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Quadruple random common fixed point results of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras

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Abstract

In this paper, we introduce the concept of cone b-metric spaces over Banach algebras and present some quadruple random coincidence points and quadruple random common fixed point theorems for nonlinear operators in such spaces.

Keywords: Quadruple random fixed point, quadruple common random fixed point, quadruple random coincidence point, cone b-metric space over Banach algebra.

2010 MSC: 47H10, 60H25.

1. Introduction

It is well-known that the classic contraction mapping principle of Banach is a fundamental result in fixed point theory. Several authors have obtained various extensions and generalizations of Banach's theorem by considering contractive mappings on many different metric spaces. For example, [3, 5, 23, 37], and others. In 1989, Bakhtin [4] introduced b-metric space as a generalization of usual metric space and proved the contraction mapping principle in b-metric space. In 2007, Huang and Zhang [10] introduced the concept of cone metric space as generalization of metric and proved some fixed theorems of contractive mapping on complete cone metric space with the assumption of normality of a cone. Fixed point theorems for contractive mapping on b-metric space and cone metric have been proved by several authors (see

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[2, 7, 14, 25–27, 29, 30, 32–34]). In 2011, Hussain and Shah [12] introduced the concept of cone b-metric space as a generalization of cone metric space and b-metric space. In 2013, Liu and Xu [21] introduced cone metric space over Banach algebra by replacing Banach spaces with Banach algebra and proved some fixed point theorems of generalized Lipschitz mapping with weaker conditions on generalized Lipschitz constants. In 2015, Huang and Radenović [9] introduced the concept of cone b-metric space over Banach algebra and proved some common fixed point theorems of generalized Lipschitz mappings in such setting without assumption of normality. Random fixed point theorems are stochastic generalizations of classical fixed point theorems. On the other hand, in 2009, Ćirić and Lakshmikantham [6] proved some coupled random coincidence and coupled random fixed points in partially ordered metric space. Afterwards, many authors have focused on random fixed point theorems in complete separable metric space and coupled random coincidence and coupled random fixed points in partially ordered metric space. A large number of works are noted in [1, 11, 16–20, 31, 35, 36] and the relevant literature therein. Recently, Jiang et al. [13] proved tripled random coincidence point and common fixed point results of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras. In this paper, we present several quadruple random common fixed points theorems with several generalized Lipschitz constants of cone b-metric spaces over Banach algebras.

2. Preliminaries

Let $\mathcal{A}$ be a Banach algebra with a unit e, and θ the zero element of $\mathcal{A}$. A nonempty closed convex subset P of $\mathcal{A}$ is called a cone if

1. $\{\theta, e\} \subset P$;
2. $P^2 = PP \subset P$, $P \cap (-P) = \{\theta\}$;
3. $\lambda P + \mu P \subset P$ for all $\lambda, \mu \geqslant 0$.

On this basis, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will indicate that $y - x \in \text{int}P$, where $\text{int}(P)$ stands for the interior of P. Write $\|\cdot\|$ as the norm on $\mathcal{A}$. A cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leqslant M\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

In the following we always suppose that $\mathcal{A}$ is a Banach algebra with a unit e. P is a cone in $\mathcal{A}$ with $\text{int}(P) \neq \emptyset$, and $\preceq$ is a partial ordering with respect to P.

Definition 2.1. Let X be a nonempty set and $\mathcal{A}$ a Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

1. $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preceq s(d(x, z) + d(z, y))$ for all $x, y, z \in X$,

where $s \geqslant 1$ is a constant. Then d is called a cone b-metric on X, and (X, d) is called a cone b-metric space over Banach algebra $\mathcal{A}$.

Definition 2.2 ([12])**.** Let (X, d) be a cone b-metric space over Banach algebra, $x \in X$ and $\{x_n\}$ a sequence in X. Then

1. $\{x_n\}$ converges to x whenever, for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.
2. $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$.
3. (X, d) is complete if every Cauchy sequence is convergent.

The following lemmas are often used (in particular when dealing with cone b-metric spaces in which the cones need not to be normal).

Lemma 2.3 ([12]). *Let (X, d) be a cone b-metric space over Banach algebra $\mathcal{A}$ and P a cone in $\mathcal{A}$. Then the following properties are often used.*

(1) *If $c \in \text{int}(P)$ and $\theta \preceq a_n \to \theta(n \to \infty)$, then there exists N such that for all $n > N$, we have $a_n \ll c$.*
(2) *If $x \preceq y$ and $y \ll z$, then $x \ll z$.*
(3) *If $\theta \preceq u \ll c$ for each $c \in \text{int}(P)$, then $u = \theta$.*
(4) *If $u \in P$ and $u \preceq ku$ for some $0 \leqslant k < 1$, then $u = \theta$.*
(5) *If $a \preceq b + c$ for each $c \in \text{int}(P)$, then $a \preceq b$.*
(6) *Let $\theta \ll c$. If $\theta \preceq d(x_n, x) \preceq b_n$ and $b_n \to \theta(n \to \infty)$, then $d(x_n, x) \ll c$, where $\{x_n\}$, x are a sequence and a given point in X, respectively.*

Lemma 2.4 ([28]). *Let $\mathcal{A}$ be a Banach algebra with a unit e and $x \in \mathcal{A}$, then $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(x)$ satisfies*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $\rho(x) <| \lambda |$, then $\lambda e - x$ is invertible in $\mathcal{A}$, moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 2.5 ([28]). *Let $\mathcal{A}$ be a Banach algebra with a unit e, and $a, b \in \mathcal{A}$. If a commutes with b, then*

$$\rho(a + b) \leqslant \rho(a) + \rho(b), \ \ \rho(ab) \leqslant \rho(a)\rho(b).$$

Lemma 2.6 ([38]). *Let P be a cone in a Banach algebra $\mathcal{A}$ and $k \in P$ be a given vector. Let $\{u_n\}$ be a sequence in P. If for each $c_1 \gg \theta$, there exists N_1 such that $u_n \ll c_1$ for all $n > N_1$, then for each $c_2 \gg \theta$, there exists N_2 such that $ku_n \ll c_2$ for all $n > N_2$.*

Lemma 2.7 ([38]). *Let $\mathcal{A}$ be a Banach algebra and $k \in \mathcal{A}$. If $\rho(k) < 1$ then $\lim_{n\to\infty} \|k^n\| = 0$.*

Lemma 2.8 ([38]). *Let $\mathcal{A}$ be a Banach algebra with a unit e, and $\{x_n\}$ a sequence in $\mathcal{A}$. If $\{x_n\}$ converges to x in $\mathcal{A}$, and for any $n \geqslant 1$, $\{x_n\}$ commutes with x, then $\rho(x_n) \to \rho(x)$ as $n \to \infty$*

Definition 2.9 ([15]). An element $(x, y, z, w) \in X^4$ is said to be a quadruple fixed point of the mapping $F : X^4 \to X$ if $F(x, y, z, w) = x$, $F(y, z, w, x) = y$, $F(z, w, x, y) = z$, and $F(w, x, y, z) = w$.

Definition 2.10 ([24]). Let X be a nonempty set and let $F : X^4 \to X$, $g : X \to X$. An element $(x, y, z, w) \in X$ is called

(1) a quadruple coincidence point of F and g if $F(x, y, z, w) = gx$, $F(y, z, w, x) = gy$, $F(z, w, x, y) = gz$, and $F(w, x, y, z) = gw$; (gx, gy, gz, gw) is said to be a quadruple point of coincidence of F and g.
(2) a quadruple common fixed point of F and g if $F(x, y, z, w) = gx = x$, $F(y, z, w, x) = gy = y$, $F(z, w, x, y) = gz = z$, and $F(w, x, y, z) = gw = w$.

Definition 2.11. The mappings $F : X^4 \to X$ and $g : X \to X$ are called w-compatible if $g(F(x, y, z, w)) = F(gx, gy, gz, gw)$ whenever $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, and $gw = F(w, x, y, z)$.

Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω and let (X, d) be a metric space. A mapping $T : \Omega \to X$ is called Σ-measurable if for any open subset U of X, $T^{-1}(U) = \{\omega : T(\omega) \in U\} \in \Sigma$ (see [8]). In what follows, when we speak of measurability we shall mean Σ-measurability. A mapping $T : \Omega \times X \to X$ is called a random operator if for any $x \in X$, $T(\cdot, x)$ is measurable. A measurable mapping $\xi : \Omega \to X$ is called a random fixed point of a random operator $T : \Omega \times X \to X$, if $\xi(\omega) = T(\omega, \xi(\omega))$ for every $\omega \in \Omega$.

Theorem 2.12 ([8]). *Let X be separable metric, Y be metric, $f : \Omega \times X \to Y$ a function measurable in ω and continuous in x, $\Gamma : \Omega \to X$ a measurable multifunction with compact values, and $g : \Omega \to Y$ a measurable function such that $g(\omega) \in f(\{\omega\} \times \Gamma(\omega))$ for all $\omega \in \Omega$. Then there exists a measurable selector $\gamma : \Omega \to X$ for Γ such that $g(\omega) = f(\omega, \gamma(\omega))$ for all $\omega \in \Omega$.*

Definition 2.13. Let (X, d) be a separable metric space and (Ω, Σ) be a measurable space. Then $F : X^4 \to X$ and $g : X \to X$ are said to be w-compatible random operators if

$$F(\omega, (g(\omega, x), g(\omega, y), g(\omega, z), g(\omega, w))) = g(\omega, F(\omega, (x, y, z, w))),$$

whenever $F(\omega, (x, y, z, w)) = g(\omega, x)$, $F(\omega, (y, z, w, x)) = g(\omega, y)$, $F(\omega, (z, w, x, y)) = g(\omega, z)$, $F(\omega, (w, x, y, z)) = g(\omega, w)$ for all $\omega \in \Omega$ and $x, y, z, w \in X$ are satisfied.

