $\alpha_i, \beta_i \in (0,\infty)$, $i = 0,1,2,3,4$, and p, q, r and s are positive integers. The initial

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conditions $y_{-s},\dots,y_{-r},\dots,y_{-q},\dots,y_{-p},\dots,y_{-1},y_0$ are arbitrary positive real numbers such that $p < q < r < s$. We illustrate our findings by considering numerical examples which represent different types of solutions of equation (1.1). In the special case when any of the coefficients α_i, β_i for $i = 0,\dots,4$ are allowed to be zero, equation (1.1) reduces to the distinct cases which have been studied by many authors. In particular, apart from [23], equation (1.1) can be considered as a generalization of that studied in [2, 10, 17]. Other related works on rational difference equation can be found in Refs. [1, 3, 4, 6–9, 11–15, 18]. See also [16, 19–22, 24].

We first recall some basic properties and definitions associated with difference equations.

Definition 1.1. A difference equation of order $(s+1)$ is of the form

$$y_{n+1} = H(y_n, y_{n-1}, \dots, y_{n-p}, \dots, y_{n-q}, \dots, y_{n-r}, \dots\dots, y_{n-s}), \quad n = 0, 1, 2, \dots \tag{1.2}$$

with $p < q < r < s$ where H is a continuous function. An equilibrium point $\tilde{y}$ of this difference equation is a point that satisfies the condition $\tilde{y} = H(\tilde{y},\tilde{y},\dots,\tilde{y})$. That is the constant sequence $\{y_n\}_{n=-s}^{\infty}$ with $y_n = \tilde{y}$ for all $n \geqslant -s$ is a solution of that equation.

Definition 1.2. Let $\tilde{y} \in (0,\infty)$ be an equilibrium point of equation (1.2). Then, we have

1. An equilibrium point $\tilde{y}$ of equation (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $y_{-s}, y_{-r}, y_{-r+1}, \dots, y_{-1}, y_0 \in (0,\infty)$ with $|y_{-s} - \tilde{y}| + |y_{-r} - \tilde{y}| + |y_{-r+1} - \tilde{y}| + \cdots + |y_{-1} - \tilde{y}| + |y_0 - \tilde{y}| < \delta$, then $|y_n - \tilde{y}| < \varepsilon$ for all $n \geqslant -s$.
2. An equilibrium point $\tilde{y}$ of equation (1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $y_{-s}, y_{-r}, y_{-r+1}, \dots, y_{-1}, y_0 \in (0,\infty)$ with $|y_{-s} - \tilde{y}| + |y_{-r} - \tilde{y}| + |y_{-r+1} - \tilde{y}| + \cdots + |y_{-1} - \tilde{y}| + |y_0 - \tilde{y}| < \gamma$, then
$$\lim_{n\to\infty} y_n = \tilde{y}.$$
3. An equilibrium point $\tilde{y}$ of equation (1.2) is called a global attractor if for every $y_{-s}, \dots, y_{-1}, y_0 \in (0,\infty)$ we have
$$\lim_{n\to\infty} y_n = \tilde{y}.$$
4. An equilibrium point $\tilde{y}$ of equation (1.2) is called globally asymptotically stable if it is locally stable and a global attractor.
5. An equilibrium point $\tilde{y}$ of equation (1.2) is called unstable if it is not locally stable.

Definition 1.3. A solution $\{y_n\}_{n=-s}^{\infty}$ of equation (1.2) is said to be periodic with period r if

$$y_{n+r} = y_n \text{ for all } n \geqslant -s.$$

Moreover, if r is the smallest positive integer having this property, then this periodic solution is said to have prime period r.

Theorem 1.4 ([5]). *Let $H : [a,b]^{s+1} \to [a,b]$ be a continuous function, where s is a positive integer, and where $[a,b]$ is an interval of real numbers. Consider the difference equation (1.2). Suppose that H satisfies the following conditions.*

1. *For each integer i with $1 \leqslant i \leqslant s+1$, the function $H(z_1, z_2, \dots, z_{s+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, \dots, z_{i-1}, z_{i+1}, \dots, z_{s+1}$.*
2. *If (m, M) is a solution of the system*
$$M = H(M_1, M_2, \dots, M_{s+1}) \quad \textit{and} \quad m = H(m_1, m_2, \dots, m_{s+1}),$$

then $M = m$, *where for each* $i = 1, 2, \ldots, s+1$, *we set*

$$M_i = \begin{cases} M, & \text{if } H \text{ is non-decreasing in } z_i, \\ m, & \text{if } H \text{ is non-increasing in } z_i, \end{cases}$$

and

$$m_i = \begin{cases} m, & \text{if } H \text{ is non-decreasing in } z_i, \\ M, & \text{if } H \text{ is non-increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium $\widetilde{y}$ *of equation* (1.2), *and every solution of equation* (1.2) *converges to* $\widetilde{y}$.

This paper is structured as follows. The local stability of the solutions of the difference equation (1.1) is introduced in Section 2. The boundedness character of the positive solution of equation (1.1) is addressed in Section 3. The periodicity of the positive solutions of equation (1.1) is investigated in Section 4. In Section 5 the global stability of the positive solution of equation (1.1) is studied. Section 6 deals with numerical experiments on the main results. Finally, we end with the conclusion in Section 7.

2. Local stability of the equilibrium solution

The local stability of the solutions of equation (1.1) is examined in this section. Assume that $\widetilde{c} = \sum_{i=0}^{4} \alpha_i$ and $\widetilde{d} = \sum_{i=0}^{4} \beta_i$. Then, one can easily check that

$$\widetilde{y} = \frac{\widetilde{c}}{\widetilde{d}}$$

is the positive equilibrium point $\widetilde{y}$ of equation (1.1). Now, let

$$H : (0, \infty)^5 \longrightarrow (0, \infty)$$

be a continuous function defined by

$$H(u_0, \ldots, u_4) = \frac{\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4}{\beta_0 u_0 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4}.$$

By taking partial derivatives of H, then the linearized equation about the positive equilibrium point $\widetilde{y}$ takes the form

$$y_{n+1} + a_4 y_n + a_3 y_{n-p} + a_2 y_{n-q} + a_1 y_{n-r} + a_0 y_{n-s} = 0,$$

where

$$\begin{aligned} a_4 &= -\frac{(\alpha_0\beta_1 - \alpha_1\beta_0) + (\alpha_0\beta_2 - \alpha_2\beta_0) + (\alpha_0\beta_3 - \alpha_3\beta_0) + (\alpha_0\beta_4 - \alpha_4\beta_0)}{\widetilde{c}\widetilde{d}}, \\ a_3 &= -\frac{(\alpha_1\beta_0 - \alpha_0\beta_1) + (\alpha_1\beta_2 - \alpha_2\beta_1) + (\alpha_1\beta_3 - \alpha_3\beta_1) + (\alpha_1\beta_4 - \alpha_4\beta_1)}{\widetilde{c}\widetilde{d}}, \\ a_2 &= -\frac{(\alpha_2\beta_0 - \alpha_0\beta_2) + (\alpha_2\beta_1 - \alpha_1\beta_2) + (\alpha_2\beta_3 - \alpha_3\beta_2) + (\alpha_2\beta_4 - \alpha_4\beta_2)}{\widetilde{c}\widetilde{d}}, \\ a_1 &= -\frac{(\alpha_3\beta_0 - \alpha_0\beta_3) + (\alpha_3\beta_1 - \alpha_1\beta_3) + (\alpha_3\beta_2 - \alpha_2\beta_3) + (\alpha_3\beta_4 - \alpha_4\beta_3)}{\widetilde{c}\widetilde{d}}, \\ a_0 &= -\frac{(\alpha_4\beta_0 - \alpha_0\beta_4) + (\alpha_4\beta_1 - \alpha_1\beta_4) + (\alpha_4\beta_2 - \alpha_2\beta_4) + (\alpha_4\beta_3 - \alpha_3\beta_4)}{\widetilde{c}\widetilde{d}}. \end{aligned} \tag{2.1}$$

Theorem 2.1 ([5]). *Assume that* $e_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. *Then,*

$$\sum_{i=1}^{k} |e_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + e_1 y_{n+k-1} + \cdots + e_k y_n = 0, \quad n = 0, 1, 2, \ldots.$$

Theorem 2.2. *Assume that*

$$\begin{aligned}
&|(\alpha_0\beta_1-\alpha_1\beta_0)+(\alpha_0\beta_2-\alpha_2\beta_0)+(\alpha_0\beta_3-\alpha_3\beta_0)+(\alpha_0\beta_4-\alpha_4\beta_0)| \\
&\quad +|(\alpha_1\beta_0-\alpha_0\beta_1)+(\alpha_1\beta_2-\alpha_2\beta_1)+(\alpha_1\beta_3-\alpha_3\beta_1)+(\alpha_1\beta_4-\alpha_4\beta_1)| \\
&\quad +|(\alpha_2\beta_0-\alpha_0\beta_2)+(\alpha_2\beta_1-\alpha_1\beta_2)+(\alpha_2\beta_3-\alpha_3\beta_2)+(\alpha_2\beta_4-\alpha_4\beta_2)| \\
&\quad +|(\alpha_3\beta_0-\alpha_0\beta_3)+(\alpha_3\beta_1-\alpha_1\beta_3)+(\alpha_3\beta_2-\alpha_2\beta_3)+(\alpha_3\beta_4-\alpha_4\beta_3)| \\
&\quad +|(\alpha_4\beta_0-\alpha_0\beta_4)+(\alpha_4\beta_1-\alpha_1\beta_4)+(\alpha_4\beta_2-\alpha_2\beta_4)+(\alpha_4\beta_3-\alpha_3\beta_4)| < \widetilde{c}\widetilde{d}.
\end{aligned}$$

Therefore the positive equilibrium point $\widetilde{y}$ of equation (1.1) is locally asymptotically stable.