3. Main results

Theorem 3.1. *Let (X, d) be a separable cone b-metric space over Banach algebra $\mathcal{A}$ and P be a cone in $\mathcal{A}$, $s \geqslant 1$ be a constant, and (Ω, Σ) be a measurable space. Suppose that the mappings $F : \Omega \times X^4 \to X$, $g : \Omega \times X \to X$ satisfy the following contractive condition:*

$$\begin{aligned} &d(F(\omega, (x_1, x_2, x_3, x_4)), F(\omega, (x_1^*, x_2^*, x_3^*, x_4^*))) \\ &\quad \preceq [a_1 d(g(\omega, x_1), F(\omega, (x_1, x_2, x_3, x_4))) + a_2 d(g(\omega, x_2), F(\omega, (x_2, x_3, x_4, x_1))) \\ &\qquad + a_3 d(g(\omega, x_3), F(\omega, (x_3, x_4, x_1, x_2))) + a_4 d(g(\omega, x_4), F(\omega, (x_4, x_1, x_2, x_3)))] \\ &\qquad + [a_5 d(g(\omega, x_1^*), F(\omega, (x_1^*, x_2^*, x_3^*, x_4^*))) + a_6 d(g(\omega, x_2^*), F(\omega, (x_2^*, x_3^*, x_4^*, x_1^*))) \\ &\qquad + a_7 d(g(\omega, x_3^*), F(\omega, (x_3^*, x_4^*, x_1^*, x_2^*))) + a_8 d(g(\omega, x_4^*), F(\omega, (x_4^*, x_1^*, x_2^*, x_3^*)))] \\ &\qquad + [a_9 d(g(\omega, x_1), F(\omega, (x_1^*, x_2^*, x_3^*, x_4^*))) + a_{10} d(g(\omega, x_2), F(\omega, (x_2^*, x_3^*, x_4^*, x_1^*))) \\ &\qquad + a_{11} d(g(\omega, x_3), F(\omega, (x_3^*, x_4^*, x_1^*, x_2^*))) + a_{12} d(g(\omega, x_4), F(\omega, (x_4^*, x_1^*, x_2^*, x_3^*)))] \\ &\qquad + [a_{13} d(g(\omega, x_1^*), F(\omega, (x_1, x_2, x_3, x_4))) + a_{14} d(g(\omega, x_2^*), F(\omega, (x_2, x_3, x_4, x_1))) \\ &\qquad + a_{15} d(g(\omega, x_3^*), F(\omega, (x_3, x_4, x_1, x_2))) + a_{16} d(g(\omega, x_4^*), F(\omega, (x_4, x_1, x_2, x_3)))] \\ &\qquad + a_{17} d(g(\omega, x_1), g(\omega, x_1^*)) + a_{18} d(g(\omega, x_2), g(\omega, x_2^*)) \\ &\qquad + a_{19} d(g(\omega, x_3), g(\omega, x_3^*)) + a_{20} d(g(\omega, x_4), g(\omega, x_4^*)) \end{aligned} \tag{3.1}$$

for all $x_1, x_2, x_3, x_4, x_1^*, x_2^*, x_3^*, x_4^*$, where $a_i \in P, a_i a_j = a_j a_i$ $(i, j = 1, \dots, 20)$ are generalized Lipschitz constants with $(s+1)\rho(a_1 + \cdots + a_8) + s(s+1)\rho(a_9 + \cdots + a_{16}) + 2s\rho(a_{17} + \cdots + a_{20}) < 2$ and $\rho(sa_1 + sa_2 + sa_3 + sa_4 + s^2a_{13} + s^2a_{14} + s^2a_{15} + s^2a_{16}) < 1$, where $s \geqslant 1$ is a constant. Let $F(\cdot, v)$, $g(\cdot, x)$ are measurable for $v \in x^4$ and $x \in X$, respectively, $F(\omega \times X^4) \subseteq g(\omega \times X)$ is complete subspace of X for each $\omega \in \Omega$, then there are mappings $\gamma, \zeta, \xi, \rho : \Omega \to X$, such that

$$F(\omega, (\gamma(\omega), \zeta(\omega), \xi(\omega), \rho(\omega))) = g(\omega, \gamma(\omega)), \qquad F(\omega, (\zeta(\omega), \xi(\omega), \rho(\omega), \gamma(\omega))) = g(\omega, \zeta(\omega)),$$
$$F(\omega, (\xi(\omega), \rho(\omega), \gamma(\omega), \zeta(\omega))) = g(\omega, \xi(\omega)), \qquad F(\omega, (\rho(\omega), \gamma(\omega), \zeta(\omega), \xi(\omega))) = g(\omega, \rho(\omega))$$

for all $\omega \in \Omega$, that is F and g have a quadruple random coincidence point.

Proof. Let $\Theta = \{\eta : \Omega \to X\}$ be a family of measurable mappings. We construct four sequences of measurable mappings $\{\gamma_n\}, \{\zeta_n\}, \{\xi_n\}, \{\rho_n\}$ in Θ and four sequences $\{g(\omega, \gamma_n(\omega))\}$, $\{g(\omega, \zeta_n(\omega))\}$, $\{g(\omega, \xi_n(\omega))\}$, $\{g(\omega, \rho_n(\omega))\}$ in X as follows. Let γ_0, ζ_0, ξ_0, $\rho_0 \in \Theta$. Since $F(\omega, (\gamma_0(\omega), \zeta_0(\omega), \xi_0(\omega), \rho_0(\omega))) \in F(\omega \times X^4) \subseteq g(\omega, X)$, by a sort of Filippov measurable implicit function theorems (see [8, 22]), there is $\gamma_1 \in \Theta$ such that

$$g(\omega, \gamma_1(\omega)) = F(\omega, (\gamma_0(\omega), \zeta_0(\omega), \xi_0(\omega), \rho_0(\omega))).$$

Similarly as $F(\omega, (\zeta_0(\omega), \xi_0(\omega), \rho_0(\omega), \gamma_0(\omega))) \subseteq g(\omega, X)$, there is $\zeta_1 \in \Theta$ such that

$$g(\omega, \zeta_1(\omega)) = F(\omega, (\zeta_0(\omega), \xi_0(\omega), \rho_0(\omega), \gamma_0(\omega))).$$

Similarly as $F(\omega, (\xi_0(\omega), \rho_0(\omega), \gamma_0(\omega), \zeta_0(\omega))) \subseteq g(\omega, X)$, there is $\xi_1 \in \Theta$ such that

$$g(\omega, \xi_1(\omega)) = F(\omega, (\xi_0(\omega), \rho_0(\omega), \gamma_0(\omega), \zeta_0(\omega))).$$

Similarly as $F(\omega, (\rho_0(\omega), \gamma_0(\omega), \zeta_0(\omega), \xi_0(\omega))) \subseteq g(\omega, X)$, there is $\rho_1 \in \Theta$ such that

$$g(\omega, \rho_1(\omega)) = F(\omega, (\rho_0(\omega), \gamma_0(\omega), \zeta_0(\omega), \xi_0(\omega))).$$

Continuing this process, we obtain four sequences $\{\gamma_n(\omega)\}$, $\{\zeta_n(\omega)\}$, $\{\xi_n(\omega)\}$, and $\{\rho_n(\omega)\}$ in X such that

$$\begin{aligned}
g(\omega,(\gamma_{n+1}(\omega))) &= F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega))),\\
g(\omega,(\zeta_{n+1}(\omega))) &= F(\omega,(\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega))),\\
g(\omega,(\xi_{n+1}(\omega))) &= F(\omega,(\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega))),\\
g(\omega,(\rho_{n+1}(\omega))) &= F(\omega,(\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega)))
\end{aligned}$$

for all $n \in \mathbb{N}$. According to (3.1), we have

$$\begin{aligned}
&d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_{n+1}(\omega)))\\
&\quad= d(F(\omega,(\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega))),F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega))))\\
&\quad\preceq [a_1 d(g(\omega,\gamma_{n-1}(\omega)),F(\omega,(\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega)))\\
&\qquad+ a_2 d(g(\omega,\zeta_{n-1}(\omega)),F(\omega,(\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega),\gamma_{n-1}(\omega)))\\
&\qquad+ a_3 d(g(\omega,\xi_{n-1}(\omega)),F(\omega,(\xi_{n-1}(\omega),\rho_{n-1}(\omega),\gamma_{n-1}(\omega),\zeta_{n-1}(\omega)))\\
&\qquad+ a_4 d(g(\omega,\rho_{n-1}(\omega)),F(\omega,(\rho_{n-1}(\omega),\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),(\xi_{n-1}(\omega)))]\\
&\qquad+ [a_5 d(g(\omega,\gamma_n(\omega)),F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega)))\\
&\qquad+ a_6 d(g(\omega,\zeta_n(\omega)),F(\omega,(\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega)))\\
&\qquad+ a_7 d(g(\omega,\xi_n(\omega)),F(\omega,(\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega)))\\
&\qquad+ a_8 d(g(\omega,\rho_n(\omega)),F(\omega,(\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega),(\xi_n(\omega)))]\\
&\qquad+ [a_9 d(g(\omega,\gamma_{n-1}(\omega)),F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega)))\\
&\qquad+ a_{10} d(g(\omega,\zeta_{n-1}(\omega)),F(\omega,(\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega)))\\
&\qquad+ a_{11} d(g(\omega,\xi_{n-1}(\omega)),F(\omega,(\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega)))\\
&\qquad+ a_{12} d(g(\omega,\rho_{n-1}(\omega)),F(\omega,(\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega)))]\\
&\qquad+ [a_{13} d(g(\omega,\gamma_n(\omega)),F(\omega,(\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega)))\\
&\qquad+ a_{14} d(g(\omega,\zeta_n(\omega)),F(\omega,(\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega),\gamma_{n-1}(\omega)))\\
&\qquad+ a_{15} d(g(\omega,\xi_n(\omega)),F(\omega,(\xi_{n-1}(\omega),\rho_{n-1}(\omega),\gamma_{n-1}(\omega),\zeta_{n-1}(\omega)))\\
&\qquad+ a_{16} d(g(\omega,\rho_n(\omega)),F(\omega,(\rho_{n-1}(\omega),\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega)))]\\
&\qquad+ [a_{17} d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega))) + a_{18} d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&\qquad+ a_{19} d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_{20} d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega)))]\\
&\quad\preceq [a_1 d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega))) + a_2 d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&\qquad+ a_3 d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_4 d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega)))\\
&\qquad+ [a_5 d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_{n+1}(\omega))) + a_6 d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&\qquad+ a_7 d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) + a_8 d(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega)))]\\
&\qquad+ [a_9 d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_{n+1}(\omega))) + a_{10} d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&\qquad+ a_{11} d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_{n+1}(\omega))) + a_{12} d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_{n+1}(\omega)))]\\
&\qquad+ [a_{13} d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_n(\omega))) + a_{14} d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_n(\omega)))\\
&\qquad+ a_{15} d(g(\omega,\xi_n(\omega)),g(\omega,\xi_n(\omega))) + a_{16} d(g(\omega,\rho_n(\omega)),g(\omega,\rho_n(\omega)))]\\
&\qquad+ a_{17} d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega))) + a_{18} d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&\qquad+ a_{19} d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_{20} d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega)))\\
&\quad\preceq [a_1 d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega))) + a_2 d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&\qquad+ a_3 d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_4 d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega)))\\
&\qquad+ [a_5 d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_{n+1}(\omega))) + a_6 d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&\qquad+ a_7 d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) + a_8 d(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega)))]\\
&\qquad+ [sa_9(d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega))) + d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_{n+1}(\omega))))\\
&\qquad+ sa_{10}(d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))) + d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))))\\
&\qquad+ sa_{11}(d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))))\\
&\qquad+ sa_{12}(d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))) + d(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))))]
\end{aligned}$$