Proof. From (2.1) and the assumption of this theorem, it is obvious that

$$\sum_{i=0}^{4} |a_i| < 1.$$

Thus, by Theorem 2.1, equation (1.1) is asymptotically stable. □

3. Boundedness of the solutions

In this section, the boundedness character of the positive solution of equation (1.1) is being studied. First recall that a sequence $\{y_n\}_{n=-s}^{\infty}$ is bounded if there exists positive constants m and M such that for all $n \geqslant -s$

$$m \leqslant y_n \leqslant M.$$

Theorem 3.1. *Every solution of equation (1.1) is bounded.*

Proof. Let

$$\begin{cases} m_0 = \min\ \alpha_i, & i = 0,\dots,4, \\ M_0 = \max\ \alpha_i, & i = 0,\dots,4, \\ l = \min\ \beta_i, & i = 0,\dots,4, \\ L = \max\ \beta_i, & i = 0,\dots,4. \end{cases}$$

We have

$$\frac{m_0\,(y_n+y_{n-p}+y_{n-q}+y_{n-r}+y_{n-s})}{L\,(y_n+y_{n-p}+y_{n-q}+y_{n-r}+y_{n-s})} \leqslant y_{n+1} \leqslant \frac{M_0\,(y_n+y_{n-p}+y_{n-q}+y_{n-r}+y_{n-s})}{l\,(y_n+y_{n-p}+y_{n-q}+y_{n-r}+y_{n-s})}$$
$$\frac{m_0}{L} \leqslant y_{n+1} \leqslant \frac{M_0}{l},$$

which implies that every solution of equation (1.1) is bounded. □

4. The periodicity of the solutions

In this section, we analyze the periodic character of the positive solution of equation (1.1).

Theorem 4.1. *If one of the following conditions holds, then equation (1.1) has no positive solutions of prime period two.*

1. *The positive integers $p, q, r,$ and s are even.*
2. *The positive integers p, q are even and the positive integers r and s are odd provided $\alpha_0+\alpha_1+\alpha_2 \geqslant \alpha_3+\alpha_4$.*
3. *The positive integers p, q are odd and the positive integers r, s are even provided $\alpha_1+\alpha_2 \geqslant \alpha_0+\alpha_3+\alpha_4$.*
4. *The positive integers p, r are even and the positive integers q, s are odd provided $\alpha_0+\alpha_1+\alpha_3 \geqslant \alpha_2+\alpha_4$.*

5. *The positive integers* q, r *are even and the positive integers* p, s *are odd provided* $\alpha_0 + \alpha_2 + \alpha_3 \geqslant \alpha_1 + \alpha_4$.
6. *The positive integers* q, r *are odd and the positive integers* p, s *are even provided* $\alpha_0 + \alpha_1 + \alpha_4 \geqslant \alpha_2 + \alpha_3$.
7. *The positive integers* p, r *are even and the positive integers* q, s *are odd provided* $\alpha_0 + \alpha_2 + \alpha_4 \geqslant \alpha_1 + \alpha_3$.
8. *The positive integers* p, q, r *and* s *are odd,* $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > \alpha_0$ *and* $\beta_1 + \beta_2 + \beta_3 + \beta_4 > \beta_0$.

Proof. Suppose that there exist positive distinctive solutions of prime period two $\ldots, A, B, A, B, \ldots$ of equation (1.1). Then the following cases are discussed.

Case 1. $p, q, r,$ and s are even positive integers. In this case $y_n = y_{n-p} = y_{n-q} = y_{n-r} = y_{n-s}$. Then there exist a positive period two solution $\{y_n\}$ such that

$$y_{2a} = A, a = -1, 0, 1, \ldots, \quad y_{2a+1} = B, a = -1, 0, 1, \ldots,$$

where $A \neq B$. From equation (1.1) we have

$$A = B = \frac{\widetilde{c}}{\widetilde{\widetilde{d}}}.$$

Thus, there is an inconsistency.

Case 2. p, q are even and the positive integers r and s are odd. In this case $y_n = y_{n-p} = y_{n-q}$ and $y_{n+1} = y_{n-r} = y_{n-s}$. From equation (1.1) we have

$$A = \frac{(\alpha_0 + \alpha_1 + \alpha_2) B + (\alpha_3 + \alpha_4) A}{(\beta_0 + \beta_1 + \beta_2) B + (\beta_3 + \beta_4) A}, \quad B = \frac{(\alpha_0 + \alpha_1 + \alpha_2) A + (\alpha_3 + \alpha_4) B}{(\beta_0 + \beta_1 + \beta_2) A + (\beta_3 + \beta_4) B}.$$

As a result, it is obtained that

$$(\alpha_0 + \alpha_1 + \alpha_2) B + (\alpha_3 + \alpha_4) A = (\beta_0 + \beta_1 + \beta_2) AB + (\beta_3 + \beta_4) A^2$$

and

$$(\alpha_0 + \alpha_1 + \alpha_2) A + (\alpha_3 + \alpha_4) B = (\beta_0 + \beta_1 + \beta_2) AB + (\beta_3 + \beta_4) B^2.$$

By subtracting, we acquire

$$A + B = -\frac{[(\alpha_0 + \alpha_1 + \alpha_2) - (\alpha_3 + \alpha_4)]}{\beta_3 + \beta_4},$$

since $\alpha_0 + \alpha_1 + \alpha_2 \geqslant \alpha_3 + \alpha_4$, we have $A + B \leqslant 0$. Thus, it leads to a contradiction.

Case 3. p, q are positive odd and the positive integers r, s are even. In this case $y_{n+1} = y_{n-p} = y_{n-q}$ and $y_n = y_{n-r} = y_{n-s}$. From equation (1.1) we have

$$A = \frac{(\alpha_1 + \alpha_2) A + (\alpha_0 + \alpha_3 + \alpha_4) B}{(\beta_1 + \beta_2) A + (\beta_0 + \beta_3 + \beta_4) B}, \quad B = \frac{(\alpha_1 + \alpha_2) B + (\alpha_0 + \alpha_3 + \alpha_4) A}{(\beta_1 + \beta_2) B + (\beta_0 + \beta_3 + \beta_4) A}.$$

Thus, we realize

$$(\alpha_1 + \alpha_2) A + (\alpha_0 + \alpha_3 + \alpha_4) B = (\beta_1 + \beta_2) A^2 + (\beta_0 + \beta_3 + \beta_4) AB$$

and

$$(\alpha_1 + \alpha_2) B + (\alpha_0 + \alpha_3 + \alpha_4) A = (\beta_1 + \beta_2) B^2 + (\beta_0 + \beta_3 + \beta_4) AB.$$

By subtracting we have

$$A + B = -\frac{[(\alpha_1 + \alpha_2) - (\alpha_0 + \alpha_3 + \alpha_4)]}{\beta_1 + \beta_2},$$

since $\alpha_1 + \alpha_2 \geqslant \alpha_0 + \alpha_3 + \alpha_4$, we have $A + B \leqslant 0$. Thus we again have a contradiction.