$$+ a_{17}d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_n(\omega))) + a_{18}d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_n(\omega)))$$
$$+ a_{19}d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) + a_{20}d(g(\omega,\rho_{n-1}(\omega)), g(\omega,\rho_n(\omega))).$$

Hence, we obtain that

$$\begin{aligned} d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))) \preceq\ & (a_1 + sa_9 + a_{17})d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_n(\omega))) \\ & + (a_2 + sa_{10} + a_{18})d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_n(\omega))) \\ & + (a_3 + sa_{11} + a_{19})d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) \\ & + (a_4 + sa_{12} + a_{20})d(g(\omega,\rho_{n-1}(\omega)), g(\omega,\rho_n(\omega))) \\ & + (a_5 + sa_9)d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))) \\ & + (a_6 + sa_{10})d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))) \\ & + (a_7 + sa_{11})d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) \\ & + (a_8 + sa_{12})d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))). \end{aligned} \tag{3.2}$$

Similarly, we can prove that

$$\begin{aligned} d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))) \preceq\ & (a_1 + sa_9 + a_{17})d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_n(\omega))) \\ & + (a_2 + sa_{10} + a_{18})d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) \\ & + (a_3 + sa_{11} + a_{19})d(g(\omega,\rho_{n-1}(\omega)), g(\omega,\rho_n(\omega))) \\ & + (a_4 + sa_{12} + a_{20})d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_n(\omega))) \\ & + (a_5 + sa_9)d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))) \\ & + (a_6 + sa_{10})d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) \\ & + (a_7 + sa_{11})d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))) \\ & + (a_8 + sa_{12})d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) \preceq\ & (a_1 + sa_9 + a_{17})d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) \\ & + (a_2 + sa_{10} + a_{18})d(g(\omega,\rho_{n-1}(\omega)), g(\omega,\rho_n(\omega))) \\ & + (a_3 + sa_{11} + a_{19})d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_n(\omega))) \\ & + (a_4 + sa_{12} + a_{20})d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_n(\omega))) \\ & + (a_5 + sa_9)d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) \\ & + (a_6 + sa_{10})d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))) \\ & + (a_7 + sa_{11})d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))) \\ & + (a_8 + sa_{12})d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))) \preceq\ & (a_1 + sa_9 + a_{17})d(g(\omega,\rho_{n-1}(\omega)), g(\omega,\rho_n(\omega))) \\ & + (a_2 + sa_{10} + a_{18})d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_n(\omega))) \\ & + (a_3 + sa_{11} + a_{19})d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_n(\omega))) \\ & + (a_4 + sa_{12} + a_{20})d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) \\ & + (a_5 + sa_9)d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))) \\ & + (a_6 + sa_{10})d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))) \\ & + (a_7 + sa_{11})d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))) \\ & + (a_8 + sa_{12})d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))). \end{aligned} \tag{3.5}$$

Put

$$d_n = d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))) + d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))) + d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) + d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))).$$

Uniting (3.2)-(3.5), ones assert that

$$\begin{aligned} d_n \preceq\ & (a_1+a_2+a_3+a_4+sa_9+sa_{10}+sa_{11}+sa_{12}+a_{17}+a_{18}+a_{19}+a_{20})d_{n-1} \\ & + (a_5+a_6+a_7+a_8+sa_9+sa_{10}+sa_{11}+sa_{12})d_n. \end{aligned} \tag{3.6}$$

Furthermore,

$$\begin{aligned}
& d(g(\omega,\gamma_{n+1}(\omega)), g(\omega,\gamma_n(\omega))) \\
&= d(F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega))), F(\omega,(\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega)))) \\
&\preceq [a_1 d(g(\omega,\gamma_n(\omega)), F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega))) \\
&\quad + a_2 d(g(\omega,\zeta_n(\omega)), F(\omega,(\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega))) \\
&\quad + a_3 d(g(\omega,\xi_n(\omega)), F(\omega,(\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega))) \\
&\quad + a_4 d(g(\omega,\rho_n(\omega)), F(\omega,(\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega)))] \\
&\quad + [a_5 d(g(\omega,\gamma_{n-1}(\omega)), F(\omega,(\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega))) \\
&\quad + a_6 d(g(\omega,\zeta_{n-1}(\omega)), F(\omega,(\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega),\gamma_{n-1}(\omega))) \\
&\quad + a_7 d(g(\omega,\xi_{n-1}(\omega)), F(\omega,(\xi_{n-1}(\omega),\rho_{n-1}(\omega),\gamma_{n-1}(\omega),\zeta_{n-1}(\omega))) \\
&\quad + a_8 d(g(\omega,\rho_{n-1}(\omega)), F(\omega,(\rho_{n-1}(\omega),\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega)))] \\
&\quad + [a_9 d(g(\omega,\gamma_n(\omega)), F(\omega,(\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega))) \\
&\quad + a_{10} d(g(\omega,\zeta_n(\omega)), F(\omega,(\zeta_{n-1}(\omega),\xi_{n-1}(\omega),\rho_{n-1}(\omega),\gamma_{n-1}(\omega))) \\
&\quad + a_{11} d(g(\omega,\xi_n(\omega)), F(\omega,(\xi_{n-1}(\omega),\rho_{n-1}(\omega),\gamma_{n-1}(\omega),\zeta_{n-1}(\omega))) \\
&\quad + a_{12} d(g(\omega,\rho_n(\omega)), F(\omega,(\rho_{n-1}(\omega),\gamma_{n-1}(\omega),\zeta_{n-1}(\omega),\xi_{n-1}(\omega)))] \\
&\quad + [a_{13} d(g(\omega,\gamma_{n-1}(\omega)), F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega))) \\
&\quad + a_{14} d(g(\omega,\zeta_{n-1}(\omega)), F(\omega,(\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega))) \\
&\quad + a_{15} d(g(\omega,\xi_{n-1}(\omega)), F(\omega,(\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega))) \\
&\quad + a_{16} d(g(\omega,\rho_{n-1}(\omega)), F(\omega,\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega)))] \\
&\quad + [a_{17} d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n-1}(\omega))) + a_{18} d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n-1}(\omega))) \\
&\quad + a_{19} d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n-1}(\omega))) + a_{20} d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n-1}(\omega)))] \\
&\preceq [a_1 d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))) + a_2 d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))) \\
&\quad + a_3 d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) + a_4 d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))) \\
&\quad + [a_5 d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_n(\omega))) + a_6 d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_n(\omega))) \\
&\quad + a_7 d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) + a_8 d(g(\omega,\rho_{n-1}(\omega)), g(\omega,\rho_n(\omega)))] \\
&\quad + [a_9 d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_n(\omega))) + a_{10} d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_n(\omega))) \\
&\quad + a_{11} d(g(\omega,\xi_n(\omega)), g(\omega,\xi_n(\omega))) + a_{12} d(g(\omega,\rho_n(\omega)), g(\omega,\rho_n(\omega)))] \\
&\quad + [a_{13} d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_{n+1}(\omega))) + a_{14} d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_{n+1}(\omega))) \\
&\quad + a_{15} d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_{n+1}(\omega))) + a_{16} d(g(\omega,\rho_{n-1}(\omega)), g(\omega,\rho_{n+1}(\omega)))] \\
&\quad + a_{17} d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n-1}(\omega))) + a_{18} d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n-1}(\omega))) \\
&\quad + a_{19} d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n-1}(\omega))) + a_{20} d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n-1}(\omega))) \\
&\preceq [a_1 d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))) + a_2 d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))) \\
&\quad + a_3 d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) + a_4 d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))) \\
&\quad + [a_5 d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_n(\omega))) + a_6 d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_n(\omega))) \\
&\quad + a_7 d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) + a_8 d(g(\omega,\rho_{n-1}(\omega)), g(\omega,\rho_n(\omega)))] \\
&\quad + [sa_{13}(d(g(\omega,\gamma_{n-1}(\omega)), g(\omega,\gamma_n(\omega))) + d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega)))) \\
&\quad + sa_{14}(d(g(\omega,\zeta_{n-1}(\omega)), g(\omega,\zeta_n(\omega))) + d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))))
\end{aligned}$$

$$
\begin{aligned}
&+ sa_{15}(d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))))\\
&+ sa_{16}(d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))) + d(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))))]\\
&+ a_{17}d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega))) + a_{18}d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&+ a_{19}d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_{20}d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))).
\end{aligned}
$$

Accordingly, it is clear that

$$
\begin{aligned}
&d(g(\omega,\gamma_{n+1}(\omega)),g(\omega,\gamma_n(\omega)))\\
&\quad\preceq (a_5 + sa_{13} + a_{17})d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega)))\\
&\qquad + (a_6 + sa_{14} + a_{18})d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&\qquad + (a_7 + sa_{15} + a_{19})d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega)))\\
&\qquad + (a_8 + sa_{16} + a_{20})d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega)))\\
&\qquad + (a_1 + sa_{13})d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_{n+1}(\omega))) + (a_2 + sa_{14})d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&\qquad + (a_3 + sa_{15})d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) + (a_4 + sa_{16})d(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))).
\end{aligned}
\tag{3.7}
$$

Similarly, we can prove that

$$
\begin{aligned}
d(g(\omega,\zeta_{n+1}(\omega)),g(\omega,\zeta_n(\omega))) \preceq\ & (a_5 + sa_{13} + a_{17})d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&+ (a_5 + sa_{14} + a_{18})d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega)))\\
&+ (a_6 + sa_{15} + a_{19})d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega)))\\
&+ (a_7 + sa_{16} + a_{20})d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega)))\\
&+ (a_1 + sa_{13})d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&+ (a_2 + sa_{14})d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega)))\\
&+ (a_3 + sa_{15})d(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega)))\\
&+ (a_4 + sa_{16})d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_{n+1}(\omega))),
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi_n(\omega))) \preceq\ & (a_5 + sa_{13} + a_{17})d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega)))\\
&+ (a_6 + sa_{14} + a_{18})d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega)))\\
&+ (a_7 + sa_{15} + a_{19})d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega)))\\
&+ (a_8 + sa_{16} + a_{20})d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&+ (a_1 + sa_{13})d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega)))\\
&+ (a_2 + sa_{14})d(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega)))\\
&+ (a_3 + sa_{15})d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_{n+1}(\omega)))\\
&+ (a_4 + sa_{16})d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))),
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
d(g(\omega,\rho_{n+1}(\omega)),g(\omega,\rho_n(\omega))) \preceq\ & (a_5 + sa_{13} + a_{17})d(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega)))\\
&+ (a_6 + sa_{14} + a_{18})d(g(\omega,\gamma_{n-1}(\omega)),g(\omega,\gamma_n(\omega)))\\
&+ (a_7 + sa_{15} + a_{19})d(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega)))\\
&+ (a_8 + sa_{16} + a_{20})d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega)))\\
&+ (a_1 + sa_{13})d(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega)))\\
&+ (a_2 + sa_{14})d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_{n+1}(\omega)))\\
&+ (a_3 + sa_{15})d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&+ (a_4 + sa_{16})d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))).
\end{aligned}
\tag{3.10}
$$

Uniting (3.7)-(3.10) one gets that

$$\begin{aligned} d_n \preceq\ & (a_5 + a_6 + a_7 + a_8 + sa_{13} + sa_{14} + sa_{15} + sa_{16} + a_{17} + a_{18} + a_{19} + a_{20})d_{n-1} \\ & + (a_1 + a_2 + a_3 + a_4 + sa_{13} + sa_{14} + sa_{15} + sa_{16})d_n. \end{aligned} \tag{3.11}$$

By using (3.6) and (3.11), it is easy to see that

$$\begin{aligned} 2d_n \preceq\ & (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + sa_9 + sa_{10} + sa_{11} + sa_{12} + sa_{13} + sa_{14} + sa_{15} \\ & + sa_{16} + 2(a_{17} + a_{18} + a_{19} + a_{20}))d_{n-1} + (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + sa_{10} \\ & + sa_{11} + sa_{12} + sa_{13} + sa_{14} + sa_{15} + sa_{16})d_n. \end{aligned}$$

Put $k_1 = a_{17} + a_{18} + a_{19} + a_{20}$ and $k = \sum_{i=1}^{8} a_i + \sum_{i=9}^{16} sa_i$, then

$$(2e - k)d_n \preceq (2k_1 + k)d_{n-1}. \tag{3.12}$$

Because of $(s+1)\rho(\sum_{i=1}^{8} a_i) + s(s+1)\rho(\sum_{i=9}^{16} sa_i) + 2s\rho(k_1) < 2$ and $s \geqslant 1$, it is clear that

$$\rho(\sum_{i=1}^{8} a_i + \sum_{i=9}^{16} sa_i) \leqslant \rho(\sum_{i=1}^{8} a_i) + s\rho(\sum_{i=9}^{16} sa_i) < 2.$$

Then by Lemma 2.4 and Lemma 2.5, it follows that $2e - k$ is invertible. Furthermore, $(2e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}$. Multiplying in both sides of (3.12) by $(2e - k)^{-1}$, we obtain

$$d_n \preceq (2e - k)^{-1}(2k_1 + k)d_{n-1}. \tag{3.13}$$

Denote $h = (2e - k)^{-1}(2k_1 + k)$, then by (3.13) we get

$$d_n \preceq hd_{n-1} \preceq \cdots \preceq h^n d_0. \tag{3.14}$$

Note by Lemma 2.4 that

$$\rho\Big(\sum_{i=0}^{n} \frac{k^i}{2^{i+1}}\Big) \leqslant \sum_{i=0}^{n} \rho\Big(\frac{k^i}{2^{i+1}}\Big) \leqslant \sum_{i=0}^{n} \Big(\frac{[\rho(k^i)]}{2^{i+1}}\Big),$$

so by Lemma 2.8 it leads to

$$\rho\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big) \leqslant \sum_{i=0}^{\infty} \Big(\frac{[\rho(k^i)]}{2^{i+1}}\Big).$$

Because $a_i a_j = a_j a_i (i, j = 1, \dots, 20)$ implies k_1 commutes k, we have

$$\begin{aligned} (2e - k)^{-1}(2k_1 + k) &= \Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big)(2k_1 + k) \\ &= 2\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big)k_1 + \Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big)k \\ &= 2k_1\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big) + k\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big) \\ &= (2k_1 + k)\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big) \\ &= (2k_1 + k)(2e - k)^{-1}, \end{aligned}$$

that is to say, $(2e-k)^{-1}$ commutes with $2k_1+k$. Note that $(s+1)\rho(\sum_{i=1}^{8} a_i) + s(s+1)\rho(\sum_{i=9}^{16} s a_i) + 2s\rho(k_1) < 2$ means $2s\rho(k_1) + (s+1)\rho(k) < 2$, then by Lemma 2.5 we again

$$\begin{aligned}
\rho(h) &= \rho((2e-k)^{-1}(2k_1+k)) \\
&\leqslant \rho((2e-k)^{-1})\rho(2k_1+k) \\
&\leqslant \rho\left(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\right)(2\rho(k_1+\rho(k))) \\
&\leqslant \sum_{i=0}^{n}\left(\frac{[\rho(k^i)]}{2^{i+1}}\right)(2\rho(k_1+\rho(k))) \\
&\leqslant \frac{1}{2-\rho(k)}(2\rho(k_1+\rho(k))) < \frac{1}{s} \leqslant 1,
\end{aligned}$$

which establishes that $e-h$ is invertible and $\|h^n\| \to 0$ as $n \to \infty$. Thus for all $m > n \geqslant 1$, one has

$$\begin{aligned}
d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_m(\omega))) &\preceq s d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_{n+1}(\omega))) \\
&\quad + s^2 d(g(\omega,\gamma_{n+1}(\omega)), g(\omega,\gamma_{n+2}(\omega))) \\
&\quad \vdots \\
&\quad + s^{m-n} d(g(\omega,\gamma_{m-1}(\omega)), g(\omega,\gamma_m(\omega))), \\
d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_m(\omega))) &\preceq s d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_{n+1}(\omega))) \\
&\quad + s^2 d(g(\omega,\zeta_{n+1}(\omega)), g(\omega,\zeta_{n+2}(\omega))) \\
&\quad \vdots \\
&\quad + s^{m-n} d(g(\omega,\zeta_{m-1}(\omega)), g(\omega,\zeta_m(\omega))), \\
d(g(\omega,\xi_n(\omega)), g(\omega,\xi_m(\omega))) &\preceq s d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) \\
&\quad + s^2 d(g(\omega,\xi_{n+1}(\omega)), g(\omega,\xi_{n+2}(\omega))) \\
&\quad \vdots \\
&\quad + s^{m-n} d(g(\omega,\xi_{m-1}(\omega)), g(\omega,\xi_m(\omega))), \\
d(g(\omega,\rho_n(\omega)), g(\omega,\rho_m(\omega))) &\preceq s d(g(\omega,\rho_n(\omega)), g(\omega,\rho_{n+1}(\omega))) \\
&\quad + s^2 d(g(\omega,\rho_{n+1}(\omega)), g(\omega,\rho_{n+2}(\omega))) \\
&\quad \vdots \\
&\quad + s^{m-n} d(g(\omega,\rho_{m-1}(\omega)), g(\omega,\rho_m(\omega))).
\end{aligned}$$

Now, by (3.14) and $s\rho(h) < 1$, it follows that

$$\begin{aligned}
& d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma_m(\omega))) + d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta_m(\omega))) \\
&\quad + d(g(\omega,\xi_n(\omega)), g(\omega,\xi_m(\omega))) + d(g(\omega,\rho_n(\omega)), g(\omega,\rho_m(\omega))) \\
&\preceq s d_n + s^2 d_{n+1} + \cdots + s^{m-n} d_{m-1} \\
&\preceq s h^n d_0 + s^2 h^{n+1} d_0 + \cdots + s^{m-n} h^{m-1} d_0 \\
&= (s h^n + s^2 h^{n+1} + \cdots + s^{m-n} h^{m-1}) d_0 \\
&= s h^n (e + sh + (sh)^2 + \cdots + (sh)^{m-n-1}) d_0 \\
&\preceq (e - sh)^{-1} s h^n d_0.
\end{aligned} \tag{3.15}$$