Case 4. p, r are positive even integers and the positive integers q, s are odd. In this case $y_n = y_{n-p} = y_{n-r}$ and $y_{n+1} = y_{n-q} = y_{n-s}$. From equation (1.1) we have

$$A = \frac{(\alpha_0 + \alpha_1 + \alpha_3)\, B + (\alpha_2 + \alpha_4)\, A}{(\beta_0 + \beta_1 + \beta_3)\, B + (\beta_2 + \beta_4)\, A}, \quad B = \frac{(\alpha_0 + \alpha_1 + \alpha_3)\, A + (\alpha_2 + \alpha_4)\, B}{(\beta_0 + \beta_1 + \beta_3)\, A + (\beta_2 + \beta_4)\, B}.$$

Therefore, it is found that

$$(\alpha_0 + \alpha_1 + \alpha_3)\, B + (\alpha_2 + \alpha_4)\, A = (\beta_0 + \beta_1 + \beta_3)\, AB + (\beta_2 + \beta_4)\, A^2$$

and

$$(\alpha_0 + \alpha_1 + \alpha_3)\, A + (\alpha_2 + \alpha_4)\, B = (\beta_0 + \beta_1 + \beta_3)\, AB + (\beta_2 + \beta_4)\, B^2.$$

By subtracting we sustain

$$A + B = -\frac{[(\alpha_0 + \alpha_1 + \alpha_3) - (\alpha_2 + \alpha_4)]}{\beta_2 + \beta_4},$$

since $\alpha_0 + \alpha_1 + \alpha_3 \geqslant \alpha_2 + \alpha_4$, we have $A + B \leqslant 0$. Thus we have a contradiction.

Case 5. q, r are positive even integers and the positive integers p, s are odd. In this case $y_n = y_{n-q} = y_{n-r}$ and $y_{n+1} = y_{n-p} = y_{n-s}$. From equation (1.1) we have

$$A = \frac{(\alpha_0 + \alpha_2 + \alpha_3)\, B + (\alpha_1 + \alpha_4)\, A}{(\beta_0 + \beta_2 + \beta_3)\, B + (\beta_1 + \beta_4)\, A}, \quad B = \frac{(\alpha_0 + \alpha_2 + \alpha_3)\, A + (\alpha_1 + \alpha_4)\, B}{(\beta_0 + \beta_2 + \beta_3)\, A + (\beta_1 + \beta_4)\, B}.$$

Consequently, we obtain

$$(\alpha_0 + \alpha_2 + \alpha_3)\, B + (\alpha_1 + \alpha_4)\, A = (\beta_0 + \beta_2 + \beta_3)\, AB + (\beta_1 + \beta_4)\, A^2$$

and

$$(\alpha_0 + \alpha_2 + \alpha_3)\, A + (\alpha_1 + \alpha_4)\, B = (\beta_0 + \beta_2 + \beta_3)\, AB + (\beta_1 + \beta_4)\, B^2.$$

By subtracting, we have

$$A + B = -\frac{[(\alpha_0 + \alpha_2 + \alpha_3) - (\alpha_1 + \alpha_4)]}{\beta_1 + \beta_4},$$

since $\alpha_0 + \alpha_2 + \alpha_3 \geqslant \alpha_1 + \alpha_4$, we have $A + B \leqslant 0$. Thus, there is also another contradiction.

Case 6. q, r are positive odd integers and the positive integers p, s are even. In this case $y_{n+1} = y_{n-q} = y_{n-r}$ and $y_n = y_{n-p} = y_{n-s}$. From equation (1.1) we have

$$A = \frac{(\alpha_0 + \alpha_1 + \alpha_4)\, B + (\alpha_2 + \alpha_3)\, A}{(\beta_0 + \beta_1 + \beta_4)\, B + (\beta_2 + \beta_3)\, A}, \quad B = \frac{(\alpha_0 + \alpha_1 + \alpha_4)\, A + (\alpha_2 + \alpha_3)\, B}{(\beta_0 + \beta_1 + \beta_4)\, A + (\beta_2 + \beta_3)\, B}.$$

As a result, it is found that

$$(\alpha_0 + \alpha_1 + \alpha_4)\, B + (\alpha_2 + \alpha_3)\, A = (\beta_0 + \beta_1 + \beta_4)\, AB + (\beta_2 + \beta_3)\, A^2$$

and

$$(\alpha_0 + \alpha_1 + \alpha_4)\, A + (\alpha_2 + \alpha_3)\, B = (\beta_0 + \beta_1 + \beta_4)\, AB + (\beta_2 + \beta_3)\, B^2.$$

By subtracting, we have

$$A + B = -\frac{[(\alpha_0 + \alpha_1 + \alpha_4) - (\alpha_2 + \alpha_3)]}{\beta_2 + \beta_3},$$

since $\alpha_0 + \alpha_1 + \alpha_4 \geqslant \alpha_2 + \alpha_3$, we have $A + B \leqslant 0$. Thus we have another contradiction.

Case 7. p, r are positive odd integers and the positive integers q, s are even. In this case $y_{n+1} = y_{n-p} = y_{n-r}$ and $y_n = y_{n-q} = y_{n-s}$. From equation (1.1) we have

$$A = \frac{(\alpha_0 + \alpha_2 + \alpha_4) B + (\alpha_1 + \alpha_3) A}{(\beta_0 + \beta_2 + \beta_4) B + (\beta_1 + \beta_3) A}, \quad B = \frac{(\alpha_0 + \alpha_2 + \alpha_4) A + (\alpha_1 + \alpha_3) B}{(\beta_0 + \beta_2 + \beta_4) A + (\beta_1 + \beta_3) B}.$$

Accordingly, it is acquired

$$(\alpha_0 + \alpha_2 + \alpha_4) B + (\alpha_1 + \alpha_3) A = (\beta_0 + \beta_2 + \beta_4) AB + (\beta_1 + \beta_3) A^2$$

and

$$(\alpha_0 + \alpha_2 + \alpha_4) A + (\alpha_1 + \alpha_3) B = (\beta_0 + \beta_2 + \beta_4) AB + (\beta_1 + \beta_3) B^2.$$

By subtracting, we sustain

$$A + B = -\frac{[(\alpha_0 + \alpha_2 + \alpha_4) - (\alpha_1 + \alpha_3)]}{\beta_1 + \beta_3},$$

since $\alpha_0 + \alpha_2 + \alpha_4 \geqslant \alpha_1 + \alpha_3$, we have $A + B \leqslant 0$. Thus we have a contradiction.

Case 8. p, q, r and the positive integer s is odd. In this case $y_{n+1} = y_{n-p} = y_{n-q} = y_{n-r} = y_{n-s}$. From equation (1.1) we have

$$A = \frac{(\alpha_0) B + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) A}{(\beta_0) B + (\beta_1 + \beta_2 + \beta_3 + \beta_4) A}, \quad B = \frac{(\alpha_0) A + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) B}{(\beta_0) A + (\beta_1 + \beta_2 + \beta_3 + \beta_4) B}.$$

Thus, we obtain

$$(\alpha_0) B + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) A = (\beta_0) AB + (\beta_1 + \beta_2 + \beta_3 + \beta_4) A^2$$

and

$$(\alpha_0) A + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) B = (\beta_0) AB + (\beta_1 + \beta_2 + \beta_3 + \beta_4) B^2.$$

By subtracting, we possess

$$A + B = \frac{[(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - \alpha_0]}{\beta_1 + \beta_2 + \beta_3 + \beta_4}.$$

By adding we obtain

$$AB = \frac{\alpha_0[(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - \alpha_0]}{(\beta_1 + \beta_2 + \beta_3 + \beta_4)\,[(\beta_1 + \beta_2 + \beta_3 + \beta_4) - \beta_0]},$$

since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > \alpha_0$ and $(\beta_1 + \beta_2 + \beta_3 + \beta_4) > \beta_0$, we have $AB < 0$. Thus, we have a contradiction. □

Theorem 4.2. *Suppose* p, q, r, *and* s *are odd,* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \alpha_0$ *and* $\beta_1 + \beta_2 + \beta_3 + \beta_4 < \beta_0$. *Then equation* (1.1) *shall have positive solutions of prime period two if and only if*

$$4\alpha_0 (\beta_1 + \beta_2 + \beta_3 + \beta_4) < [(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - \alpha_0][\beta_0 - (\beta_1 + \beta_2 + \beta_3 + \beta_4)]. \tag{4.1}$$

Proof. Suppose that there exist positive distinctive solutions of prime period two

$$\ldots, A, B, A, B, \ldots.$$

of equation (1.1). Since p, q, r, and s are odd, we have $y_{n+1} = y_{n-p} = y_{n-q} = y_{n-r} = y_{n-s}$. From equation (1.1) we have

$$A = \frac{(\alpha_0) B + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) A}{(\beta_0) B + (\beta_1 + \beta_2 + \beta_3 + \beta_4) A}, \quad B = \frac{(\alpha_0) A + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) B}{(\beta_0) A + (\beta_1 + \beta_2 + \beta_3 + \beta_4) B},$$

subsequently, we obtain

$$(\alpha_0)\,B + (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\,A = (\beta_0)\,AB + (\beta_1+\beta_2+\beta_3+\beta_4)\,AA^2$$

and

$$(\alpha_0)\,A + (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\,B = (\beta_0)\,AB + (\beta_1+\beta_2+\beta_3+\beta_4)\,AB^2.$$

By subtracting, it is sustained

$$A+B = \frac{[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{\beta_1+\beta_2+\beta_3+\beta_4}.$$

By adding up, we acquire

$$AB = \frac{\alpha_0[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{(\beta_1+\beta_2+\beta_3+\beta_4)\,[\beta_0-(\beta_1+\beta_2+\beta_3+\beta_4)]},$$

where $(\alpha_1+\alpha_2+\alpha_3+\alpha_4) > \alpha_0$ and $\beta_0 > (\beta_1+\beta_2+\beta_3+\beta_4)$. Assume that A and B are two positive distinct real roots of the quadratic equation

$$t^2-(A+B)\,t+AB=0. \tag{4.2}$$

Then, we deduce that

$$\left(\frac{[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{(\beta_1+\beta_2+\beta_3+\beta_4)}\right)^2 > \frac{4\alpha_0[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{(\beta_1+\beta_2+\beta_3+\beta_4)\,[\beta_0-(\beta_1+\beta_2+\beta_3+\beta_4)]}. \tag{4.3}$$

From equation (4.3), it is obtained

$$4\alpha_0\,(\beta_1+\beta_2+\beta_3+\beta_4) < [(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0][\beta_0-(q\beta_1+\beta_2+\beta_3+\beta_4)].$$

Hence, the condition (4.1) is valid. Contrariwise, presume that the condition (4.1) is valid where $(\alpha_1+\alpha_2+\alpha_3+\alpha_4) > \alpha_0$ and $\beta_0 > (\beta_1+\beta_2+\beta_3+\beta_4)$. Then, it is immediately deduced from (4.1) that the inequality (4.3) holds. There exist two positive distinctive real numbers A and B demonstrating two positive roots of equation (4.2) such that

$$A = \frac{[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]+\delta}{2(\beta_1+\beta_2+\beta_3+\beta_4)}, \tag{4.4}$$

and

$$B = \frac{[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]-\delta}{2(\beta_1+\beta_2+\beta_3+\beta_4)}, \tag{4.5}$$

where

$$\delta^2 = [(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]^2 - \frac{4\alpha_0(\beta_1+\beta_2+\beta_3+\beta_4)[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{[\beta_0-(\beta_1+\beta_2+\beta_3+\beta_4)]}.$$

A and B are to be proven as positive solutions of prime period two of equation (1.1). To this end, it can be assumed that $y_{-s}=B,\dots,y_{-r}=B,\dots,y_{-q}=B,\dots,y_{-p}=B,\dots,y_{-1}=B$ and $y_0=A$. Now, we are going to show that $y_1=B$ and $y_2=A$. From equation (1.1) we deduce that

$$y_1 = \frac{\alpha_0y_0+\alpha_1y_{-p}+\alpha_2y_{-q}+\alpha_3y_{-r}+\alpha_4y_{-s}}{\beta_0y_0+\beta_1y_{-p}+\beta_2y_{-q}+\beta_3y_{-r}+\beta_4y_{-s}} = \frac{\alpha_0A+(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\,B}{\beta_0A+(\beta_1+\beta_2+\beta_3+\beta_4)\,B}. \tag{4.6}$$

Substituting (4.4) and (4.5) into (4.6) we deduce that

$$y_1-B = \frac{\alpha_0A+(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\,B}{\beta_0A+(\beta_1+\beta_2+\beta_3+\beta_4)\,B} - \frac{\beta_0AB+(\beta_1+\beta_2+\beta_3+\beta_4)\,B^2}{\beta_0A+(\beta_1+\beta_2+\beta_3+\beta_4)\,B} = \frac{E-F}{G}, \tag{4.7}$$

where,

$$E = \frac{\alpha_0\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]+\delta\right)}{2(\beta_1+\beta_2+\beta_3+\beta_4)} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]-\delta\right)}{2(\beta_1+\beta_2+\beta_3+\beta_4)},$$

$$F = \left(\frac{\alpha_0\beta_0[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{(\beta_1+\beta_2+\beta_3+\beta_4)\left[\beta_0-(\beta_1+\beta_2+\beta_3+\beta_4)\right]}\right) + (\beta_1+\beta_2+\beta_3+\beta_4)\left(\frac{[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]-\delta}{2(\beta_1+\beta_2+\beta_3+\beta_4)}\right)^2,$$

$$G = \frac{\beta_0\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]+\delta\right)}{2(\beta_1+\beta_2+\beta_3+\beta_4)} + \frac{(\beta_1+\beta_2+\beta_3+\beta_4)\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]-\delta\right)}{2(\beta_1+\beta_2+\beta_3+\beta_4)}.$$

Multiplying the denominator and numerator of (4.7) by $4(\beta_1+\beta_2+\beta_3+\beta_4)^2$ we get

$$\begin{aligned}
y_1 - B &= \frac{2\alpha_0(\beta_1+\beta_2+\beta_3+\beta_4)\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]+\delta\right)}{G_1}, \\
&+ \frac{2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\beta_4)\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]-\delta\right)}{G_1} \\
&- \frac{\left(\frac{4\alpha_0\beta_0(\beta_1+\beta_2+\beta_3+\beta_4)[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{[\beta_0-(\beta_1+\beta_2+\beta_3+\beta_4)]}\right)}{G_1} \\
&- \frac{(\beta_1+\beta_2+\beta_3+\beta_4)\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]-\delta\right)^2}{G_1}, \\
y_1 - B &= \frac{2\alpha_0(\beta_1+\beta_2+\beta_3+\beta_4)[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{G_1} \\
&+ \frac{2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\beta_4)[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{G_1} \\
&- \frac{(\beta_1+\beta_2+\beta_3+\beta_4)[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]^2+(\beta_1+\beta_2+\beta_3+\beta_4)\delta^2}{G_1} \\
&- \frac{\left(\frac{4\alpha_0\beta_0(\beta_1+\beta_2+\beta_3+\beta_4)[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]}{[\beta_0-(\beta_1+\beta_2+\beta_3+\beta_4)]}\right)}{G_1} \\
&+ \frac{2\alpha_0(\beta_1+\beta_2+\beta_3+\beta_4)\delta-2\alpha_0(\beta_1+\beta_2+\beta_3+\beta_4)\delta}{G_1} \\
&+ \frac{2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\beta_4)\delta-2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\beta_4)\delta}{G_1} = 0,
\end{aligned}$$

where

$$\begin{aligned}
G_1 = {} & 2\beta_0(\beta_1+\beta_2+\beta_3+\beta_4)\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]+\delta\right) \\
& + 2(\beta_1+\beta_2+\beta_3+\beta_4)^2\left([(\alpha_1+\alpha_2+\alpha_3+\alpha_4)-\alpha_0]-\delta\right).
\end{aligned}$$

Similarly, we can show that

$$y_2 = \frac{\alpha_0 y_1+\alpha_1 y_{-p+1}+\alpha_2 y_{-q+1}+\alpha_3 y_{-r+1}+\alpha_4 y_{-s+1}}{\beta_0 y_1+\beta_1 y_{-p+1}+\beta_2 y_{-q+1}+\beta_3 y_{-r+1}+\beta_4 y_{-s+1}} = \frac{\alpha_0 B+(\alpha_1+\alpha_2+\alpha_3+\alpha_4)A}{\beta_0 B+(\beta_1+\beta_2+\beta_3+\beta_4)A} = A.$$

By using the mathematical induction, we have $y_n = B$ and $y_{n+1} = A, \quad n \geqslant -s.$ □

5. Global stability

In this section, the global asymptotic stability of the positive solutions of equation (1.1) is analyzed.

Theorem 5.1. *For $\alpha_i, \beta_i \in (0,\infty)$, $i = 0,1,2,3,4$, the positive equilibrium point $\widetilde{y}$ of equation (1.1) is a global attractor if the conditions*

$$\alpha_0\beta_1 \geqslant \alpha_1\beta_0,\ \alpha_0\beta_2 \geqslant \alpha_2\beta_0,\ \alpha_0\beta_3 \geqslant \alpha_3\beta_0,\ \alpha_0\beta_4 \geqslant \alpha_4\beta_0,\ \alpha_1\beta_2 \geqslant \alpha_2\beta_1,\ \alpha_1\beta_3 \geqslant \alpha_3\beta_1,$$
$$\alpha_1\beta_4 \geqslant \alpha_4\beta_1, \alpha_2\beta_3 \geqslant \alpha_3\beta_2,\ \alpha_3\beta_4 \geqslant \alpha_4\beta_2,\ \alpha_3\beta_4 \geqslant \alpha_4\beta_3,\ \text{and}\ \ \alpha_4 \geqslant (\alpha_0+\alpha_1+\alpha_2+\alpha_3)$$

hold.