Owing to

$$\|(e-sh)^{-1} s h^n d_0\| \leqslant \|(e-sh)^{-1}\| s \|h^n\| \|d_0\| \to 0, (n \to \infty),$$

we have $e - sh^{-1}sh^n d_0 \to 0$ $(n \to \infty)$. According to Lemma 2.6, and for any $c \gg \theta$, there exists N_0 such that for all $n > N_0$, $(e - sh)^{-1}sh^n d_0 \ll c$. Furthermore, from (3.15) and for any $m > n > N_0$, it follows that

$$\begin{aligned} &d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_m(\omega))) + d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_m(\omega))) \\ &\quad + d(g(\omega,\xi_n(\omega)),g(\omega,\xi_m(\omega))) + d(g(\omega,\rho_n(\omega)),g(\omega,\rho_m(\omega))) \ll c, \end{aligned}$$

which implies that

$$\begin{aligned} &d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma_m(\omega))) \ll c, && d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_m(\omega))) \ll c, \\ &d(g(\omega,\xi_n(\omega)),g(\omega,\xi_m(\omega))) \ll c, && d(g(\omega,\rho_n(\omega)),g(\omega,\rho_m(\omega))) \ll c. \end{aligned}$$

Hence, $\{g(\omega,\gamma_n(\omega))\}$, $\{g(\omega,\zeta_n(\omega))\}$, $\{g(\omega,\xi_n(\omega))\}$, $\{g(\omega,\rho_n(\omega))\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist $\gamma^*(\omega)$, $\zeta^*(\omega)$, $\xi^*(\omega)$ and $\rho^*(\omega) \in X$ for all $\omega \in \Omega$ such that

$$\begin{aligned} &g(\omega,(\gamma_n(\omega))) \to g(\omega,(\gamma^*(\omega))), && g(\omega,(\zeta_n(\omega))) \to g(\omega,(\zeta^*(\omega))), \\ &g(\omega,(\xi_n(\omega))) \to g(\omega,(\xi^*(\omega))), && g(\omega,(\rho_n(\omega))) \to g(\omega,(\rho^*(\omega))), \end{aligned}$$

as $n \to \infty$. Moreover, note that

$$\begin{aligned} &d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),g(\omega,\gamma^*(\omega))) \\ &\quad \preceq s(d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),g(\omega,\gamma_{n+1}(\omega))) + d(g(\omega,\gamma_{n+1}(\omega)),g(\omega,\gamma^*(\omega)))) \\ &\quad = s(d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega)))) \\ &\qquad + sd(g(\omega,\gamma_{n+1}(\omega)),g(\omega,\gamma^*(\omega)))) \\ &\quad \preceq s[a_1 d(g(\omega,\gamma^*(\omega)),F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))) \\ &\qquad + a_2 d(g(\omega,\zeta^*(\omega)),F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))) \\ &\qquad + a_3 d(g(\omega,\xi^*(\omega)),F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))) \\ &\qquad + a_4 d(g(\omega,\rho^*(\omega)),F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega)))] \\ &\qquad + s[a_5 d(g(\omega,\gamma_n(\omega)),F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega))) \\ &\qquad + a_6 d(g(\omega,\zeta_n(\omega)),F(\omega,(\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega))) \\ &\qquad + a_7 d(g(\omega,\xi_n(\omega)),F(\omega,(\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega))) \\ &\qquad + a_8 d(g(\omega,\rho_n(\omega)),F(\omega,(\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega)))] \\ &\qquad + s[a_9 d(g(\omega,\gamma^*(\omega)),F(\omega,(\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega))) \\ &\qquad + a_{10} d(g(\omega,\zeta^*(\omega)),F(\omega,(\zeta_n(\omega),\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega))) \\ &\qquad + a_{11} d(g(\omega,\xi^*(\omega)),F(\omega,(\xi_n(\omega),\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega))) \\ &\qquad + a_{12} d(g(\omega,\rho^*(\omega)),F(\omega,(\rho_n(\omega),\gamma_n(\omega),\zeta_n(\omega),\xi_n(\omega)))] \\ &\qquad + s[a_{13} d(g(\omega,\gamma_n(\omega)),F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))) \\ &\qquad + a_{14} d(g(\omega,\zeta_n(\omega)),F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))) \\ &\qquad + a_{15} d(g(\omega,\xi_n(\omega)),F(\omega,(\xi^*\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))) \\ &\qquad + a_{16} d(g(\omega,\rho_n(\omega)),F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega)))] \\ &\qquad + s[a_{17} d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_n(\omega))) + a_{18} d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_n(\omega))) \\ &\qquad + a_{19} d(g(\omega,\xi^*(\omega)),g(\omega,\xi_n(\omega))) + a_{20} d(g(\omega,\rho^*(\omega)),g(\omega,\rho_n(\omega)))] \\ &\qquad + sd(g(\omega,\gamma_{n+1}(\omega)),g(\omega,\gamma^*(\omega)))) \\ &\quad \preceq s[a_1 d(g(\omega,\gamma^*(\omega)),F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))) \\ &\qquad + a_2 d(g(\omega,\zeta^*(\omega)),F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))) \\ &\qquad + a_3 d(g(\omega,\xi^*(\omega)),F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))) \\ &\qquad + a_4 d(g(\omega,\rho^*(\omega)),F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega)))] \\ &\qquad + s[sa_5 d(g(\omega,\gamma_n(\omega)),g(\omega,\gamma^*(\omega))) + sa_5 d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_{n+1}(\omega))) \\ &\qquad + sa_6 d(g(\omega,\zeta_n(\omega)),g(\omega,\zeta^*(\omega))) + sa_6 d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_{n+1}(\omega))) \end{aligned}$$

$$
\begin{aligned}
&+ sa_7 d(g(\omega,\xi_n(\omega)), g(\omega,\xi^*(\omega))) + sa_7 d(g(\omega,\xi^*(\omega)), g(\omega,\xi_{n+1}(\omega)))\\
&+ sa_8 d(g(\omega,\rho_n(\omega)), g(\omega,\rho^*(\omega))) + sa_8 d(g(\omega,\rho^*(\omega)), g(\omega,\rho_{n+1}(\omega)))]\\
&+ s[a_9 d(g(\omega,\gamma^*(\omega)), g(\omega,\gamma_{n+1}(\omega))) + a_{10} d(g(\omega,\zeta^*(\omega)), g(\omega,\zeta_{n+1}(\omega)))\\
&+ a_{11} d(g(\omega,\xi^*(\omega)), g(\omega,\xi_{n+1}(\omega))) + a_{12} d(g(\omega,\rho^*(\omega)), g(\omega,\rho_{n+1}(\omega)))]\\
&+ s[sa_{13} d(g(\omega,\gamma_n(\omega)), g(\omega,\gamma^*(\omega))) + sa_{13} d(g(\omega,\gamma^*(\omega)), F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega)))\\
&+ sa_{14} d(g(\omega,\zeta_n(\omega)), g(\omega,\zeta^*(\omega))) + sa_{14} d(g(\omega,\zeta^*(\omega)), F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega)))\\
&+ sa_{15} d(g(\omega,\xi_n(\omega)), g(\omega,\xi^*(\omega))) + sa_{15} d(g(\omega,\xi^*(\omega)), F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega)))\\
&+ sa_{16} d(g(\omega,\rho_n(\omega)), g(\omega,\rho^*(\omega))) + sa_{16} d(g(\omega,\rho^*(\omega)), F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega)))]\\
&+ s[a_{17} d(g(\omega,\gamma^*(\omega)), g(\omega,\gamma_n(\omega))) + a_{18} d(g(\omega,\zeta^*(\omega)), g(\omega,\zeta_n(\omega)))\\
&+ a_{19} d(g(\omega,\xi^*(\omega)), g(\omega,\xi_n(\omega))) + a_{20} d(g(\omega,\rho^*(\omega)), g(\omega,\rho_n(\omega)))]\\
&+ sd(g(\omega,\gamma_{n+1}(\omega)), g(\omega,\gamma^*(\omega)))).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))), g(\omega,\gamma^*(\omega)))\\
&\quad\preceq (sa_1 + s^2a_{13}) d(g(\omega,\gamma^*(\omega)), F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))))\\
&\qquad + (sa_2 + s^2a_{14}) d(g(\omega,\zeta^*(\omega)), F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))))\\
&\qquad + (sa_3 + s^2a_{15}) d(g(\omega,\xi^*(\omega)), F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))))\\
&\qquad + (sa_4 + s^2a_{16}) d(g(\omega,\rho^*(\omega)), F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))))\\
&\qquad + (s^2a_5 + s^2a_{13} + sa_{17}) d(g(\omega,\gamma^*(\omega)), g(\omega,\gamma_n(\omega)))\\
&\qquad + (s^2a_6 + s^2a_{14} + sa_{18}) d(g(\omega,\zeta^*(\omega)), g(\omega,\zeta_n(\omega)))\\
&\qquad + (s^2a_7 + s^2a_{15} + sa_{19}) d(g(\omega,\xi^*(\omega)), g(\omega,\xi_n(\omega)))\\
&\qquad + (s^2a_8 + s^2a_{16} + sa_{20}) d(g(\omega,\rho^*(\omega)), g(\omega,\rho_n(\omega)))\\
&\qquad + (s^2a_5 + sa_9 + s) d(g(\omega,\gamma^*(\omega)), g(\omega,\gamma_{n+1}(\omega)))\\
&\qquad + (s^2a_6 + sa_{10}) d(g(\omega,\zeta^*(\omega)), g(\omega,\zeta_{n+1}(\omega)))\\
&\qquad + (s^2a_7 + sa_{11}) d(g(\omega,\xi^*(\omega)), g(\omega,\xi_{n+1}(\omega)))\\
&\qquad + (s^2a_8 + sa_{12}) d(g(\omega,\rho^*(\omega)), g(\omega,\rho_{n+1}(\omega))).
\end{aligned}
\tag{3.16}
$$

Similarly,

$$
\begin{aligned}
&d(F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))), g(\omega,\zeta^*(\omega)))\\
&\quad\preceq (sa_1 + s^2a_{13}) d(g(\omega,\zeta^*(\omega)), F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))))\\
&\qquad + (sa_2 + s^2a_{14}) d(g(\omega,\xi^*(\omega)), F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))))\\
&\qquad + (sa_3 + s^2a_{15}) d(g(\omega,\rho^*(\omega)), F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))))\\
&\qquad + (sa_4 + s^2a_{16}) d(g(\omega,\gamma^*(\omega)), F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))))\\
&\qquad + (s^2a_5 + s^2a_{13} + sa_{17}) d(g(\omega,\zeta^*(\omega)), g(\omega,\zeta_n(\omega)))\\
&\qquad + (s^2a_6 + s^2a_{14} + sa_{18}) d(g(\omega,\xi^*(\omega)), g(\omega,\xi_n(\omega)))\\
&\qquad + (s^2a_7 + s^2a_{15} + sa_{19}) d(g(\omega,\rho^*(\omega)), g(\omega,\rho_n(\omega)))\\
&\qquad + (s^2a_8 + s^2a_{16} + sa_{20}) d(g(\omega,\gamma^*(\omega)), g(\omega,\gamma_n(\omega)))\\
&\qquad + (s^2a_5 + sa_9 + s) d(g(\omega,\zeta^*(\omega)), g(\omega,\zeta_{n+1}(\omega)))\\
&\qquad + (s^2a_6 + sa_{10}) d(g(\omega,\xi^*(\omega)), g(\omega,\xi_{n+1}(\omega)))\\
&\qquad + (s^2a_7 + sa_{11}) d(g(\omega,\rho^*(\omega)), g(\omega,\rho_{n+1}(\omega))\\
&\qquad + (s^2a_8 + sa_{12}) d(g(\omega,\gamma^*(\omega)), g(\omega,\gamma_{n+1}(\omega))).
\end{aligned}
\tag{3.17}
$$

Similarly,

$$\begin{aligned}
&d(F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))),g(\omega,\xi^*(\omega)))\\
&\preceq (sa_1+s^2a_{13})d(g(\omega,\xi^*(\omega)),F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))))\\
&\quad+(sa_2+s^2a_{14})d(g(\omega,\rho^*(\omega)),F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))))\\
&\quad+(sa_3+s^2a_{15})d(g(\omega,\gamma^*(\omega)),F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))))\\
&\quad+(sa_4+s^2a_{16})d(g(\omega,\zeta^*(\omega)),F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))))\\
&\quad+(s^2a_5+s^2a_{13}+sa_{17})d(g(\omega,\xi^*(\omega)),g(\omega,\xi_n(\omega)))\\
&\quad+(s^2a_6+s^2a_{14}+sa_{18})d(g(\omega,\rho^*(\omega)),g(\omega,\rho_n(\omega)))\\
&\quad+(s^2a_7+s^2a_{15}+sa_{19})d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_n(\omega)))\\
&\quad+(s^2a_8+s^2a_{16}+sa_{20})d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_n(\omega)))\\
&\quad+(s^2a_5+sa_9+s)d(g(\omega,\xi^*(\omega)),g(\omega,\xi_{n+1}(\omega)))\\
&\quad+(s^2a_6+sa_{10})d(g(\omega,\rho^*(\omega)),g(\omega,\rho_{n+1}(\omega))\\
&\quad+(s^2a_7+sa_{11})d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_{n+1}(\omega)))\\
&\quad+(s^2a_8+sa_{12})d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_{n+1}(\omega))),
\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}
&d(F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))),g(\omega,\rho^*(\omega)))\\
&\preceq (sa_1+s^2a_{13})d(g(\omega,\rho^*(\omega)),F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))))\\
&\quad+(sa_2+s^2a_{14})d(g(\omega,\gamma^*(\omega)),F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))))\\
&\quad+(sa_3+s^2a_{15})d(g(\omega,\zeta^*(\omega)),F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))))\\
&\quad+(sa_4+s^2a_{16})d(g(\omega,\xi^*(\omega)),F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))))\\
&\quad+(s^2a_5+s^2a_{13}+sa_{17})d(g(\omega,\rho^*(\omega)),g(\omega,\rho_n(\omega)))\\
&\quad+(s^2a_6+s^2a_{14}+sa_{18})d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_n(\omega)))\\
&\quad+(s^2a_7+s^2a_{15}+sa_{19})d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_n(\omega)))\\
&\quad+(s^2a_8+s^2a_{16}+sa_{20})d(g(\omega,\xi^*(\omega)),g(\omega,\xi_n(\omega)))\\
&\quad+(s^2a_5+sa_9+s)d(g(\omega,\rho^*(\omega)),g(\omega,\rho_{n+1}(\omega)))\\
&\quad+(s^2a_6+sa_{10})d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_{n+1}(\omega)))\\
&\quad+(s^2a_7+sa_{11})d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&\quad+(s^2a_8+sa_{12})d(g(\omega,\xi^*(\omega)),g(\omega,\xi_{n+1}(\omega))).
\end{aligned}\tag{3.19}$$

Put

$$\begin{aligned}
\delta = {}&d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),g(\omega,\gamma^*(\omega)))\\
&+d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),g(\omega,\gamma^*(\omega)))\\
&+d(F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))),g(\omega,\xi^*(\omega)))\\
&+d(F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))),g(\omega,\rho^*(\omega))).
\end{aligned}$$

On view of (3.16)-(3.19), we get

$$\begin{aligned}
\delta &\preceq (sa_1+sa_2+sa_3+sa_4+s^2a_{13}+s^2a_{14}+s^2a_{15}+s^2a_{16})\delta\\
&\quad+(s^2a_5+s^2a_6+s^2a_7+s^2a_8+s^2a_{13}+s^2a_{14}+s^2a_{15}+s^2a_{16}+sa_{17}+sa_{18}+sa_{19}+sa_{20})\\
&\quad\times d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_n(\omega)))\\
&\quad+(s^2a_5+s^2a_6+s^2a_7+s^2a_8+s^2a_{13}+s^2a_{14}+s^2a_{15}+s^2a_{16}+sa_{17}+sa_{18}+sa_{19}+sa_{20})
\end{aligned}$$

$$
\begin{aligned}
&\times d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_n(\omega)))\\
&+(s^2a_5+s^2a_6+s^2a_7+s^2a_8+s^2a_{13}+s^2a_{14}+s^2a_{15}+s^2a_{16}+sa_{17}+sa_{18}+sa_{19}+sa_{20})\\
&\times d(g(\omega,\xi^*(\omega)),g(\omega,\xi_n(\omega)))\\
&+(s^2a_5+s^2a_6+s^2a_7+s^2a_8+s^2a_{13}+s^2a_{14}+s^2a_{15}+s^2a_{16}+sa_{17}+sa_{18}+sa_{19}+sa_{20})\\
&\times d(g(\omega,\rho^*(\omega)),g(\omega,\rho_n(\omega)))\\
&+(s^2a_5+s^2a_6+s^2a_7+s^2a_8+sa_9+sa_{10}+sa_{11}+sa_{12}+s)d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_{n+1}(\omega)))\\
&+(s^2a_5+s^2a_6+s^2a_7+s^2a_8+sa_9+sa_{10}+sa_{11}+sa_{12}+s)d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&+(s^2a_5+s^2a_6+s^2a_7+s^2a_8+sa_9+sa_{10}+sa_{11}+sa_{12}+s)d(g(\omega,\xi^*(\omega)),g(\omega,\xi_{n+1}(\omega)))\\
&+(s^2a_5+s^2a_6+s^2a_7+s^2a_8+sa_9+sa_{10}+sa_{11}+sa_{12}+s)d(g(\omega,\rho^*(\omega)),g(\omega,\rho_{n+1}(\omega))).
\end{aligned}
$$

Then

$$
\begin{aligned}
\delta \preceq\ & \frac{B}{e-A}d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_n(\omega)))+\frac{B}{e-A}d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_n(\omega)))\\
&+\frac{B}{e-A}d(g(\omega,\xi^*(\omega)),g(\omega,\xi_n(\omega)))+\frac{B}{e-A}d(g(\omega,\rho^*(\omega)),g(\omega,\rho_n(\omega)))\\
&+\frac{C}{e-A}d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_{n+1}(\omega))+\frac{C}{e-A}d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_{n+1}(\omega)))\\
&+\frac{C}{e-A}d(g(\omega,\xi^*(\omega)),g(\omega,\xi_{n+1}(\omega)))+\frac{C}{e-A}d(g(\omega,\rho^*(\omega)),g(\omega,\rho_{n+1}(\omega))),
\end{aligned}
$$

where $A=sa_1+sa_2+sa_3+sa_4+s^2a_{13}+s^2a_{14}+s^2a_{15}+s^2a_{16}$, $B=s^2a_5+s^2a_6+s^2a_7+s^2a_8+s^2a_{13}+s^2a_{14}+s^2a_{15}+s^2a_{16}+sa_{17}+sa_{18}+sa_{19}+sa_{20}$, $C=s^2a_5+s^2a_6+s^2a_7+s^2a_8+sa_9+sa_{10}+sa_{11}+sa_{12}+s$, $\rho(A)<1$. Since $g(\omega,(\gamma_n(\omega))\to g(\omega,(\gamma^*(\omega))$, $g(\omega,(\zeta_n(\omega))\to g(\omega,(\zeta^*(\omega))$, $g(\omega,(\xi_n(\omega))\to g(\omega,(\xi^*(\omega))$, $g(\omega,(\rho_n(\omega))\to g(\omega,(\rho^*(\omega))$, then Lemma 2.7 follows that for any $c\gg\theta$, there exists N_0 such that for $n>N_0$, we have

$$
\begin{aligned}
&\frac{B}{e-A}d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_n(\omega)))\ll\frac{c}{8}, && \frac{B}{e-A}d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_n(\omega)))\ll\frac{c}{8},\\
&\frac{B}{e-A}d(g(\omega,\xi^*(\omega)),g(\omega,\xi_n(\omega)))\ll\frac{c}{8}, && \frac{B}{e-A}d(g(\omega,\rho^*(\omega)),g(\omega,\rho_n(\omega)))\ll\frac{c}{8},\\
&\frac{C}{e-A}d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma_{n+1}(\omega)))\ll\frac{c}{8}, && \frac{C}{e-A}d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta_{n+1}(\omega)))\ll\frac{c}{8},\\
&\frac{C}{e-A}d(g(\omega,\xi^*(\omega)),g(\omega,\xi_{n+1}(\omega)))\ll\frac{c}{8}, && \frac{C}{e-A}d(g(\omega,\rho^*(\omega)),g(\omega,\rho_{n+1}(\omega)))\ll\frac{c}{8}.
\end{aligned}
$$