Proof. Let $\{y_n\}_{n=-s}^{\infty}$ be a positive solution of equation (1.1) and let $H : (0,\infty)^5 \to (0,\infty)$ be a continuous function defined by

$$H(u_0,\ldots,u_4) = \frac{\alpha_0u_0+\alpha_1u_1+\alpha_2u_2+\alpha_3u_3+\alpha_4u_4}{\beta_0u_0+\beta_1u_1+\beta_2u_2+\beta_3u_3+\beta_4u_4}.$$

By differentiating the function $H(u_0,\ldots,u_4)$, it can be realized that

$$H_{u_0} = \frac{(\alpha_0\beta_1-\alpha_1\beta_0)\,u_1+(\alpha_0\beta_2-\alpha_2\beta_0)\,u_2+(\alpha_0\beta_3-\alpha_3\beta_0)\,u_3+L_1u_4}{(\beta_0u_0+\beta_1u_1+\beta_2u_2+\beta_3u_3+\beta_4u_4)^2},$$

where $L_1 = (\alpha_0\beta_4-\alpha_4\beta_0)$ and

$$H_{u_1} = \frac{-(\alpha_0\beta_1-\alpha_1\beta_0)\,u_0+(\alpha_1\beta_2-\alpha_2\beta_1)\,u_2+(\alpha_1\beta_3-\alpha_3\beta_1)\,u_3+L_2u_4}{(\beta_0u_0+\beta_1u_1+\beta_2u_2+\beta_3u_3+\beta_4u_4)^2},$$

where $L_2 = (\alpha_1\beta_4-\alpha_4\beta_1)$ and

$$H_{u_2} = \frac{-(\alpha_0\beta_2-\alpha_2\beta_0)\,u_0-(\alpha_1\beta_2-\alpha_2\beta_1)\,u_1+(\alpha_2\beta_3-\alpha_3\beta_2)\,u_3+L_3u_4}{(\beta_0u_0+\beta_1u_1+\beta_2u_2+\beta_3u_3+\beta_4u_4)^2},$$

where $L_3 = (\alpha_3\beta_4-\alpha_4\beta_2)$ and

$$H_{u_3} = \frac{-(\alpha_0\beta_3-\alpha_3\beta_0)\,u_0-(\alpha_1\beta_3-\alpha_3\beta_1)\,u_1-(\alpha_2\beta_3-\alpha_3\beta_2)\,u_2+L_4u_4}{(\beta_0u_0+\beta_1u_1+\beta_2u_2+\beta_3u_3+\beta_4u_4)^2},$$

where $L_4 = (\alpha_3\beta_4-\alpha_4\beta_3)$ and

$$H_{u_4} = \frac{-(\alpha_0\beta_4-\alpha_4\beta_0)\,u_0-(\alpha_1\beta_4-\alpha_4\beta_1)\,u_1-(\alpha_2\beta_4-\alpha_4\beta_2)\,u_2-L_5u_4}{(\beta_0u_0+\beta_1u_1+\beta_2u_2+\beta_3u_3+\beta_4u_4)^2},$$

where $L_5 = (\alpha_3\beta_4-\alpha_4\beta_3)$.

It is observed that the function $H(u_0,u_1,u_2,u_3,u_4)$ is non-decreasing in u_0 and non-increasing in u_4. Now, we consider four cases.

Case (1). Let the function $H(u_0,\ldots,u_4)$ be non-decreasing in u_0,u_1,u_2,u_3 and non-increasing in u_4. Suppose that (m,M) is a solution of the system

$$M = H(M,M,M,M,m) \quad \text{and} \quad m = H(m,m,m,m,M).$$

Then from equation (1.1), we get that

$$M = \frac{\alpha_0M+\alpha_1M+\alpha_2M+\alpha_3M+\alpha_4m}{\beta_0M+\beta_1M+\beta_2M+\beta_3M+\beta_4m} \quad \text{and} \quad m = \frac{\alpha_0m+\alpha_1m+\alpha_2m+\alpha_3m+\alpha_4M}{\beta_0m+\beta_1m+\beta_2m+\beta_3m+\beta_4M}.$$

Thus

$$M = \frac{(\alpha_0+\alpha_1+\alpha_2+\alpha_3)\,M+\alpha_4m}{(\beta_0+\beta_1+\beta_2+\beta_3)\,M+\beta_4m} \quad \text{and} \quad m = \frac{(\alpha_0+\alpha_1+\alpha_2+\alpha_3)\,m+\alpha_4M}{(\beta_0+\beta_1+\beta_2+\beta_3)\,m+\beta_4M}.$$

From which we have

$$(\alpha_0+\alpha_1+\alpha_2+\alpha_3)\,M+\alpha_4 m-(\beta_0+\beta_1+\beta_2+\beta_3)\,M^2=\beta_4 Mm \tag{5.1}$$

and

$$(\alpha_0+\alpha_1+\alpha_2+\alpha_3)\,m+\alpha_4 M-(\beta_0+\beta_1+\beta_2+\beta_3)\,m^2=\beta_4 Mm. \tag{5.2}$$

By subtracting (5.1) and (5.2), we obtain

$$(M-m)\{[(\alpha_0+\alpha_1+\alpha_2+\alpha_3)-\alpha_4]-(\beta_0+\beta_1+\beta_2+\beta_3)\,(M+m)\}=0. \tag{5.3}$$

Since $\alpha_4 \geqslant (\alpha_0+\alpha_1+\alpha_2+\alpha_3)$, we deduce from (5.3) that

$$M=m.$$

It follows by Theorem 1.4, that $\widetilde{y}$ is a global attractor of equation (1.1).

Case (2). Let the function $H(u_0,\dots,u_4)$ be non-decreasing in u_0,u_1 and non-increasing in u_2, u_3, u_4. Suppose that (M,m) is a solution of the system

$$M=H(M,M,m,m,m) \quad \text{and} \quad m=H(m,m,M,M,M).$$

Then from equation (1.1), we get that

$$M=\frac{\alpha_0 M+\alpha_1 M+\alpha_2 m+\alpha_3 m+\alpha_4 m}{\beta_0 M+\beta_1 M+\beta_2 m+\beta_3 m+\beta_4 m} \quad \text{and} \quad m=\frac{\alpha_0 m+\alpha_1 m+\alpha_2 M+\alpha_3 M+\alpha_4 M}{\beta_0 m+\beta_1 m+\beta_2 M+\beta_3 M+\beta_4 M}.$$

Thus

$$M=\frac{(\alpha_0+\alpha_1)M+(\alpha_2+\alpha_3+\alpha_4)\,m}{(\beta_0+\beta_1)M+(\beta_2+\beta_3+\beta_4)\,m} \quad \text{and} \quad m=\frac{(\alpha_0+\alpha_1)m+(\alpha_2+\alpha_3+\alpha_4)\,M}{(\beta_0+\beta_1)m+(\beta_2+\beta_3+\beta_4)\,M}.$$

From which we have

$$(\alpha_0+\alpha_1)M+(\alpha_2+\alpha_3+\alpha_4)\,m-(\beta_0+\beta_1)M^2=(\beta_2+\beta_3+\beta_4)\,Mm \tag{5.4}$$

and

$$(\alpha_0+\alpha_1)m+(\alpha_2+\alpha_3+\alpha_4)\,M-(\beta_0+\beta_1)m^2=(\beta_2+\beta_3+\beta_4)\,Mm. \tag{5.5}$$

From (5.4) and (5.5), we obtain

$$(M-m)\{[(\alpha_0+\alpha_1)-(\alpha_2+\alpha_3+\alpha_4)]-(\beta_0+\beta_1)\,(M+m)\}=0. \tag{5.6}$$

Since $(\alpha_2+\alpha_3+\alpha_4) \geqslant (\alpha_0+\alpha_1)$, we deduce from (5.6) that

$$M=m.$$

It follows by Theorem 1.4 that $\widetilde{y}$ is a global attractor of equation (1.1).