Hence,

$$
\delta\ll\frac{c}{8}+\frac{c}{8}+\frac{c}{8}+\frac{c}{8}+\frac{c}{8}+\frac{c}{8}+\frac{c}{8}+\frac{c}{8}=c.
$$

Now, according to Lemma 2.4, it follows that $\delta=\theta$, that is,

$$
\begin{aligned}
\theta=\ & d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),g(\omega,\gamma^*(\omega)))\\
&+d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),g(\omega,\gamma^*(\omega)))\\
&+d(F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))),g(\omega,\xi^*(\omega)))\\
&+d(F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))),g(\omega,\rho^*(\omega))),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),g(\omega,\gamma^*(\omega)))=\theta,\\
&d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),g(\omega,\gamma^*(\omega)))=\theta,
\end{aligned}
$$

$$d(F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))),g(\omega,\xi^*(\omega)))=\theta,$$
$$d(F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))),g(\omega,\rho^*(\omega)))=\theta.$$

Thus,

$$\begin{aligned}
d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega)))) &= g(\omega,\gamma^*(\omega)),\\
d(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega)))) &= g(\omega,\zeta^*(\omega)),\\
d(F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega)))) &= g(\omega,\xi^*(\omega)),\\
d(F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega)))) &= g(\omega,\rho^*(\omega)).
\end{aligned}$$

Therefore $(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))$ is a quadruple random coincidence point of F and g for all $\omega\in\Omega$. □

Corollary 3.2. *Let (X,d) be a separable cone b-metric space over Banach algebra $\mathcal{A}$ and P be a cone in $\mathcal{A}$, $s\geqslant 1$ be a constant, and (Ω,Σ) be a measurable space. Suppose that the mappings $F:\Omega\times X^4\to X$, $g:\Omega\times X\to X$ satisfy the following contractive condition:*

$$\begin{aligned}
d(F(\omega,(x_1,x_2,x_3,x_4)),F(\omega,(x_1^*,x_2^*,x_3^*,x_4^*))) \preceq\ & a_1 d(g(\omega,x_1),g(\omega,x_1^*))+a_2 d(g(\omega,x_2),g(\omega,x_2^*))\\
&+a_3 d(g(\omega,x_3),g(\omega,x_3^*))+a_3 d(g(\omega,x_4),g(\omega,x_4^*))
\end{aligned}$$

for all $x_1,x_2,x_3,x_4,x_1^*,x_2^*,x_3^*,x_4^*$, where $a_1,a_2,a_3,a_4\in P$ are generalized Lipschitz constants with $p(a_1+a_2+a_3+a_4)<\frac{1}{s}$, $F(\cdot,v)$, $g(\cdot,x)$ are measurable for $v\in x^4$ and $x\in X$, respectively, and $F(\omega\times X^4)\subseteq g(\omega\times X)$ is complete subspace of X for each $\omega\in\Omega$, then there are mappings $\gamma,\zeta,\xi,\rho:\Omega\to X$ such that

$$\begin{aligned}
F(\omega,(\gamma(\omega),\zeta(\omega),\xi(\omega),\rho(\omega))) &= g(\omega,\gamma(\omega)),\\
F(\omega,(\zeta(\omega),\xi(\omega),\rho(\omega),\gamma(\omega))) &= g(\omega,\zeta(\omega)),\\
F(\omega,(\xi(\omega),\rho(\omega),\gamma(\omega),\zeta(\omega))) &= g(\omega,\xi(\omega)),\\
F(\omega,(\rho(\omega),\gamma(\omega),\zeta(\omega),\xi(\omega))) &= g(\omega,\rho(\omega))
\end{aligned}$$

for all $\omega\in\Omega$, that is F and g have a quadruple random coincidence point.

Theorem 3.3. *In addition to hypotheses of Theorem 3.1, if F and g are w-compatible, then F and g have a unique quadruple common fixed point. Moreover, a quadruple random common fixed point of F and g is of the form $(\gamma^*(\omega),\gamma^*(\omega),\gamma^*(\omega),\gamma^*(\omega))\in X$ for all $\omega\in\Omega$.*

Proof. By Theorem 3.1, F and g have quadruple random coincidence point $(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))$. Then $(g(\omega,\gamma^*(\omega)),g(\omega,\zeta^*(\omega)),g(\omega,\xi^*(\omega)),g(\omega,\rho^*(\omega)))$ is quadruple random point of coincidence of F and g such that

$$\begin{aligned}
g(\omega,\gamma^*(\omega)) &= F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))),\\
g(\omega,\zeta^*(\omega)) &= F(\omega,(\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega))),\\
g(\omega,\xi^*(\omega)) &= F(\omega,(\xi^*(\omega),\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega))),\\
g(\omega,\rho^*(\omega)) &= F(\omega,(\rho^*(\omega),\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega))).
\end{aligned}$$

First, we shall show that the quadruple random point of coincidence is unique. Suppose that F and g have another quadruple random point of coincidence

$$(g(\omega,\gamma^{**}(\omega)),g(\omega,\zeta^{**}(\omega)),g(\omega,\xi^{**}(\omega)),g(\omega,\rho^{**}(\omega))),$$

such that

$$g(\omega,\gamma^{**}(\omega))=F(\omega,(\gamma^{**}(\omega),\zeta^{**}(\omega),\xi^{**}(\omega),\rho^{**}(\omega))),$$

$$\begin{aligned}
g(\omega,\zeta^{**}(\omega)) &= F(\omega,(\zeta^{**}(\omega),\xi^{**}(\omega),\rho^{**}(\omega),\gamma^{**}(\omega))),\\
g(\omega,\xi^{**}(\omega)) &= F(\omega,(\xi^{**}(\omega),\rho^{**}(\omega),\gamma^{**}(\omega),\zeta^{**}(\omega))),\\
g(\omega,\rho^{**}(\omega)) &= F(\omega,(\rho^{**}(\omega),\gamma^{**}(\omega),\zeta^{**}(\omega),\xi^{**}(\omega))),
\end{aligned}$$

where $(\gamma^{**}(\omega),\zeta^{**}(\omega),\xi^{**}(\omega),\rho^{**}(\omega)) \in X^4$ for all $\omega \in \Omega$. Then we have

$$\begin{aligned}
&d(g(\omega,\gamma^{*}(\omega)),g(\omega,\gamma^{**}(\omega)))\\
&\quad= d(F(\omega,(\gamma^{*}(\omega),\zeta^{*}(\omega),\xi^{*}(\omega),\rho^{*}(\omega))),F(\omega,(\gamma^{**}(\omega),\zeta^{**}(\omega),\xi^{**}(\omega),\rho^{**}(\omega))))\\
&\quad\preceq [a_1 d(g(\omega,\gamma^{*}(\omega)),F(\omega,(\gamma^{*}(\omega),\zeta^{*}(\omega),\xi^{*}(\omega),\rho^{*}(\omega)))\\
&\qquad + a_2 d(g(\omega,\zeta^{*}(\omega)),F(\omega,(\zeta^{*}(\omega),\xi^{*}(\omega),\rho^{*}(\omega),\gamma^{*}(\omega)))\\
&\qquad + a_3 d(g(\omega,\xi^{*}(\omega)),F(\omega,(\xi^{*}(\omega),\rho^{*}(\omega),\gamma^{*}(\omega),\zeta^{*}(\omega)))\\
&\qquad + a_4 d(g(\omega,\rho^{*}(\omega)),F(\omega,(\rho^{*}(\omega),\gamma^{*}(\omega),\zeta^{*}(\omega),\xi^{*}(\omega)))]\\
&\qquad + [a_5 d(g(\omega,\gamma^{**}(\omega)),F(\omega,(\gamma^{**}(\omega),\zeta^{**}(\omega),\xi^{**}(\omega),\rho^{**}(\omega)))\\
&\qquad + a_6 d(g(\omega,\zeta^{**}(\omega)),F(\omega,(\zeta^{**}(\omega),\xi^{**}(\omega),\rho^{**}(\omega),\gamma^{**}(\omega)))\\
&\qquad + a_7 d(g(\omega,\xi^{**}(\omega)),F(\omega,(\xi^{**}(\omega),\rho^{**}(\omega),\gamma^{**}(\omega),\zeta^{**}(\omega)))\\
&\qquad + a_8 d(g(\omega,\rho^{**}(\omega)),F(\omega,(\rho^{**}(\omega),\gamma^{**}(\omega),\zeta^{**}(\omega),\xi^{**}(\omega)))]\\
&\qquad + [a_9 d(g(\omega,\gamma^{*}(\omega)),F(\omega,(\gamma^{**}(\omega),\zeta^{**}(\omega),\xi^{**}(\omega),\rho^{**}(\omega)))\\
&\qquad + a_{10} d(g(\omega,\zeta^{*}(\omega)),F(\omega,(\zeta^{**}(\omega),\xi^{**}(\omega),\rho^{**}(\omega),\gamma^{**}(\omega)))\\
&\qquad + a_{11} d(g(\omega,\xi^{*}(\omega)),F(\omega,(\xi^{**}(\omega),\rho^{**}(\omega),\gamma^{**}(\omega),\zeta^{**}(\omega)))\\
&\qquad + a_{12} d(g(\omega,\rho^{*}(\omega)),F(\omega,(\rho^{**}(\omega),\gamma^{**}(\omega),\zeta^{**}(\omega),\xi^{**}(\omega)))]\\
&\qquad + [a_{13} d(g(\omega,\gamma^{**}(\omega)),F(\omega,(\gamma^{*}(\omega),\zeta^{*}(\omega),\xi^{*}(\omega),\rho^{*}(\omega)))\\
&\qquad + a_{14} d(g(\omega,\zeta^{**}(\omega)),F(\omega,(\zeta^{*}(\omega),\xi^{*}(\omega),\rho^{*}(\omega),\gamma^{*}(\omega)))\\
&\qquad + a_{15} d(g(\omega,\xi^{**}(\omega)),F(\omega,(\xi^{*}(\omega),\rho^{*}(\omega),\gamma^{*}(\omega),\zeta^{*}(\omega)))\\
&\qquad + a_{16} d(g(\omega,\rho^{**}(\omega)),F(\omega,(\rho^{*}(\omega),\gamma^{*}(\omega),\zeta^{*}(\omega),\xi^{*}(\omega)))]\\
&\qquad + [a_{17} d(g(\omega,\gamma^{*}(\omega)),g(\omega,\gamma^{**}(\omega))) + a_{18} d(g(\omega,\zeta^{*}(\omega)),g(\omega,\zeta^{**}(\omega)))\\
&\qquad + a_{19} d(g(\omega,\xi^{*}(\omega)),g(\omega,\xi^{**}(\omega))) + a_{20} d(g(\omega,\rho^{*}(\omega)),g(\omega,\rho^{**}(\omega)))]\\
&\quad= [a_1 d(g(\omega,\gamma^{*}(\omega)),g(\omega,\gamma^{*}(\omega))) + a_2 d(g(\omega,\zeta^{*}(\omega)),g(\omega,\zeta^{*}(\omega)))\\
&\qquad + a_3 d(g(\omega,\xi^{*}(\omega)),g(\omega,\xi^{*}(\omega)) + a_4 d(g(\omega,\rho^{*}(\omega)),g(\omega,\rho^{*}(\omega)))]\\
&\qquad + [a_5 d(g(\omega,\gamma^{**}(\omega)),g(\omega,\gamma^{**}(\omega))) + a_6 d(g(\omega,\zeta^{**}(\omega)),g(\omega,\zeta^{**}(\omega)))\\
&\qquad + a_7 d(g(\omega,\xi^{**}(\omega)),g(\omega,\xi^{**}(\omega))) + a_8 d(g(\omega,\rho^{**}(\omega)),g(\omega,\rho^{**}(\omega)))]\\
&\qquad + [a_9 d(g(\omega,\gamma^{*}(\omega)),g(\omega,\gamma^{**}(\omega))) + a_{10} d(g(\omega,\zeta^{*}(\omega)),g(\omega,\zeta^{**}(\omega)))\\
&\qquad + a_{11} d(g(\omega,\xi^{*}(\omega)),g(\omega,\xi^{**}(\omega))) + a_{12} d(g(\omega,\rho^{*}(\omega)),g(\omega,\rho^{**}(\omega)))]\\
&\qquad + [a_{13} d(g(\omega,\gamma^{**}(\omega)),g(\omega,\gamma^{*}(\omega))) + a_{14} d(g(\omega,\zeta^{**}(\omega)),g(\omega,\zeta^{*}(\omega)))\\
&\qquad + a_{15} d(g(\omega,\xi^{**}(\omega)),g(\omega,\xi^{*}(\omega))) + a_{16} d(g(\omega,\rho^{**}(\omega)),g(\omega,\rho^{*}(\omega)))]\\
&\qquad + [a_{17} d(g(\omega,\gamma^{*}(\omega)),g(\omega,\gamma^{**}(\omega))) + a_{18} d(g(\omega,\zeta^{*}(\omega)),g(\omega,\zeta^{**}(\omega)))\\
&\qquad + a_{19} d(g(\omega,\xi^{*}(\omega)),g(\omega,\xi^{**}(\omega))) + a_{20} d(g(\omega,\rho^{*}(\omega)),g(\omega,\rho^{**}(\omega)))].
\end{aligned}$$

Hence,

$$\begin{aligned}
d(g(\omega,\gamma^{*}(\omega)),g(\omega,\gamma^{**}(\omega))) \preceq{}& (a_9+a_{13}+a_{17}) d(g(\omega,\gamma^{*}(\omega)),g(\omega,\gamma^{**}(\omega)))\\
&+ (a_{10}+a_{14}+a_{18}) d(g(\omega,\zeta^{*}(\omega)),g(\omega,\zeta^{**}(\omega)))\\
&+ (a_{11}+a_{15}+a_{19}) d(g(\omega,\xi^{*}(\omega)),g(\omega,\xi^{**}(\omega)))\\
&+ (a_{12}+a_{16}+a_{20}) d(g(\omega,\rho^{*}(\omega)),g(\omega,\rho^{**}(\omega))).
\end{aligned} \tag{3.20}$$

Similarly, we can prove that

$$\begin{aligned} d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega))) \preceq\, &(a_9+a_{13}+a_{17})d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega))) \\ &+(a_{10}+a_{14}+a_{18})d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) \\ &+(a_{11}+a_{15}+a_{19})d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega))) \\ &+(a_{12}+a_{16}+a_{20})d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma^{**}(\omega))), \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) \preceq\, &(a_9+a_{13}+a_{17})d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) \\ &+(a_{10}+a_{14}+a_{18})d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega))) \\ &+(a_{11}+a_{15}+a_{19})d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma^{**}(\omega))) \\ &+(a_{12}+a_{16}+a_{20})d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega))), \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega))) \preceq\, &(a_9+a_{13}+a_{17})d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega))) \\ &+(a_{10}+a_{14}+a_{18})d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma^{**}(\omega))) \\ &+(a_{11}+a_{15}+a_{19})d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega))) \\ &+(a_{12}+a_{16}+a_{20})d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))). \end{aligned} \tag{3.23}$$

By combining (3.20)-(3.23), we get

$$\begin{aligned} &d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma^{**}(\omega)))+d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega))) \\ &\quad +d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega)))+d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega))) \\ &\quad \preceq (a_9+a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+a_{15}+a_{16}+a_{17}+a_{18}+a_{19}+a_{20})(d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma^{**}(\omega))) \\ &\quad +d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega)))+d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega)))+d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega)))). \end{aligned}$$

Set $\lambda = a_9+a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+a_{15}+a_{16}+a_{17}+a_{18}+a_{19}+a_{20}$, and

$$\begin{aligned} \beta &= d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma^{**}(\omega)))+d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega))) \\ &\quad +d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega)))+d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega))). \end{aligned}$$

We have $\beta \preceq \lambda\beta \preceq \cdots \preceq \lambda^n\beta$. Now we get that

$$\rho(\lambda) \leqslant \rho(a_9+\cdots+a_{16})+\rho(a_{17}+a_{18}+a_{19}+a_{20}) < 1,$$

which leads to $\lambda^n \to \theta$ ($n \to \infty$, we claim that, for each $c \gg \theta$, there exists $n_0(c)$ such that $\lambda^n \ll c$ ($n > n_0(c)$). Consequently by Lemma 2.6,

$$\begin{aligned} &d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma^{**}(\omega)))+d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega))) \\ &\quad +d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega)))+d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega)))=\theta. \end{aligned}$$

Hence,

$$\begin{aligned} d(g(\omega,\gamma^*(\omega)),g(\omega,\gamma^{**}(\omega)))&=\theta, & d(g(\omega,\zeta^*(\omega)),g(\omega,\zeta^{**}(\omega)))&=\theta, \\ d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega)))&=\theta, & d(g(\omega,\rho^*(\omega)),g(\omega,\rho^{**}(\omega)))&=\theta, \end{aligned}$$

that is,

$$\begin{aligned} g(\omega,\gamma^*(\omega))&=g(\omega,\gamma^{**}(\omega)), & g(\omega,\zeta^*(\omega))&=g(\omega,\zeta^{**}(\omega)), \\ g(\omega,\xi^*(\omega))&=g(\omega,\xi^{**}(\omega)), & g(\omega,\rho^*(\omega))&=g(\omega,\rho^{**}(\omega)), \end{aligned} \tag{3.24}$$

which implies the uniqueness of the quadruple random point of coincidence of F and g. By a similar way, someone can prove that

$$\begin{aligned} g(\omega,\gamma^*(\omega)) &= g(\omega,\zeta^{**}(\omega)), \quad g(\omega,\zeta^*(\omega)) = g(\omega,\xi^{**}(\omega)),\\ g(\omega,\xi^*(\omega)) &= g(\omega,\rho^{**}(\omega)), \quad g(\omega,\rho^*(\omega)) = g(\omega,\gamma^{**}(\omega)), \end{aligned} \tag{3.25}$$

$$\begin{aligned} g(\omega,\gamma^*(\omega)) &= g(\omega,\xi^{**}(\omega)), \quad g(\omega,\zeta^*(\omega)) = g(\omega,\rho^{**}(\omega)),\\ g(\omega,\xi^*(\omega)) &= g(\omega,\gamma^{**}(\omega)), \quad g(\omega,\rho^*(\omega)) = g(\omega,\zeta^{**}(\omega)), \end{aligned} \tag{3.26}$$

$$\begin{aligned} g(\omega,\gamma^*(\omega)) &= g(\omega,\rho^{**}(\omega)), \quad g(\omega,\zeta^*(\omega)) = g(\omega,\gamma^{**}(\omega))\\ g(\omega,\xi^*(\omega)) &= g(\omega,\zeta^{**}(\omega)), \quad g(\omega,\rho^*(\omega)) = g(\omega,\xi^{**}(\omega)), \end{aligned} \tag{3.27}$$

In view of (3.24)-(3.27), one can assert

$$g(\omega,\gamma^*(\omega)) = g(\omega,\zeta^*(\omega)) = g(\omega,\xi^*(\omega)) = g(\omega,\rho^*(\omega)).$$

In other words, the unique quadruple random point of coincidence of F and g is $(g(\omega,\gamma^*(\omega)), g(\omega,\zeta^*(\omega)), g(\omega,\xi^*(\omega)), g(\omega,\rho^*(\omega)))$.

Let $u(\omega) = g(\omega,\gamma^*) = F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega)))$. Since F and g are w-compatible, then we have

$$\begin{aligned} g(\omega,u(\omega)) &= g(g(\omega,\gamma^*(\omega)))\\ &= g(F(\omega,(\gamma^*(\omega),\zeta^*(\omega),\xi^*(\omega),\rho^*(\omega))))\\ &= F(\omega,(g(\omega,\gamma^*(\omega)),g(\omega,\zeta^*(\omega)),g(\omega,\xi^*(\omega)),g(\omega,\rho^*(\omega))))\\ &= F(\omega,(u(\omega),u(\omega),u(\omega),u(\omega))). \end{aligned}$$

Hence $(u(\omega),u(\omega),u(\omega),u(\omega))$ is the unique quadruple common random fixed point of F and g for all $\omega \in \Omega$. This completes the proof. □

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