Case (3). Let the function $H(u_0,\dots,u_4)$ be non-decreasing in u_0, u_1, u_2 and non-increasing in u_3, u_4. Suppose that (M,m) is a solution of the system

$$M=H(M,M,M,m,m) \quad \text{and} \quad m=H(m,m,m,M,M).$$

Then from equation (1.1), we get that

$$M=\frac{\alpha_0 M+\alpha_1 M+\alpha_2 M+\alpha_3 m+\alpha_4 m}{\beta_0 M+\beta_1 M+\beta_2 M+\beta_3 m+\beta_4 m} \quad \text{and} \quad m=\frac{\alpha_0 m+\alpha_1 m+\alpha_2 m+\alpha_3 M+\alpha_4 M}{\beta_0 m+\beta_1 m+\beta_2 m+\beta_3 M+\beta_4 M}.$$

Thus
$$M=\frac{(\alpha_0+\alpha_1+\alpha_2)M+(\alpha_3+\alpha_4)\,m}{(\beta_0+\beta_1+\beta_2)M+(\beta_3+\beta_4)\,m} \quad \text{and} \quad m=\frac{(\alpha_0+\alpha_1+\alpha_2)m+(\alpha_3+\alpha_4)\,M}{(\beta_0+\beta_1+\beta_2)m+(\beta_3+\beta_4)\,M}.$$
From which we have
$$(\alpha_0+\alpha_1+\alpha_2)M+(\alpha_3+\alpha_4)\,m-(\beta_0+\beta_1+\beta_2)M^2=(\beta_3+\beta_4)\,Mm \tag{5.7}$$
and
$$(\alpha_0+\alpha_1+\alpha_2)m+(\alpha_3+\alpha_4)\,M-(\beta_0+\beta_1+\beta_2)m^2=(\beta_3+\beta_4)\,Mm. \tag{5.8}$$
From (5.7) and (5.8), we obtain
$$(M-m)\{[(\alpha_0+\alpha_1+\alpha_2)-(\alpha_3+\alpha_4)]-(\beta_0+\beta_1+\beta_2)\,(M+m)\}=0. \tag{5.9}$$
Since $(\alpha_3+\alpha_4)\geqslant(\alpha_0+\alpha_1+\alpha_2)$, we deduce from (5.9) that
$$M=m.$$
It follows by Theorem 1.4, that $\widetilde{y}$ is a global attractor of equation (1.1).

Case (4). Let the function $H(u_0,\dots,u_4)$ be non-decreasing in u_0,u_1,u_3 and non-increasing in u_2,u_4. Suppose that (M,m) is a solution of the system
$$M=H(M,M,m,M,m) \quad \text{and} \quad m=H(m,m,M,m,M).$$
Then from equation (1.1), we get that
$$M=\frac{\alpha_0M+\alpha_1M+\alpha_2m+\alpha_3M+\alpha_4m}{\beta_0M+\beta_1M+\beta_2m+\beta_3M+\beta_4m} \quad \text{and} \quad m=\frac{\alpha_0m+\alpha_1m+\alpha_2M+\alpha_3m+\alpha_4M}{\beta_0m+\beta_1m+\beta_2M+\beta_3m+\beta_4M}.$$
Thus
$$M=\frac{(\alpha_0+\alpha_1+\alpha_3)M+(\alpha_2+\alpha_4)\,m}{(\beta_0+\beta_1+\beta_3)M+(\beta_2+\beta_4)\,m} \quad \text{and} \quad m=\frac{(\alpha_0+\alpha_1+\alpha_3)m+(\alpha_2+\alpha_4+)\,M}{(\beta_0+\beta_1+\beta_3)m+(\beta_2+\beta_4)\,M}.$$
From which we have
$$(\alpha_0+\alpha_1+\alpha_3)M+(\alpha_2+\alpha_4)\,m-(\beta_0+\beta_1+\beta_3)M^2=(\beta_2+\beta_4)\,Mm \tag{5.10}$$
and
$$(\alpha_0+\alpha_1+\alpha_3)m+(\alpha_2+\alpha_4)\,M-(\beta_0+\beta_1+\beta_3)m^2=(\beta_2+\beta_4)\,Mm. \tag{5.11}$$
From (5.10) and (5.11), we obtain
$$(M-m)\{[(\alpha_0+\alpha_1+\alpha_3)-(\alpha_2+\alpha_4)]-(\beta_0+\beta_1+\beta_3)\,(M+m)\}=0. \tag{5.12}$$
Since $(\alpha_2+\alpha_4)\geqslant(\alpha_0+\alpha_1+\alpha_3)$, we deduce from (5.12) that
$$M=m.$$
It follows by Theorem 1.4, that $\widetilde{y}$ is a global attractor of equation (1.1) and then the proof is completed. □

6. Numerical examples on the main results

Several interesting numerical examples shall be considered in an attempt to exhibit the results of the previous sections and to support the theoretical discussions in this section. Various types of qualitative behavior of solutions to the nonlinear difference equation (1.1) are presented in these examples.

Example 6.1 (Theorem 4.1 (Case 1))**.** Figure 1 shows that equation (1.1) has no prime period two solution if p, q, r, s are even. Choose $p = 2,\ q = 4,\ \ r = 6,\ s = 8,\ \ y_{-8} = 1,\ y_{-7} = 2,\ y_{-6} = 3,\ y_{-5} = 4,\ y_{-4} = 5,$ $y_{-3} = 6,\ y_{-2} = 7,\ y_{-1} = 8,\ y_0 = 9,\ \alpha_0 = 2,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 15,\ \alpha_4 = 25,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5,\ \beta_4 = 6.$

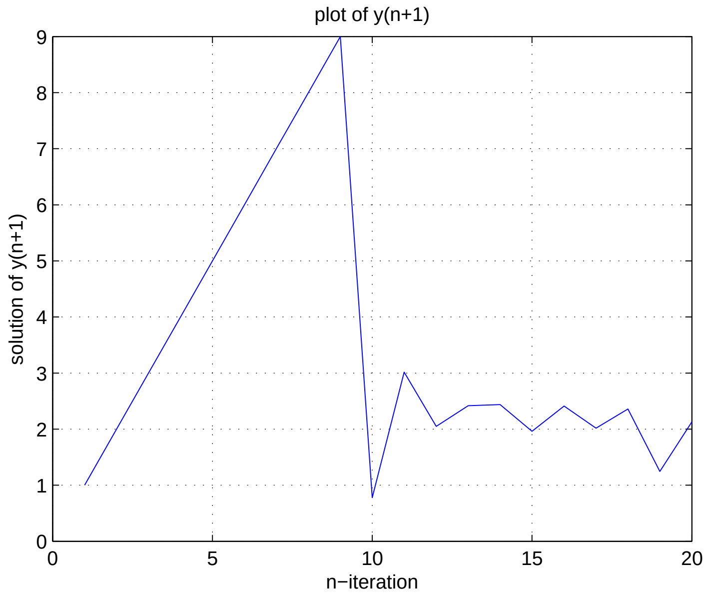

Figure 1

Example 6.2 (Theorem 4.1 (Case 2))**.** Figure 2 shows that equation (1.1) has no prime period two solution if p, q are even and r, s are odd. Choose $p = 2,\ q = 4,\ \ r = 5,\ s = 7,\ \ y_{-7} = 1,\ y_{-6} = 2,\ y_{-5} = 3,\ y_{-4} = 4,\ y_{-3} = 5,\ y_{-2} = 6,\ y_{-1} = 7,\ y_0 = 8,\ \ \alpha_0 = 2,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 5,\ \alpha_4 = 4,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5,\ \beta_4 = 6.$

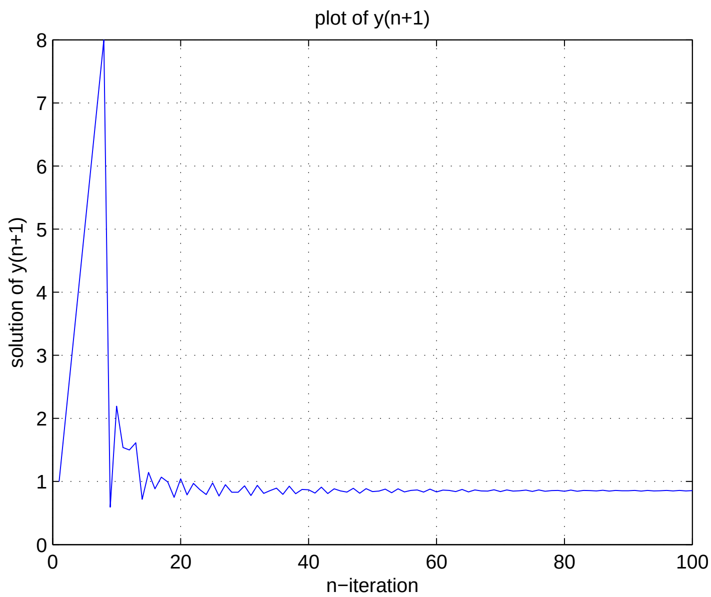

Figure 2

Example 6.3 (Theorem 4.1 (Case 3))**.** Figure 3 shows that equation (1.1) has no prime period two solution if p, q are odd and r, s are even. Choose $p = 1,\ q = 3,\ r = 4,\ s = 6,\ y_{-6} = 1,\ y_{-5} = 2,\ y_{-4} = 3,\ y_{-3} = 4,\ y_{-2} = 5,\ y_{-1} = 6,\ y_0 = 7,\ \alpha_0 = 2,\ \alpha_1 = 100,\ \alpha_2 = 200,\ \alpha_3 = 5,\ \alpha_4 = 40,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5,\ \beta_4 = 6.$

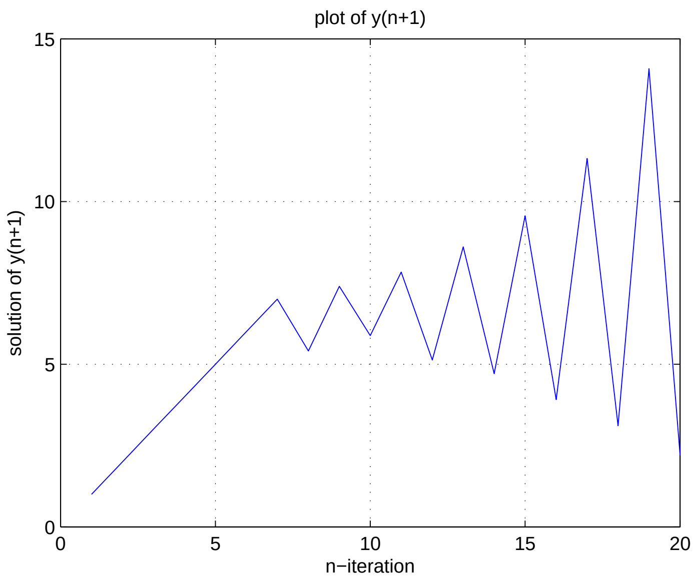

Figure 3

Example 6.4 (Theorem 4.1 (Case 4))**.** Figure 4 shows that equation (1.1) has no prime period two solution if p, r are even and q, s are odd. Choose $p = 2,\ q = 3,\ r = 4,\ s = 5,\ y_{-5} = 1,\ y_{-4} = 2,\ y_{-3} = 3,\ y_{-2} = 4,\ y_{-1} = 5,\ y_0 = 6,\ \alpha_0 = 2,\ \alpha_1 = 100,\ \alpha_2 = 20,\ \alpha_3 = 500,\ \alpha_4 = 4,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5,\ \beta_4 = 6.$

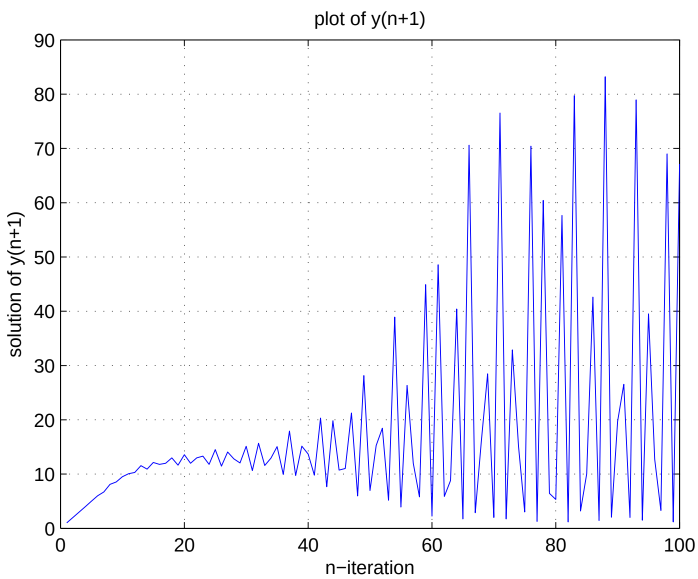

Figure 4

Example 6.5 (Theorem 4.1 (Case 5))**.** Figure 5 shows that equation (1.1) has no prime period two solution if q, r are even and p, s are odd. Choose $p = 1,\ q = 2,\ r = 4,\ s = 5,\ y_{-5} = 1,\ y_{-4} = 2,\ y_{-3} = 3,\ y_{-2} = 4,\ y_{-1} = 5,\ y_0 = 6,\ \alpha_0 = 100,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 15,\ \alpha_4 = 4,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5,\ \beta_4 = 6.$

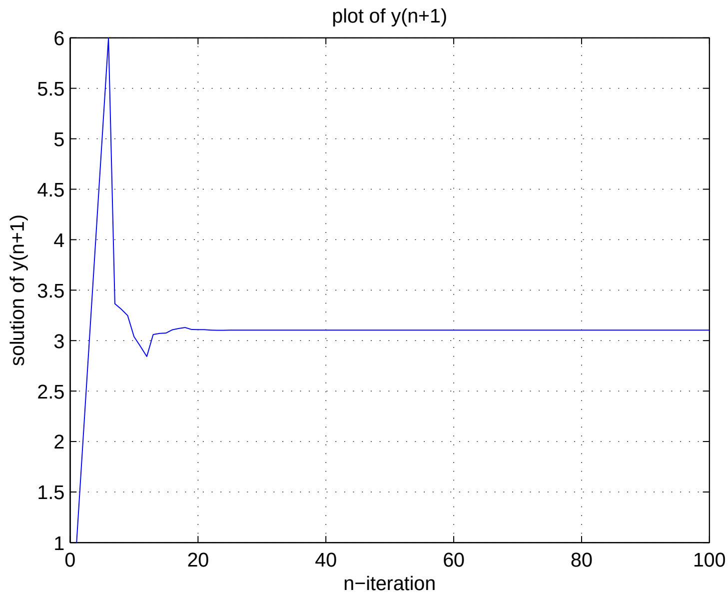

Figure 5

Example 6.6 (Theorem 4.1 (Case 6))**.** Figure 6 shows that equation (1.1) has no prime period two solution if q, r are odd and p, s are even. Choose $p = 2,\ q = 3,\ \ r = 5,\ s = 6,\ y_{-6} = 1,\ y_{-5} = 2,\ y_{-4} = 3,\ y_{-3} = 4,\ y_{-2} = 5,\ y_{-1} = 6,\ y_0 = 7,\ \alpha_0 = 2,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 5,\ \alpha_4 = 400,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5,\ \beta_4 = 6.$

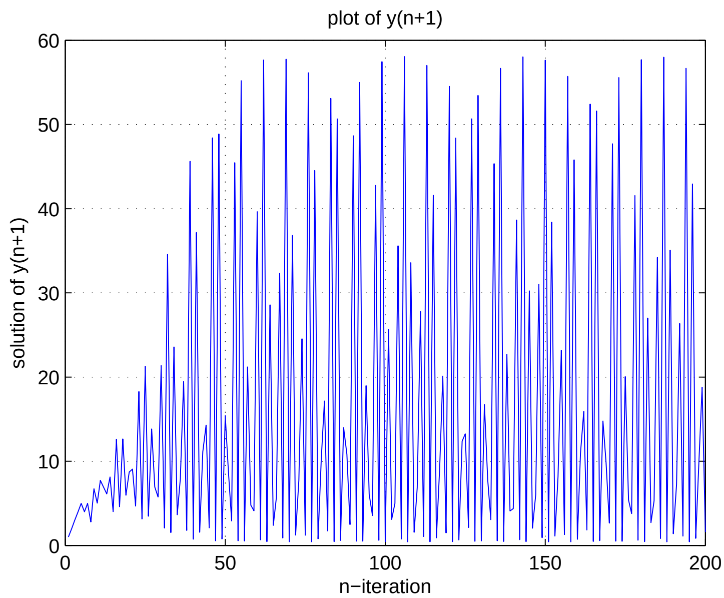

Figure 6

Example 6.7 (Theorem 4.1 (Case 7))**.** Figure 7 shows that equation (1.1) has no prime period two solution if p, r are odd and q, s are even. Choose $p = 1,\ q = 2,\ r = 3,\ s = 4,\ y_{-4} = 1,\ y_{-3} = 2,\ y_{-2} = 3,\ y_{-1} = 4,\ y_0 = 5,\ \alpha_0 = 100,\ \alpha_1 = 10,\ \alpha_2 = 200,\ \alpha_3 = 5,\ \alpha_4 = 400,\ \beta_0 = 30,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5,\ \beta_4 = 6.$

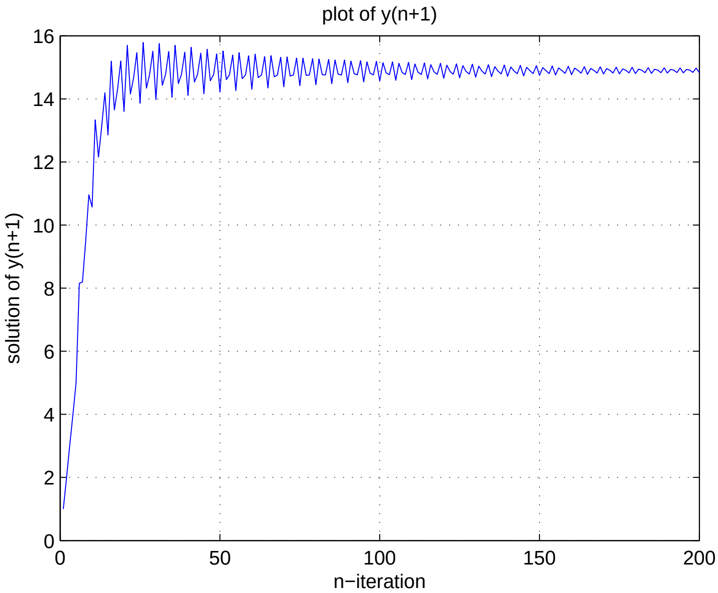

Figure 7

Example 6.8 (Theorem 4.1 (Case 8))**.** Figure 8 shows that (1.1) has no prime period two solution if p, q, r, s are odd. Choose $p = 1,\ q = 3,\ r = 5,\ s = 7,\ y_{-7} = 1,\ y_{-6} = 2,\ y_{-5} = 3,\ y_{-4} = 4,\ y_{-3} = 5,\ y_{-2} = 6,$ $y_{-1} = 7,\ y_0 = 8,\ \alpha_0 = 2,\ \alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 15,\ \alpha_4 = 25,\ \beta_0 = 2,\ \beta_1 = 3,\ \beta_2 = 4,\ \beta_3 = 5,\ \beta_4 = 6.$

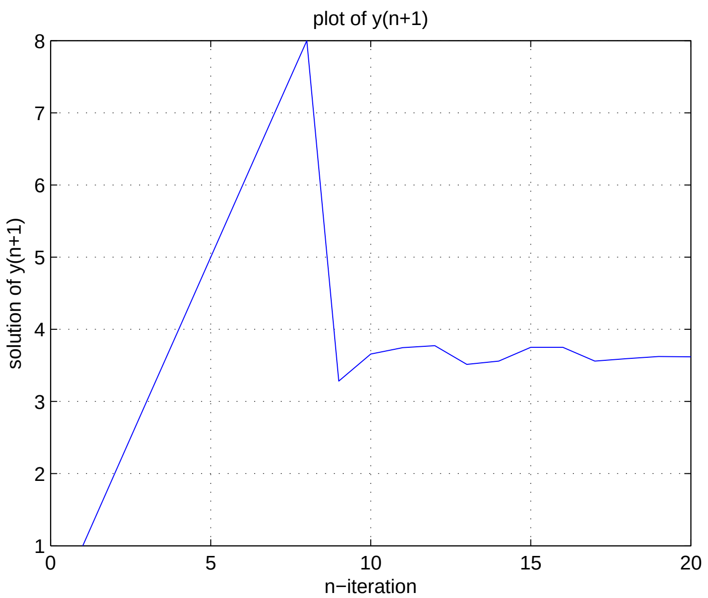

Figure 8

Example 6.9. Figure 9 shows that equation (1.1) has prime period two solution and $p < q < r < s$. Choose $p = 1,\ q = 3,\ r = 5,\ s = 7,\ p = \max\{p, q, r, s\} = 7,\ y_{-7} = y_{-5} = y_{-3} = y_{-1} = y_1 \simeq 0.46,\ y_{-6} = y_{-4} = y_{-2} = y_0 = y_2 \simeq 0.031,\ \alpha_0 = 10,\ \alpha_1 = 3,\ \alpha_2 = 30,\ \alpha_3 = 8,\ \alpha_4 = 45,\ \beta_0 = 500,\ \beta_1 = 5,\ \beta_2 = 40,\ \beta_3 = 9,\ \beta_4 = 100.$

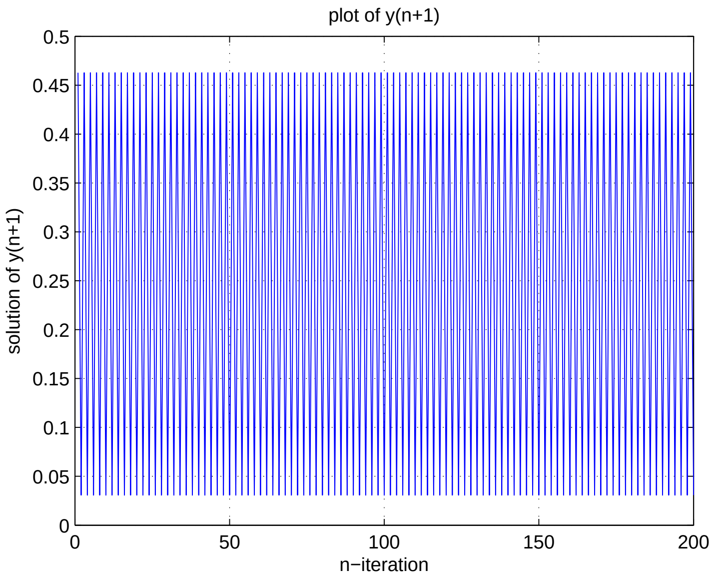

Figure 9

Example 6.10. Figure 10 shows that the solution of equation (1.1) has global stability and $p < q < r < s$. Choose $p = 2,\ q = 4,\ r = 6,\ s = 8,\ y_{-8} = 1,\ y_{-7} = 2,\ y_{-6} = 3,\ y_{-5} = 4,\ y_{-4} = 5,\ y_{-3} = 6,\ y_{-2} = 7,\ y_{-1} = 8,\ y_0 = 9,\ \alpha_0 = 0.5,\ \alpha_1 = 0.25,\ \alpha_2 = 1,\ \alpha_3 = 2,\ \alpha_4 = 0.1,\ \beta_0 = 3,\ \beta_1 = 2,\ \beta_2 = 10,\ \beta_3 = 25,\ \beta_4 = 3.$

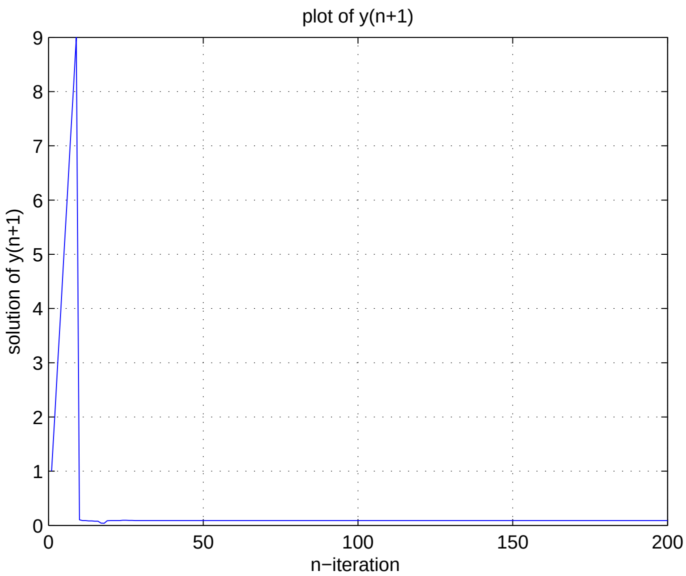

Figure 10

7. Conclusion

It has been discussed that certain properties of the nonlinear rational deference equation (1.1), particularly the periodicity, the boundedness and the global stability of the positive solutions for this equation. Some figures were given to illustrate the behavior of these solutions. The result shown can be considered as a more generalization than the results retrieved in Refs. [2, 10, 17, 23]. As indicated, Examples 6.1-6.8 verify Theorem 4.1 that illustrated equation (1.1) has no prime period two solution, while Example 6.9

verifies Theorem 4.2 which shows that equation (1.1) has prime period two solution. Whereas Example 6.10 verifies Theorems 2.2 and 5.1, which shows that the solution of equation (1.1) is globally asymptotic stable.

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